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A NOTE TO THE THEORY OF PERIODIC SOLUTIONS OF A PARABOLIC EQUATION

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V. Šťastnová and O. Vejvoda [1] investigated the existence of an ω-periodic solution to the problem

$$u_t = u_{xx} + cu + g(t, x)\,, \tag{1}$$

$$u(t, 0) = h_0(t)\,, \quad u(t, \pi) = h_1(t)\,, \tag{2}$$

where g, h_0, h_1 are ω-periodic in t.

Here a little more general problem, namely that of the existence of an ω-periodic solution to the equation

$$u_t = a(t)\, u_{xx} + c(t)\, u + g(t, x)\,, \quad (t, x) \in \mathbb{R} \times \langle 0, \pi \rangle\,, \tag{1'}$$

with boundary conditions (2) will be studied.

Let us assume that

$$g(t, x)\,, \quad g_x(t, x) \quad \text{are continuous on} \quad \mathbb{R} \times \langle 0, \pi \rangle\,, \tag{3}$$

$$h_0(t)\,, \quad h_1(t)\,, \quad a(t)\,, \quad c(t) \quad \text{are continuous on} \quad \mathbb{R}\,,$$

$$a(t)\,, \quad c(t)\,, \quad g(t, x)\,, \quad h_0(t)\,, \quad h_1(t) \quad \text{are } \omega\text{-periodic in} \quad t\,, \tag{4}$$

$$a(t) > 0\,, \quad t \in \mathbb{R}\,. \tag{5}$$

Performing successively the transformations

$$u(t, x) = v(t, x) \exp\left(\int_0^t c(s)\, \mathrm{d}s\right) \tag{6}$$

and

$$v(t, x) = w(A(t), x)\,, \tag{7}$$

where

$$A(t) = \int_0^t a(s)\,\mathrm{d}s\,,$$

and putting $\tau = A(t)$, we find that the problem (1′), (2) is equivalent to the problem

$$w_\tau = w_{xx} + \hat{g}(\tau, x) \exp\left(-\int_0^{A^{-1}(\tau)} c(s)\,\mathrm{d}s\right), \tag{8}$$

$$\tau \in \mathbb{R}\,, \quad x \in \langle 0, \pi \rangle\,,$$

$$w(\tau, 0) = \tilde{h}_0(\tau) \exp\left(-\int_0^{A^{-1}(\tau)} c(s)\,\mathrm{d}s\right), \tag{9}$$

$$w(\tau, \pi) = \tilde{h}_1(\tau) \exp\left(-\int_0^{A^{-1}(\tau)} c(s)\,\mathrm{d}s\right),$$

$$w(0, x) = w(\hat{\omega}, x) \exp\left(\int_0^{\omega} c(s)\,\mathrm{d}s\right) \tag{10}$$

with

$$\hat{g}(\tau, x) = g(A^{-1}(\tau), x)/a(A^{-1}(\tau))\,,$$

$$\hat{\omega} = A(\omega)\,,$$

$$\tilde{h}_i(\tau) = h_i(A^{-1}(\tau))\,, \quad i = 0, 1\,,$$

where A^{-1} is the inverse to A.

This problem, given by (8), (9), (10), differs from the problem investigated in [1] only in the equation (10). But this difference is unessential and we can proceed quite analogously as in [1]. We have to distinguish two cases:

$$\int_0^{\omega} [c(s) - k^2\, a(s)]\,\mathrm{d}s \neq 0 \quad \text{for all natural} \quad k\text{'s}\,, \tag{11}$$

$$\int_0^{\omega} [c(s) - k_0^2\, a(s)]\,\mathrm{d}s = 0 \quad \text{for some natural} \quad k_0\,. \tag{12}$$

In the former case we obtain the following theorem:

Theorem 1. *Let the assumptions* (3), (4), (5), (11) *be satisfied. Then the problem given by* (1′), (2) *has a unique ω-periodic classical solution, given by*

$$u(t, x) = \int_0^{\pi} \int_0^t [\theta(A(t) - A(\sigma), x - \xi) - \theta(A(t) - A(\sigma), x + \xi)]\,. \tag{13}$$

$$.\, g(\sigma, \xi) \exp\left(\int_\sigma^t c(s)\,\mathrm{d}s\right) \mathrm{d}\sigma\,\mathrm{d}\xi + \int_0^{\pi} [\theta(A(t), x - \xi) - \theta(A(t), x + \xi)]\,.$$

$$. \varphi(\xi) \exp\left(\int_0^t c(s)\,\mathrm{d}s\right) \mathrm{d}\xi - 2\int_0^t h_0(\sigma) \exp\left(\int_\sigma^t c(s)\,\mathrm{d}s\right) \frac{\partial}{\partial x}\theta(A(t) - A(\sigma), x) \,.$$

$$. \, a(\sigma)\,\mathrm{d}\sigma + 2\int_0^t h_1(\sigma) \exp\left(\int_\sigma^t c(s)\,\mathrm{d}s\right) \frac{\partial}{\partial x}\theta(A(t) - A(\sigma), \pi - x)\, a(\sigma)\,\mathrm{d}\sigma \,,$$

where φ is the only solution of the equation

$$(14) \qquad \varphi(x) = \int_0^\pi \int_0^\omega [\theta(A(\omega) - A(\sigma), x - \xi) - \theta(A(\omega) - A(\sigma), x + \xi)] \,.$$

$$. \, g(\sigma, \xi) \exp\left(\int_\sigma^\omega c(s)\,\mathrm{d}s\right) \mathrm{d}\sigma\,\mathrm{d}\xi + \int_0^\pi [\theta(A(\omega), x - \xi) - \theta(A(\omega), x + \xi)] \,.$$

$$. \, \varphi(\xi) \exp\left(\int_0^\omega c(s)\,\mathrm{d}s\right) \mathrm{d}\xi - 2\int_0^\omega h_0(\sigma) \exp\left(\int_0^\omega c(s)\,\mathrm{d}s\right) \frac{\partial}{\partial x}\theta(A(\omega) - A(\sigma), x) \,.$$

$$. \, a(\sigma)\,\mathrm{d}\sigma + 2\int_0^\omega h_1(\sigma) \exp\left(\int_\sigma^\omega c(s)\,\mathrm{d}s\right) \frac{\partial}{\partial x}\theta(A(\omega) - A(\sigma), \pi - x)\, a(\sigma)\,\mathrm{d}\sigma \,,$$

and $\theta(t, x) = \sum_{n=-\infty}^{\infty} \exp\left(-(x + 2n\pi)^2/4t\right)/\sqrt{(4\pi t)}$.

In the latter case we find, again using the same method as in [1], that the ω-periodic solution exists if and only if

$$(15) \qquad 0 = \int_0^\pi \int_0^\omega \exp\left(\int_0^t [a(s)\,k_0^2 - c(s)]\,\mathrm{d}s\right) g(t, x) \sin(k_0 x)\,\mathrm{d}t\,\mathrm{d}x \,+$$

$$+ \, k_0 \int_0^\omega [h_0(t) + (-1)^{k_0+1}\, h_1(t)] \exp\left(\int_0^t [a(s)\,k_0^2 - c(s)]\,\mathrm{d}s\right) a(t)\,\mathrm{d}t \,.$$

We can also derive the following theorem:

Theorem 2. *Let the assumptions* (3), (4), (5), (12) *be satisfied. Then the problem given by* (1′), (2) *has a solution if and only if the condition* (15) *is fulfilled. If this condition is satisfied, then the one-parametric family of solutions is given by* (13), *where* $\varphi(x) = d \sin(k_0 x) + \psi(x)$, $\psi(x)$ *is a particular solution of* (14) *and* d *is an arbitrary constant.*

Remark. Similarly as in [1], the weakly nonlinear problem corresponding to (1′), (2) may be dealt with. Let us only note that then condition (15) immediately yields the form of the bifurcation equation for the constant d.

References

[1] *V. Šťastnová* and *O. Vejvoda:* Periodic solutions of the first boundary value problem for a linear and weakly nonlinear heat equation. Aplikace matematiky 13 (1968), 466—477, 14 (1969), 241.

Souhrn

POZNÁMKA K TEORII PERIODICKÝCH ŘEŠENÍ PARABOLICKÉ ROVNICE

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V této poznámce se vyšetřuje existence ω-periodického klasického řešení rovnice (1′) s okrajovou podmínkou (2) za předpokladu, že funkce g, h_0, h_1, a, c jsou ω-periodické a splňují předpoklady (3), (5). Tato úloha je zobecněním úlohy řešené v [1], a proto i výsledky jsou obdobné. Jestliže $\int_0^\omega [c(s) - k^2 a(s)] \, ds \neq 0$ pro všechna k přirozená, pak uvedená úloha má jediné ω-periodické klasické řešení. Jestliže $\int_0^\omega [c(s) - k_0^2 a(s)] \, . \, ds = 0$ pro nějaké přirozené k_0, pak úloha má řešení právě když platí (15). Analogickým způsobem jako v [1], za použití transformací (6), (7), lze stanovit nutné a postačující podmínky pro existenci ω-periodického řešení i slabě nelineární úlohy.

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