Florian-Alexandru Potra

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AN APPLICATION OF THE INDUCTION METHOD OF V. PTÁK TO THE STUDY OF REGULA FALSI

FLORIAN ALEXANDRU POTRA

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Let $\boldsymbol{R}^p$ denote the p-dimensional Euclidean space, i.e. the set of all vectors $\boldsymbol{r}$ of p real components r_i, $i = 1, \ldots, p$. In $\boldsymbol{R}^p$ we shall consider the relation "$\leqq$" induced by the cone $\boldsymbol{R}^p_+ = \{\boldsymbol{r} \in \boldsymbol{R}^p;\ r_i \geqq 0,\ i = 1, \ldots, p\}$. Let T be a p-dimensional interval of the form $T = \{\boldsymbol{r} \in \boldsymbol{R}^p;\ 0 < r_i < a_i\}$ $i = 1, 2, \ldots, p$, where the numbers a_i $i = 1, \ldots, P$, are supposed to satisfy the inequalities $a_1 \geqq a_2 \geqq \ldots \geqq a_p < 0$. Some of these numbers may be infinite. If ω is a real function of p real variables which maps the p-dimensional interval T into the interval $]0, a_p[= \{r \in \boldsymbol{R};\ 0 < r < a_p\}$, then we may define inductively:

$$\omega^0(r_1, \ldots, r_p) = r_p,\ \omega^{n+1}(r_1, \ldots, r_p) = \omega^n(r_2, \ldots, r_p, \omega(r_1, \ldots, r_p)) \quad n = 0, 1, 2, \ldots;\quad \boldsymbol{r} \in T. \tag{1}$$

Using the above notation we can define the notion of p-dimensional rate of convergence which generalizes the notion of rate of convergence given in [5], [6].

Definition 1. *A function $\omega : T \to\ ]0, a_p[$ is called a p-dimensional rate of convergence if the series $\sigma(r) = \sum_{n=0}^{\infty} \omega^n(r)$ is convergent for any $\boldsymbol{r} \in T$.*

To the p-dimensional rate of convergence ω, let us attach a vector function $\boldsymbol{\omega} : T \to T$ defined by

$$\boldsymbol{\omega}(r_1, \ldots, r_p) = (r_2, \ldots, r_p, \omega(r_1, \ldots, r_p)), \quad \boldsymbol{r} \in T. \tag{2}$$

Then the iterates $\boldsymbol{\omega}^n$ of $\boldsymbol{\omega}$ are defined as follows:

$$\boldsymbol{\omega}^0(\boldsymbol{r}) = \boldsymbol{r}, \quad \boldsymbol{\omega}^{n+1}(\boldsymbol{r}) = \boldsymbol{\omega}(\boldsymbol{\omega}^n(\boldsymbol{r})), \quad n = 0, 1, 2, \ldots, \boldsymbol{r} \in T. \tag{3}$$

As ω is a p-dimensional rate of convergence, then for any $\boldsymbol{r} \in T$, the following series is convergent in $\boldsymbol{R}^p$:

$$\boldsymbol{\sigma}(\boldsymbol{r}) = \sum_{n=0}^{\infty} \boldsymbol{\omega}^n(\boldsymbol{r}). \tag{4}$$

The above introduced vector functions $\boldsymbol{\omega}$ and $\boldsymbol{\sigma}$ are evidently connected by the following relation:

$$\boldsymbol{\sigma}(\boldsymbol{r}) = \boldsymbol{\sigma}(\boldsymbol{\omega}(\boldsymbol{r})) + \boldsymbol{r}\,, \quad \boldsymbol{r} \in T\,. \tag{5}$$

Let us denote by $\sigma_1, \ldots, \sigma_p$ the components of the vector function $\boldsymbol{\sigma}$. Then we obviously have the following relations:

$$\sigma_p(\boldsymbol{r}) = \sigma(\boldsymbol{r}) = \sum_{n=0}^{\infty} \omega^n(\boldsymbol{r})\,, \quad \boldsymbol{r} \in T\,, \tag{6}$$

$$\sigma_{p-k}(\boldsymbol{r}) = \sigma(\boldsymbol{r}) + \sum_{j=p-k}^{p-1} r_j\,, \quad k = 1, 2, \ldots, p-1\,, \quad \boldsymbol{r} \in T\,. \tag{7}$$

Now, let (X, d) be a complete metric space, x an element of X, and A a subset of X. We denote by $d(x, A)$ the g.l.b. of the set $\{d(x, y);\, y \in A\}$. If A is a subset of X^p, then A_i will denote the set $\{x_i \in \boldsymbol{R};\, (x_1, \ldots, x_p) \in A\}$. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an element of X^p. We denote by $\boldsymbol{d}(\boldsymbol{x}, A)$ the vector from $\boldsymbol{R}^p_+$ with components $d(x_i, A_i)$, $i = 1, 2, \ldots, p$. We shall use the following notation:

$$U(A; \boldsymbol{r}) = \{\boldsymbol{x} \in X^p;\, \boldsymbol{d}(\boldsymbol{x}, A) \leqq \boldsymbol{r}\}\,. \tag{8}$$

If $\boldsymbol{x}$ is an element of X^p, we shall write $U(\boldsymbol{x}, \boldsymbol{r})$ instead of $U(\{\boldsymbol{x}\}, \boldsymbol{r})$ for simplicity. Let $\{Z(\boldsymbol{r})\}_{\boldsymbol{r} \in T}$ be a family of subsets of the space X^p. We define the limit of this family by

$$Z(0) = \bigcap_{\boldsymbol{s} \in T} \Big(\bigcup_{\boldsymbol{t} \leqq \boldsymbol{s}} Z(\boldsymbol{t})^- \Big)\,. \tag{9}$$

We can state now the following generalisation of the Induction Theorem of V. Pták [4]:

Theorem 1. *If*

$$Z(\boldsymbol{r}) \subset U(Z(\boldsymbol{\omega}(\boldsymbol{r})), \boldsymbol{r})\,, \tag{10}$$

for each $\boldsymbol{r} \in T$, then

$$Z(\boldsymbol{r}) \subset U(Z(0), \boldsymbol{\sigma}(\boldsymbol{r})) \tag{11}$$

for each $\boldsymbol{r} \in T$.

Proof. If $\boldsymbol{x}_0 \in Z(\boldsymbol{r})$, then by (10) there exists an $\boldsymbol{x}_1 \in U(\boldsymbol{x}_0, \boldsymbol{r}) \cap Z(\boldsymbol{\omega}(\boldsymbol{r}))$. Now, using again (10), there exists an $\boldsymbol{x}_2 \in U(\boldsymbol{x}_1, \boldsymbol{\omega}(\boldsymbol{r})) \cap Z(\boldsymbol{\omega}^2(\boldsymbol{r}))$. We infer by induction that for any $n \in \{0, 1, 2, \ldots\}$, there exists an $\boldsymbol{x}_{n+1} \in U(\boldsymbol{x}_n, \boldsymbol{\omega}^n(\boldsymbol{r})) \cap Z(\boldsymbol{\omega}^{n+1}(\boldsymbol{r}))$. Because of the convergence of the series (4) it follows that the sequence $(\boldsymbol{x}_n)_{n=1}^{\infty}$ is a Cauchy sequence in X^p. Hence it has a limit $\boldsymbol{x}_\infty$. Now, as $\boldsymbol{x}_n \in Z(\boldsymbol{\omega}^n(\boldsymbol{r}))$ and $\lim_{n \to \infty} \boldsymbol{\omega}^n(\boldsymbol{r}) = 0$ it follows that $\boldsymbol{x}_\infty \in Z(0)$.

On the other hand, $\boldsymbol{d}(\boldsymbol{x}_0, \boldsymbol{x}_\infty) \leqq \sum_{n=0}^{\infty} \boldsymbol{d}(\boldsymbol{x}_{n+1}, \boldsymbol{x}_n) \leqq \sum_{n=0}^{\infty} \boldsymbol{\omega}^n(\boldsymbol{r}) = \boldsymbol{\sigma}(\boldsymbol{r})$ and thus the proof of the theorem is complete. ■

In the following we shall show how Theorem 1 can be applied to the study of convergence of iterative procedure of the form

$$x_{n+1} = F(x_{n-p+1}, x_{n-p+2}, \ldots, x_n), \quad n = 0, 1, 2, \ldots, \tag{12}$$

where F is a mapping of X^p into X and $\boldsymbol{x}_0 = (x_{-p+1}, x_{-p+2}, \ldots, x_0)$ is a fixed element of X^p. Suppose we can attach to the pair $(F, \boldsymbol{x}_0)$ a family of sets $\{Z(\boldsymbol{r})\}_{\boldsymbol{r}\in T} \subset$ $\subset X^p$ and a p-dimensional rate of convergence ω such that the following conditions are satisfied:

$$\boldsymbol{x}_0 \in Z(\boldsymbol{r}_0) \quad \text{for a certain} \quad \boldsymbol{r}_0 \in T. \tag{13}$$

$$\text{If} \quad \boldsymbol{r} \in T \quad \text{and} \quad \boldsymbol{y} = (y_1, \ldots, y_p) \in Z(\boldsymbol{r}), \quad \text{then} \tag{14}$$

$$(y_2, \ldots, y_p, F(y)) \in Z(\omega(\boldsymbol{r})) \cap U(y, \boldsymbol{r}).$$

The above conditions imply, according to Theorem 1, that $Z(0)$ is not void. Moreover, it follows that via the iterative procedure (12) one obtains a sequence $\{x_n\}_{n=0}^{\infty}$ which converges to an element $x^* \in X$ with $(x^*, \ldots, x^*) \in Z(0)$, and such that for any $n \in \{0, 1, 2, \ldots\}$ the following relations are satisfied:

$$\boldsymbol{x}_n = (x_{n-p+1}, x_{n-p+2}, \ldots, x_n) \in Z(\omega^n(\boldsymbol{r}_0)), \tag{15}$$

$$d(x_{n+1}, x_n) \leqq \omega^n(\boldsymbol{r}_0), \tag{16}$$

$$d(x_n, x_0) \leqq \sigma(\boldsymbol{r}_0) - \sigma(\omega^n(\boldsymbol{r}_0)), \tag{17}$$

$$d(x_n, x^*) \leqq \sigma(\omega^n(\boldsymbol{r}_0)). \tag{18}$$

The inequality (18) will be called an apriori estimate of the distance between the elements of the sequence $\{x_n\}_{n=0}^{\infty}$ and x^*. The name of "apriori estimate" is justified by the fact that the right hand side of (18) can be computed before obtaining $x_1, x_2, \ldots$ $\ldots, x_n$ via the iterative procedure. Let us now suppose that for a certain $n \in \{1, 2, \ldots\}$ one has already obtained $x_1, x_2, \ldots, x_n$. If

$$\boldsymbol{x}_{n-1} \in Z(\boldsymbol{d}(\boldsymbol{x}_n, \boldsymbol{x}_{n-1})) \tag{19}$$

then taking $\boldsymbol{x}_{n-1}$ instead of $\boldsymbol{x}_0$ and $\boldsymbol{d}(\boldsymbol{x}_n, \boldsymbol{x}_{n-1})$ instead of $\boldsymbol{r}_0$, we infer, like in (18), that

$$d(x_n, x^*) \leqq \sigma(\omega(\boldsymbol{d}(\boldsymbol{x}_n, \boldsymbol{x}_{n-1})) = \sigma(\boldsymbol{d}(\boldsymbol{x}_n, \boldsymbol{x}_{n-1})) - d(x_n, x_{n-1}). \tag{20}$$

This inequality is called an "aposteriori estimate", because it can be computed only after obtaining $x_1, x_2, \ldots, x_n$ via the iterative procedure.

Summing up what we have stated above, we obtain the following.

Corollary 1. *If the conditions* (13) *and* (14) *hold, then via the iterative procedure* (12) *one obtains a sequence* $\{x_n\}_{n=0}^{\infty}$ *which converges to an element* $x^* \in X$ *such that the relations* (15)–(18) *are satisfied. If, in addition, for some* $n \in \{1, 2, \ldots\}$ *the condition* (19) *is fulfilled, then for this n the inequality* (20) *is also satisfied.*

In the sequel we shall apply Corollary 1 to the study of convergence of the secant method. First we shall state a lemma, whose proof is mainly based on the convergence of the secant method in a very particular case:

Lemma 1. *If d, H, q_0 and r_0 are positive constants satisfying the inequality*

$$(\sqrt{(r_0)} + \sqrt{(q_0 + r_0)})^2 \leqq \frac{d}{H}, \tag{22}$$

then the function

$$\omega(q, r) = \frac{r(q + r)}{r + 2\sqrt{(r(q + r) + a^2)}} \tag{23}$$

is a 2-dimensional rate of convergence on the interval $T = \{(q, r); 0 < q < \infty, 0 < r < \infty\}$, *and the corresponding function* σ *is given by*

$$\sigma(q, r) = r - a + \sqrt{(r(q + r) + a^2)}, \tag{24}$$

where

$$a = \frac{1}{2H}\sqrt{((d - Hq_0)^2 - 4Hdr_0)}. \tag{25}$$

Proof. First let us remark that the condition (22) implies that $(d - Hq_0)^2 \geqq \geqq 4Hdr_0$, so that the formula (25) makes sense. Let us consider the real polynomial

$$f(x) = H(x^2 - a^2) \tag{26}$$

and let us denote by $x^* = a$ its positive root. Let us consider for each pair $(q, r) \in T$, the points

$$x_0 = r + \sqrt{(r(q + r) + a^2)}, \quad x_{-1} = x_0 + q. \tag{27}$$

It is easy to prove that by the algorithm

$$x_{n+1} = x_n - \frac{x_{n-1} - x_n}{f(x_{n-1}) - f(x_n)} f(x_n), \quad n = 0, 1, 2, \ldots,$$

one obtains a sequence $\{x_n\}_{n=0}^{\infty}$ which converges to x^*. We have evidently $x_{-1} - x_0 = q$ and $x_0 - x_1 = r$. Taking $\omega(q, r) = [(x_0 - x_1)/(f(x_0) - f(x_1))] f(x_1)$ and $\sigma(q, r) = x_0 - x^*$, one obtains exactly the expressions (23) and (24). The fact that the series $\sum_{n=0}^{\infty} \omega^n(q, r)$ is convergent and that its sum equals $\sigma(q, r)$ is obvious. Moreover, for any $n \in \{0, 1, 2, \ldots\}$ we have:

$$x_n - x_{n+1} = \omega^n(q, r), \tag{28}$$

$$x_n - x_0 = \sigma(q, r) - \sigma(\omega^n(q, r)), \tag{29}$$

$$x_n - x^* = \sigma(\omega^n(q, r)), \tag{30}$$

where, as in (5), we have denoted

$$\omega^0(q, r) = (q, r), \quad \omega^n(q, r) = (\omega^{n-1}(q, r), \quad \omega^n(q, r)), \quad n = 1, 2, \ldots \tag{31}$$ ■

The generalization of the secant method that we will study below is based on the notion of divided difference of an operator. This notion was introduced by J. Schröder [8] and represents a generalization of the usual notion of divided difference of a function [3], in the same sense in which the Fréchet derivative [2] represents a generalization of the classical notion of derivative.

Let f be a (nonlinear) operator which maps a Banach space E into a Banach space F and let x and y by two distinct points of its domain. Let us denote by $L(E, F)$ the Banach space of all bounded linear operators defined on E and with values in F.

Definition 2. *A linear operator $A \in L(E, F)$ is called a divided difference of the operator f on the points x and y, if the following equality holds:*

$$A(x - y) = f(x) - f(y). \tag{32}$$

Concerning the existence of the divided differences of an operator, see [1]. Concerning examples in some particular spaces, see [10].

Using the above defined notion A. Sergeev [9], and J. Schmidt [7] generalized the secant method, obtaining an iterative procedure for solving nonlinear equations in Banach spaces. Let the closed sphere $U = U(x_0, m)$ be included in the domain of f and let D denote the set $\{(x, y) \in U \times U; x \neq y\}$. We may consider a mapping $D \ni (x, y) \to [x, y; f] \in L(E, F)$, where $[x, y; f]$ represents a divided difference of the operator f at the points x and y, i.e.

$$[x, y; f](x - y) = f(x) - f(y). \tag{33}$$

In [9] the author supposes that the mapping $(x, y) \to [x, y; f]$ is symmetric (i.e, $[x, y; f] = [y, x : f]$), while in [7] this assumption is no longer required. In both of the above cited papers one supposes that the mapping $(x, y) \to [x, y; f]$ satisfies a Lipschitz condition. We shall write this condition in the form

$$\|[x, y; f] - [u, v; f]\| \leqq H(\|x - u\| + \|y - v\|). \tag{34}$$

It is easy to prove that if the above inequality holds for all x, y, u, v of U with $x \neq y$ and $u \neq v$, then the limit $\lim_{y \to x} [x, y; f]$ exists for any $x \in U$, and it equals the Fréchet derivative $f'(x)$. Thus the mapping $(x, y) \to [x, y; f]$ can be extended from D to $U \times U$ by taking $[x, x; f] = f'(x)$. Let now x_{-1} be a point of U such that the divided difference $[x_{-1}, x_0; f]$ is boundedly invertible. The generalized secant method is described by the following algorithm:

$$x_{n+1} = x_n - [x_{n-1}, x_n; f]^{-1} f(x_n), \quad n = 0, 1, 2, \ldots. \tag{35}$$

The above iterative procedure makes sense if at each step the operator $[x_{n-1}, x_n; f]$

is invertible and the point x_{n+1} obtained lies in the domain of f. In the following, we shall apply Corollary 1 and Lemma 1 to the study of convergence of the iterative procedure (35). We shall give sufficient conditions for the convergence of the sequence $\{x_n\}_{n=0}^{\infty}$ to a root x^* of the equation $f(x) = 0$, and shall obtain sharp estimates for the distances $\|x_n - x^*\|$.

Theorem 2. *If the conditions* (33) *and* (34) *are satisfied for all* $x, y, u, v \in U = U(x_0, m)$ *and if the following inequalities are fulfilled*:

$$\|x_0 - x_{-1}\| \leqq q_0 , \tag{36}$$

$$(\|[x_{-1}, x_0; f]^{-1}\|)^{-1} \geqq d , \tag{37}$$

$$\|[x_{-1}, x_0; f]^{-1} f(x_0)\| \leqq r_0 , \tag{38}$$

$$m \geqq \sigma(q_0, r_0) , \tag{39}$$

$$(\sqrt{(r_0)} + \sqrt{(q_0 + r_0)})^2 \leqq \frac{d}{H} , \tag{40}$$

then the iterative procedure (35) *makes sense and the sequence* $\{x_n\}_{n=0}^{\infty}$ *obtained by it converges to a root* x^* *of the equation* $f(x) = 0$, *so that the following inequalities hold*:

$$\|x_n - x_0\| \leqq \sigma(q_0, r_0) - \sigma(\omega^n(q_0, r_0)) , \quad n = 0, 1, 2, \ldots, \tag{41}$$

$$\|x_n - x^*\| \leqq \sigma(\omega^n(q_0, r_0)) , \quad n = 0, 1, 2, \ldots, \tag{42}$$

$$\|x_n - x^*\| \leqq \sigma(\|x_{n-1} - x_{n-2}\| , \ \|x_n - x_{n-1}\|) - \|x_n - x_{n-1}\| , \quad n = 1, 2, \ldots, \tag{43}$$

where ω *and* σ *are given by* (23) *and* (24) *and* $\boldsymbol{\omega}$ *is related to* ω *as in* (31).

Proof. For any pair of positive numbers (q, r) we consider the set

$$Z(q, r) = \{(x, y) \in U \times U; \|x - y\| \leqq q, \|y - x_0\| \leqq \sigma(q_0, r_0) - \sigma(q, r) , \\ \|[x, y; f]^{-1}\|^{-1} \geqq h(q, r), \|[x, y; f]^{-1} f(y)\| \leqq r\} ; \tag{44}$$

where we denote

$$h(q, r) = 2a + H(q + 2\sigma(q, r)) . \tag{45}$$

It is easy to verify that $h(q_0, r_0) = d$. This relation together with the inequalities (36)–(39) implies that $(x_{-1}, x_0) \in Z(q_0, r_0)$. Thus the condition (13) of Corollary 1 is fulfilled. Now, let us suppose that $(x, y) \in Z(q, r)$. Denoting

$$z = y - [x, y; f]^{-1} f(y) . \tag{46}$$

we have to prove that $(y, z) \in Z(r, \omega(q, r))$. The condition $\|z - y\| \leqq r$ is obvious.

Taking into account the fact that (see (5))

$$\sigma(q, r) - r = \sigma(r, \omega(q, r)), \tag{47}$$

we infer that $\|z - x_0\| \leqq \sigma(q_0, r_0) - \sigma(r, \omega(q, r))$.

This relation implies that z belongs to U. In order to prove the invertibility of $[y, z; f]$, we shall use the fact if A and B are two linear operators belonging to $L(E, F)$ such that A is boundedly invertible and $\|A - B\| < \|A^{-1}\|^{-1}$, then B is also boundedly invertible and $\|B^{-1}\|^{-1} \geq \|A^{-1}\|^{-1} - \|A - B\|$. According to (34), (44) and (45) we have

$$\|[x, y; f] - [y, z; f]\| \leqq H(q + r) < h(q, r) \leqq \|[x, y; f]^{-1}\|^{-1},$$

so that $[y, z; f]$ is boundedly invertible and we have

$$\|[y, z; f]^{-1}\|^{-1} \geqq h(q, r) - H(q + r) = h(r, \omega(q, r)). \tag{48}$$

From (46) we infer that

$$f(z) = f(z) - f(y) - [x, y; f](z - y) = ([z, y; f] - [x, y; f])(z - y). \tag{49}$$

Using (34), (45) and (48), from the above equality we obtain

$$\|[y, z; f]^{-1} f(z)\| \leqq [h(r, \omega(q, r))]^{-1} H(q + r) r = \omega(q, r),$$

and the proof of the fact that $(y, z) \in Z(r, \omega(q, r))$ is complete. Thus the condition (14) of Corollary 1 is also fulfilled. According to this Corollary the sequence $\{x_n\}_{n=0}^{\infty}$ obtained by (35) converges to a point x^* so that the inequalities (41) and (42) hold (see (17) and (18)). As in (49) we infer that

$$f(x_{n+1}) = ([x_{n+1}, x_n; f] - [x_{n-1}, x_n; f])(x_{n+1} - x_n)$$

and, passing to the limit, we obtain $f(x^*) = 0$.

In order to complete the proof of the theorem, we still have to demonstrate the inequality (43). For this purpose, according to Corollary 1, it is sufficient to prove that

$$(x_{n-2}, x_{n-1}) \in Z(\|x_{n-2} - x_{n-1}\|, \|x_{n-1} - x_n\|),$$

for every $n \in \{1, 2, \ldots\}$ (see (19)). The first and the last condition from the definition (44) of $Z(q, r)$ are obviously fulfilled in this case. From (24) and (45) it follows that the functions σ and h are increasing in the sense that if $q \leqq q_1$ and $r \leqq r_1$, then $\sigma(q, r) \leqq \sigma(q_1, r_1)$ and $h(q, r) \leqq h(q_1, r_1)$. According to (15) we have

$$\|x_{n-2} - x_{n-1}\| \leqq \omega^{n-2}(q_0, r_0) \quad \text{and} \quad \|x_{n-1} - x_n\| \leqq \omega^{n-1}(q_0, r_0) \tag{50}$$

$$\text{for} \quad n = 1, 2, \ldots,$$

where for $n = 1$ one has to take $\omega^{-1}(q_0, r_0) = q_0$.

The above inequalities imply that

$$(51) \qquad \sigma(\|x_{n-2} - x_{n-1}\|, \|x_{n-1} - x_n\|) \leqq \sigma(\omega^{n-1}(q_0, r_0)) \text{ and}$$
$$h(\|x_{n-2} - x_{n-1}\|, \|x_{n-1} - x_n\|) \leqq h(\omega^{n-1}(q_0, r_0)), \quad n = 1, 2, \ldots.$$

From (15) it follows that $(x_{n-2}, x_{n-1}) \in Z(\omega^{n-1}(q_0, r_0))$ for $n = 1, 2, \ldots$, so that we have

$$(52) \qquad \|x_{n-1} - x_0\| \leqq \sigma(q_0, r_0) - \sigma(\omega^{n-1}(q_0, r_0)) \quad \text{and}$$
$$\|[x_{n-2}, x_{n-1}; f]^{-1}\|^{-1} \geqq h(\omega^{n-1}(q_0, r_0)), \quad n = 1, 2, \ldots$$

Finally, (51) and (52) imply that the second and the third condition of (44) are also satisfied in our case. ∎

Let us add some remarks concerning the hypotheses of the above theorem. The constant q_0 appearing in (36) can be taken as small as desired, because having an initial approximation x_0, we can take x_{-1} to be a small perturbation of x_0, for example $x_{-1} = (1 + \varepsilon) x_0$. The crucial hypothesis of Theorem 2 is the inequality (40). This inequality is satisfied only if r_0 is small enough, which means that the initial approximation x_0 is close enough to the root x^*. However, we shall show that the condition (40) is, in a sense, the weakest possible.

More precisely, we hawe

Proposition 1. *For any positive constants d, H, q_0 and r_0 with $H(\sqrt{(r_0)} + \sqrt{(q_0 + r_0)})^2 > d$, there exist a function $f : \mathbf{R} \to \mathbf{R}$ and two points x_0 and x_{-1} such that* (34) *holds for all $x, y, u, v \in \mathbf{R}$, the conditions* (36)–(38) *are satisfied, but the equation $f(x) = 0$ has no solution.*

Proof. If $H(\sqrt{(q_0 + r_0)} + \sqrt{(r_0)})^2 > d > H(\sqrt{(q_0 + r_0)} - \sqrt{(r_0)})^2$, take

$$f(x) = Hx^2 + dr_0 - \frac{1}{4H}(d - Hq_0)^2, \quad x_0 = \frac{d - Hq_0}{2H}, \quad x_{-1} = \frac{d + Hq_0}{2H}.$$

If $H(\sqrt{(q_0 + r_0)} - \sqrt{(r_0)})^2 \geqq d$, take $f(x) = (d/q_0)\, x^2 + r_0$, $x_0 = 0$, $x_{-1} = q_0$. ∎

In the following proposition, we shall prove that the estimates (42) and (43) are in a sense, the best possible:

Proposition 2. *For any positive constants d, H, q_0 and r_0 with $H(\sqrt{(r_0)} + \sqrt{(q_0 + r_0)})^2 \leqq d$ there exist a function $f : \mathbf{R} \to \mathbf{R}$ and two points x_0 and x_{-1} which satisfy the hypotheses of Theorem* 2, *and for which the inequalities* (41)–(43) *are verified with the signs of equality.*

Proof. The proof of this proposition is a consequence of the proof of Lemma 1; indeed, for f given by (26) and x_0, x_{-1} given by (27) with $q = q_0$ and $r = r_0$ we have

$$\frac{f(x_{-1}) - f(x_0)}{x_{-1} - x_0} = d \quad \text{and} \quad \frac{f(x_0)}{d} = r_0\,. \quad ∎$$

Finally, we shall try to answer the question concerning the uniqueness of the solution of the equation $f(x) = 0$. From (41) it follows that $\|x^* - x_0\| \leqq \sigma(r_0)$. Let $\mathring{V}$ denote the open sphere with center x_0 and radius $\mu = \sigma(q_0, r_0) + 2a$.

Proposition 3. *If the inequality* (40) *from Theorem* 2 *is strict, then the root* x^*, *whose existence is guaranteed by the same theorem, is the unique solution of the equation* $f(x) = 0$ *in the set* $U \cap \mathring{V}$.

Proof. First, we note that the inequality (40) is equivalent to the inequality

$$\frac{d}{H} \geqq (q_0 + 2r_0) + \sqrt{(r_0(q_0 + r_0))}\,. \tag{53}$$

If either (40) or (53) is strict, then $a > 0$. Let y^* be an element of $U \cap \mathring{V}$ such that $f(y^*) = 0$. Using (33) we obtain the relation

$$x^* - y^* = [x_{-1}, x_0; f]^{-1} ([x_{-1}, x_0; f] - [x^*, y^*; f]) (x^* - y^*)\,. \tag{54}$$

Now, taking into account (34) we obtain

$$\|x^* - y^*\| \leqq \frac{H}{d} (\|x^* - x_{-1}\| + \|y^* - x_0\|) \|x^* - y^*\|\,. \tag{55}$$

On the other hand, from (24), (36) and (53) we infer that

$$\frac{H}{d} (\|x^* - x_{-1}\| + \|y^* - x_0\|) < \frac{H}{d} (2\sigma(r_0) + 2a + q_0) = 1\,. \tag{56}$$

Finally, the inequalities (55) and (56) imply that $x^* = y^*$, so that the proof of the proposition is complete. ■

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Souhrn

APLIKACE INDUKČNÍ METODY V. PTÁKA NA VYŠETŘOVÁNÍ METODY REGULE FALSI

FLORIAN ALEXANDRU POTRA

V článku se zavádí pojem "p — rozměrné rychlosti konvergence", který zobecňuje pojem rychlosti konvergence, zavedený V. Ptákem [5], [6]. S použitím tohoto pojmu je zobecněna jeho indukční věta [4] V. Ptáka, což umožňuje vyšetření iteračních procesů tvaru

$$x_{n+1} = (F(x_{n-p+1}, x_{n-p+2}, \ldots, x_n), \quad n = 0, 1, 2, \ldots.$$

Výsledky jsou ilustrovány na příkladě konvergence metody sečen a jsou odvozeny ostré odhady pro chybu každého kroku iteračního procesu.

Author's address: Dr. *F. A. Potra*, Department of Mathematics, INCREST, Bd. Păcii 220, 77538 Bucharest, Romania.