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NONLINEAR ELLIPTIC PROBLEMS WITH JUMPING NONLINEARITIES NEAR THE FIRST EIGENVALUE

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1. INTRODUCTION

Let Ω be a bounded open subset of $\mathbb{R}^N$ with a boundary $\partial\Omega$. Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be continuous function satisfying Carathéodory's conditions and a certain type of the growth condition, let $a_{\alpha\beta} = a_{\beta\alpha} \in L^\infty(\Omega)$ and let μ, ν be two numbers, $\mu\nu > 0$. We are concerned with the weak solvability of the Dirichlet problem

$$(1)\qquad \sum_{|\alpha|=|\beta|=m} -(-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x)\, D^\beta u(x)) + \lambda_1 u(x) + \mu u^+(x) + \nu u^-(x) + g(x, u(x)) = f(x) \quad \text{on} \quad \Omega\,,$$
$$Bu = 0 \quad \text{on} \quad \partial\Omega$$

(where $u^+(x) = \max\{u(x), 0\}$, $u^-(x) = \max\{-u(x), 0\}$ and B denotes the Dirichlet boundary conditions) for a given real-valued right hand side $f \in L^2(\Omega)$ under the assumption that λ_1 is the first eigenvalue of the linear boundary value problem

$$(2)\qquad \sum_{|\alpha|=|\beta|=m} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x)\, D^\beta u(x)) - \lambda u(x) = 0 \quad \text{on} \quad \Omega\,,$$
$$Bu = 0 \quad \text{on} \quad \partial\Omega\,,$$

and there is one and only one normed nonnegative eigenfunction $v_1 \neq 0$ corresponding to λ_1.

In the present paper we prove a result about the weak solvability of (1) analogous to that in [2] but under less restrictive conditions upon μ, ν and g than in Fučík's paper [2]. The proof is based on the variational characterization of the eigenvalues of (2). A similar method is used in [1].

2. PRELIMINARIES

We will denote by $\|u\|_m$ the norm in $\mathbf{E} = W_0^{m,2}(\Omega)$, $m \geqq 1$ is an integer and the usual Sobolev space notation is employed; $\|u\|_0$ is the norm in $L^2(\Omega)$. The inner product in $\mathbf{E}$ will be denoted by $(u, v)_m$ while $(u, v)_0$ stands for the inner product in $L^2(\Omega)$.

Let us consider a formal differential operator

$$\mathscr{L} = - \sum_{|\alpha| = |\beta| = m} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha\beta}(x) D^{\beta}) .$$

In what follows we shall assume

(3) $$a_{\alpha\beta}(x) = a_{\beta\alpha}(x) \in L^{\infty}(\Omega) ;$$

there exists $c > 0$ such that

(4) $$\sum_{|\alpha| = |\beta| = m} a_{\alpha\beta} \xi^{\alpha} \xi^{\beta} > c|\xi|^{2m}$$

for all $\xi \in \mathbb{R}^N$.

For $u, v \in \mathbf{E}$, set

$$((u, v)) = \int_{\Omega} \sum_{|\alpha| = |\beta| = m} a_{\alpha\beta} D^{\alpha} u D^{\beta} v .$$

Remark 1. $\mathscr{L}$, together with the Dirichlet boundary condition $Bu = 0$ on $\partial\Omega$, defines by putting $(Lu, v)_m = -((u, v))$ a linear bounded self-adjoint operator L from $\mathbf{E}$ into $\mathbf{E}$, with a countable set of eigenvalues $0 < \lambda_1 \leqq \lambda_2 \leqq \ldots$ and a corresponding complete orthogonal set of eigenfunctions $v_1, v_2, \ldots$ (see e.g. [5]). We recall that λ_k can be determined as follows:

$$\lambda_k = \min \left\{ \frac{((v, v))}{\|v\|_0^2} ; \ v \in \mathbf{E}, \ (v, v_i)_0 = 0, \ i = 1, 2, \ldots, k - 1 \right\}.$$

Let us denote by $L_k : \mathbf{E} \to \mathbf{E}$ the linear operator defined by

$$(L_k u, v)_m = (Lu, v)_m + \lambda_k (u, v)_0 .$$

Let $g(x, s) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that

(5) g is measurable in $x \in \Omega$ for all $s \in \mathbb{R}$, and g is continuous in s for almost all $x \in \Omega$; $g(x, 0) \in L^2(\Omega)$.

Moreover, let us suppose that there exists $c_1 > 0$ such that

(6) $$|g(x, s_1) - g(x, s_2)| \leqq c_1 |s_1 - s_2|$$

for all $s_1, s_2 \in \mathbb{R}$ and almost all $x \in \Omega$. Let us remark that (6) implies

$$|g(x, s)| \leqq |g(x, 0)| + c_1 |s|$$

for all $s \in \mathbb{R}$ and almost all $x \in \Omega$.

Define the mappings

$$N : \mathbf{E} \to \mathbf{E} , \quad G : \mathbf{E} \to \mathbf{E} , \quad F : \mathbf{E} \to \mathbf{E}$$

by the relations

(7) $$(N(u), v)_m = \mu \int_{\Omega} u^{+}(x) v(x) \, dx + \nu \int_{\Omega} u^{-}(x) v(x) \, dx ,$$

(8) $$(G(u), v)_m = \int_{\Omega} g(x, u(x)) v(x) \, dx ,$$

(9) $$(F(f), v)_m = \int_\Omega f(x)\, v(x)\, \mathrm{d}x$$

for all $u, v \in \mathbf{E}$, $f \in L^2(\Omega)$.

Definition. *A function $u \in \mathbf{E}$ is said to be a weak solution of the boundary value problem* (1) *if*

(10) $$L_1(u) + N(u) + G(u) = F(f)\,.$$

Remark 2. It is easy to see that the mapping N defined by (7) is lipschitzian with the constant $\max\{|\mu|, |\nu|\}$. Indeed,

$$\|N(u_1) - N(u_2)\|_m = \sup_{\|v\|_m = 1} (N(u_1) - N(u_2), v)_m \leqq$$

$$\leqq \left\| \frac{\mu - \nu}{2} (u_1 - u_2) + \frac{\mu + \nu}{2} (u_1 - u_2) \right\|_0 \leqq$$

$$\leqq \left| \frac{\mu - \nu}{2} \right| \|u_1 - u_2\|_0 + \left| \frac{\mu + \nu}{2} \right| \|u_1 - u_0\|_0 \leqq$$

$$\leqq \max\{|\mu|, |\nu|\} \|u_1 - u_2\|_0 \leqq \max\{|\mu|, |\nu|\} \|u_1 - u_2\|_m\,.$$

If $\|F\|$ means the norm of the linear mapping F defined by (9) then $\|F\| \leqq 1$ (see [2], Remark 4).

3. MAIN RESULT

Denote by P the orthogonal projection from $\mathbf{E}$ onto $\operatorname{Ker} L_1$ and put $P^c(x) = x - P(x)$, $x \in \mathbf{E}$. Suppose that

(11) $\operatorname{Ker} L_1$ is the linear hull of v_1, $v_1 \in \mathbf{E}$, $v_1 \geqq 0$ almost everywhere, $v_1 \neq 0$.

The restriction L_1 of the operator L_1 onto $\operatorname{Im} L_1$ is a one-to-one mapping and there exists a continuous mapping $K : \operatorname{Im} L_1 \to \operatorname{Im} L_1$ which is called the right inverse of L_1. Thus for each $x \in \operatorname{Im} L_1$ we have $x = KLx$.

Lemma 1. *Suppose* (3)–(6), (11). *Let*

(12) $$\mu\nu > 0\,, \quad \max\{|\mu|, |\nu|\} + c_1 < \lambda_2 - \lambda_1\,.$$

Then for an arbitrary $t \in \mathbb{R}$ and $f \in L^2(\Omega)$ there exists exactly one $v_{t,f} \in \operatorname{Im} L_1$ satisfying

(13) $$L_1(v_{t,f}) + P^c N(tv_1 + v_{t,f}) + P^c G(tv_1 + v_{t,f}) = P^c F(f)\,.$$

Proof. Let $f \in L^2(\Omega)$ be fixed and for every $w \in \operatorname{Im} L_1$, $t \in \mathbb{R}$ let us denote

$$\Phi_t(w) = L_1(w) + P^c N(tv_1 + w) + P^c G(tv_1 + w)\,.$$

We shall prove the lemma by showing that Φ_t is a strictly monotone mapping in $\operatorname{Im} L_1$. For $w_1, w_2 \in \operatorname{Im} L_1$ we have

$$(\Phi_t(w_1) - \Phi_t(w_2), w_1 - w_2)_m = -((w_1 - w_2, w_1 - w_2)) +$$
$$+ \lambda_1 \|w_1 - w_2\|_0^2 + \mu \int_\Omega ((t\, v_1(x) + w_1(x))^+ - (t\, v_1(x) + w_2(x))^+) \,.$$
$$. (w_1(x) - w_2(x))\,\mathrm{d}x + \nu \int_\Omega ((t\, v_1(x) + w_1(x))^- - (t\, v_1(x) + w_2(x))^-) \,.$$
$$. (w_1(x) - w_2(x))\,\mathrm{d}x + \int_\Omega (g(x, t\, v_1(x) + w_1(x)) - g(x, t\, v_1(x) + w_2(x))) \,.$$
$$. (w_1(x) - w_2(x))\,\mathrm{d}x \,.$$

The inequalities (12) imply the existence of such an $\varepsilon > 0$ that

$$(\Phi_t(w_1) - \Phi_t(w_2), w_1 - w_2)_m \leqq -((w_1 - w_2, w_1 - w_2)) +$$
$$+ \lambda_1 \|w_1 - w_2\|_0^2 + (\lambda_2 - \lambda_1 - \varepsilon)\, \|w_1 - w_2\|_0^2 \,.$$

The variational characterization of λ_2 implies

$$\|w_1 - w_2\|_0^2 \leqq \frac{((w_1 - w_2, w_1 - w_2))}{\lambda_2} \,.$$

Using this fact we obtain

$$(\Phi_t(w_1) - \Phi_t(w_2), w_1 - w_2)_m \leqq -\frac{\varepsilon}{\lambda_2}\,((w_1 - w_2, w_1 - w_2)) \,.$$

Since $((z, z))^{1/2}$ is a norm in $\boldsymbol{E}$ equivalent to $\|z\|_m$, the result follows from a well known lemma of Minty (see [4]).

Remark 3. Let us denote

(14) $$\Psi_f : t \mapsto PN(tv_1 + v_{t,f}) + PG(tv_1 + v_{t,f}) \,.$$

It is proved in [2], Lemma 2 that the equation (10) has a solution $u_0 \in \boldsymbol{E}$ if and only if there exists $t_0 \in \mathbb{R}$ such that

(15) $$\Psi_f(t_0) = P\, F(f)$$

and, moreover, $u_0 = t_0 v_1 + v_{t_0,f}$.

Remark 4. As in [2], instead of (15) we can consider an equivalent equation

(16) $$\psi_f(t) = (F(f), v_1)_m \,,$$

where $\psi_f(t)$ is a real-valued function defined by

(17) $$\psi_f(t) = (\Psi_f(t), v_1)_m \,.$$

Using the notation from Section 2 we have

$$(18) \qquad \psi_f(t) = \mu \int_\Omega (t\, v_1(x) + v_{t,f}(x))^+ \, v_1(x) \,\mathrm{d}x + $$

$$+ \nu \int_\Omega (t\, v_1(x) + v_{t,f}(x))^- \, v_1(x) \,\mathrm{d}x + \int_\Omega g(x, t\, v_1(x) + v_{t,f}(x))\, v_1(x) \,\mathrm{d}x \,.$$

Remark 5. We shall assume in the sequel that the function g or $-g$ is bounded from below by a sublinear function in the case $\mu > 0$ and $\nu > 0$ or $\mu < 0$ and $\nu < 0$, respectively; this means that there exist a function g_1 defined on $\Omega \times \mathbb{R}$ and $c_2 > 0$, $\delta \in \langle 0, 1)$, $r(x) \in L^2(\Omega)$ such that

$$|g_1(x, s)| \leqq r(x) + c_2 |s|^\delta$$

and

$$g(x, s) \geqq g_1(x, s) \quad \text{or} \quad -g(x, s) \geqq g_1(x, s)$$

for almost all $x \in \Omega$ and all $s \in \mathbb{R}$.

Lemma 2. *For a fixed $f \in L^2(\Omega)$ the function ψ_f is continuous on $\mathbb{R}$ and*

$$\lim_{|t| \to +\infty} \psi_f(t) = +\infty$$

if $\mu > 0$, $\nu > 0$ and

$$\lim_{|t| \to \infty} \psi_f(t) = -\infty$$

in the opposite case.

Proof. Fix $f \in L^2(\Omega)$ and suppose

$$\lim_{n \to \infty} |t_n - t_0| = 0 \,.$$

According to the proof of Lemma 1 we have

$$\Phi_{t_n}(v_{t_n,f}) = \Phi_{t_0}(v_{t_0,f}) = P^c F(f) \,.$$

This fact implies

$$\|\Phi_{t_n}(v_{t_n,f}) - \Phi_{t_n}(v_{t_0,f})\|_m = \|\Phi_{t_n}(v_{t_0,f}) - \Phi_{t_0}(v_{t_0,f})\|_m =$$

$$= \|P^c N(t_n v_1 + v_{t_0,f}) - P^c N(t_0 v_1 + v_{t_0,f}) + P^c G(t_n v_1 + v_{t_0,f}) -$$

$$- P^c G(t_0 v_1 + v_{t_0,f})\|_m \leqq (\max\{|\mu|, |\nu|\} + c_1)\, |t_n - t_0| \,.$$

Analogously as in the proof of Lemma 1 we obtain

$$|t_n - t_0| \geqq \frac{\varepsilon}{\lambda_2(\max\{|\mu|, |\nu|\} + c_1)} \cdot \frac{((v_{t_n,f} - v_{t_0,f}, v_{t_n,f} - v_{t_0,f}))}{\|v_{t_n,f} - v_{t_0,f}\|_m},$$

which implies $\lim_{n \to \infty} \|v_{t_n,f} - v_{t_0,f}\|_m = 0$.

The continuity of the function $\psi_f(t)$ defined by (18) now follows from the necessary and sufficient condition for the continuity of Němickij's operator in the space $L^2(\Omega)$ (see e.g. [3]).

Let us suppose $\mu > 0$, $\nu > 0$ (the proof of the opposite case is analogous). Suppose on the contrary that there exists $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$, $|t_n| \to \infty$ such that

$$\lim_{|t_n|\to\infty} \int_\Omega \left(v_1(x) + \frac{v_{t_n,f}(x)}{t_n} \right)^+ v_1(x)\,\mathrm{d}x = 0, \tag{19}$$

$$\lim_{|t_n|\to\infty} \int_\Omega \left(v_1(x) + \frac{v_{t_n,f}(x)}{t_n} \right)^- v_1(x)\,\mathrm{d}x = 0 \tag{20}$$

hold simultaneously. Then

$$0 \leftarrow \int_\Omega \left(v_1(x) + \frac{v_{t_n,f}(x)}{t_n} \right) v_1(x)\,\mathrm{d}x = \int_\Omega v_1^2(x)\,\mathrm{d}x$$

because $v_{t_n,f}/t_n \in \operatorname{Im} L_1$. This is a contradiction with the assumption made at the beginning of this section. The assertion follows from (18) because the function g is bounded from below by a sublinear function.

Theorem. *Let all the assumptions of Lemma* 1 *and Remark* 5 *be fulfilled,* $\mu > 0$ *and* $\nu > 0$ *Then there exists a lower semicontinuous function* $\Gamma : \operatorname{Im} L_1 \to \mathbb{R}$ *such that* $\inf_{F(f)\in \operatorname{Im} L_1} \Gamma(f) > -\infty$ *for* $f \in L^2(\Omega)$ *and*

(i) *the boundary value problem* (1) *has a weak solution for the right hand side* $f \in L^2(\Omega)$ *if and only if* $f \in \mathbf{A}$, *where*

$$\mathbf{A} = \left\{ f \in L^2(\Omega);\ \int_\Omega f(x)\, v_1(x)\,\mathrm{d}x \geqq \Gamma(Q^c(f)) \right\};$$

(ii) *the boundary value problem* (1) *has at least two weak solutions for the right hand side* $f \in L^2(\Omega)$ *if and only if* $f \in \mathbf{B}$,

$$\mathbf{B} = \left\{ f \in L^2(\Omega);\ \int_\Omega f(x)\, v_1(x)\,\mathrm{d}x > \Gamma(Q^c(f)) \right\},$$

where Q *is the orthogonal projection from* $L^2(\Omega)$ *onto* $\mathbf{X} = \{f \in L^2(\Omega);$ $F(f) \in \operatorname{Ker} L_1\}$.

Proof. If we put

$$\Gamma(f) = \min_{t\in R} \psi_f(t)$$

for $f \in L^2(\Omega)$, $F(f) \in \operatorname{Im} L_1$ then the inequalities

$$\begin{aligned} \|f_1 - f_2\|_m \|v_{t,f_1} - v_{t,f_2}\|_m &\geqq \|P^c F(f_1) - P^c F(f_2)\|_m \|v_{t,f_1} - v_{t,f_2}\|_m \geqq \\ &\geqq -(\Phi_t(v_{t,f_1}) - \Phi_t(v_{t,f_2}), v_{t,f_1} - v_{t,f_2})_m \geqq \\ &\geqq \frac{\varepsilon}{\lambda_2} ((v_{t,f_1} - v_{t,f_2}, v_{t,f_1} - v_{t,f_2})), \quad t \in \mathbb{R}, \quad F(f_i) \in \operatorname{Im} L_1, \quad i = 1, 2, \end{aligned}$$

imply the lower semicontinuity of Γ. The other assertions of Theorem follow from the previous lemmas and remarks.

Remark 6. Theorem is presented for the case $\mu > 0$ and $\nu > 0$. In the opposite case $\Gamma = \max_{t \in R} \psi_f(t)$ will be an upper semicontinuous function and the inequalities in (i) and (ii) will be converse.

Remark 7. Let us consider the following simple boundary value problem:

$$u''(x) + \lambda\, u(x) = 0\,, \quad x \in (0, \pi)\,,$$
$$u(0) = u(\pi) = 0\,.$$

Let us denote by

$$(u, v)_1 = \int_0^\pi u'(x)\, v'(x)\, \mathrm{d}x + \int_0^\pi u(x)\, v(x)\, \mathrm{d}x\,,$$
$$(u, v)_0 = \int_0^\pi u(x)\, v(x)\, \mathrm{d}x$$

the inner products in $W_0^{1,2}(0, \pi)$ and $L^2(0, \pi)$, respectively. In this case we have

$$\frac{1}{\|K\|} = \inf_{\|w\|_1 = 1} \sup_{\|u\|_1 = 1} \left[-\int_0^\pi w'(x)\, u'(x)\, \mathrm{d}x + \int_0^\pi w(x)\, u(x)\, \mathrm{d}x \right] \leqq$$
$$\leqq \inf_{\|w\|_1 = 1} \sup_{\|u\|_1 = 1} \left[-\int_0^\pi w'(x)\, u'(x)\, \mathrm{d}x + 4\int_0^\pi w(x)\, u(x)\, \mathrm{d}x - 3\int_0^\pi w(x)\, u(x)\, \mathrm{d}x \right] \leqq$$
$$\leqq 3\sqrt{\frac{2}{5\pi}} \left[\int_0^\pi (\sin 2x)^2\, \mathrm{d}x \right]^{1/2} = \frac{3}{\sqrt{5}}\,, \quad \text{where} \quad w \in \operatorname{Im} L_1 \quad \text{and} \quad u \in W_0^{1,2}(0, \pi)\,.$$

On the other hand, $\lambda_2 - \lambda_1 = 3$. This fact shows that the condition (12) is more general than the condition $\|K\| \max\{\mu, \nu\} < 1$ from the paper [2].

Remark 8. We put $\mathbf{E} = W^{m,2}(\Omega)$ if B denotes the Neumann boundary conditions.

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Souhrn

NELINEÁRNÍ ELIPTICKÉ PROBLÉMY SE SKÁKAJÍCÍ NELINEARITOU V OKOLÍ PRVNÍHO VLASTNÍHO ČÍSLA

PAVEL DRÁBEK

V článku je proveden rozbor existence a násobnosti řešení nelineárního eliptického problému

$$\mathscr{L}u + \lambda_1 u + \mu u^+ + \nu u^- + g(x, u) = f \text{ v } \Omega$$

$$Bu = 0 \text{ na } \partial\Omega\,,$$

kde parametry μ a ν se pohybují v okolí prvního vlastního čísla λ_1. Uvedené postačující podmínky jsou obecnější než v práci [2].

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