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CONTACT BETWEEN ELASTIC BODIES — III. DUAL FINITE ELEMENT ANALYSIS

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In Part I and II of this paper, the primary variational formulation of the contact problem – i.e., in terms of displacements – has been studied. In the present part, we use a dual variational formulation of the same problem, – i.e., in terms of stresses. For simplicity, we restrict ourselves to the problems with a bounded contact zone and with polygonal boundaries.

The dual formulation – i.e. principle of minimum complementary energy is deduced in Section 1 via the theory of a saddle-point. In contradistinction to the primary principle – i.e. minimum of potential energy – we obtain a uniquely solvable formulation even in case that the primary formulation has an infinity of solutions.

In section 2 we define approximations to the dual variational problem on the basis of self-equilibriated triangular block-elements, which have been proposed by Watwood and Hartz [3] and studied in [4] and [5]. An a priori L_2-error estimate is proved under a hypothesis that the solution is sufficiently regular. An algorithm for the approximations is presented in Section 3.

1. DUAL VARIATIONAL FORMULATION OF THE CONTACT PROBLEM WITH A BOUNDED CONTACT ZONE AND ZERO FRICTION

Starting with the primary variational formulation – the principle of minimum potential energy – we derive a dual formulation – the principle of minimum complementary energy – via theory of a saddle-point. To this end, we recall first the concept of the saddle-point.

Let $\mathscr{A}, \mathscr{B}$ be two non-empty sets, $\mathscr{H}$ a real functional, defined on $\mathscr{A} \times \mathscr{B}$. The pair $\{u, \lambda\}$ will be called a *saddle point* of $\mathscr{H}$ on $\mathscr{A} \times \mathscr{B}$, if

$$(1.1) \qquad \mathscr{H}(u, \mu) \leqq \mathscr{H}(u, \lambda) \leqq \mathscr{H}(v, \lambda) \quad \forall v \in \mathscr{A}, \quad \forall \mu \in \mathscr{B}.$$

Lemma 1.1. *Let $\{u, \lambda\}$ be a saddle point of $\mathscr{H}$ on $\mathscr{A} \times \mathscr{B}$. Then*

$$\mathscr{H}(u, \lambda) = \inf_{v \in \mathscr{A}} \sup_{\mu \in \mathscr{B}} \mathscr{H}(v, \mu) = \sup_{\mu \in \mathscr{B}} \inf_{v \in \mathscr{A}} \mathscr{H}(v, \mu) .$$

For the proof – see e.g. [2] – chpt. 5.

Let us recall the primary variational formulation of the contact problem (cf. [1] – Section 1). We introduced the following notation

$$W = \{\mathbf{u} \mid \mathbf{u} = (\mathbf{u}', \mathbf{u}'') \in [H^1(\Omega')]^2 \times [H^1(\Omega'')]^2\} ,$$

$$V = \{\mathbf{u} \in W \mid \mathbf{u}' = 0 \text{ on } \Gamma_u,\ u_n'' = 0 \text{ on } \Gamma_0\} ,$$

$$K = \{\mathbf{v} \in V \mid v_n' + v_n'' \leqq 0 \text{ on } \Gamma_K\} ,$$

$$A(\mathbf{u}, \mathbf{v}) = \int_{\Omega' \cup \Omega''} c_{ijkm}\, e_{ij}(\mathbf{u})\, e_{km}(\mathbf{v})\, \mathrm{d}\mathbf{x} ,$$

$$L(\mathbf{v}) = \int_{\Omega' \cup \Omega''} F_i v_i \,\mathrm{d}\mathbf{x} + \int_{\Gamma_\tau' \cup \Gamma_\tau''} P_i v_i \,\mathrm{d}s ,$$

$$\mathscr{L}(\mathbf{v}) = \tfrac{1}{2} A(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}) .$$

The primary problem is to find $\mathbf{u} \in K$ such that

$$\mathscr{L}(\mathbf{u}) \leqq \mathscr{L}(\mathbf{v}) \quad \forall \mathbf{v} \in K . \tag{1.2}$$

Let us introduce a set S of new parameters as follows:

$$S = \{\mathscr{N} = (\mathscr{N}_{ij});\ i, j = 1, 2,\ \mathscr{N}_{ij} \in L_2(\Omega),\ \mathscr{N}_{12} = \mathscr{N}_{21}\} ,$$
$$\Omega = \Omega' \cup \Omega'' .$$

Setting

$$\mathscr{N}_{ij} = e_{ij}(\mathbf{v}) , \quad i, j = 1, 2 , \tag{1.3}$$

we may write

$$\mathscr{L}(\mathbf{v}) = \frac{1}{2} \int_\Omega c_{ijkl} \mathscr{N}_{ij} \mathscr{N}_{kl} \,\mathrm{d}\mathbf{x} - L(\mathbf{v}) = \mathscr{L}_1(\mathscr{N}, \mathbf{v})$$

and the primary problem (1.2) is equivalent to the minimization of $\mathscr{L}_1(\mathscr{N}, \mathbf{v})$ with the constraints (1.3). The latter problem can be solved by means of Lagrange multipliers λ and the following Lagrangian $\mathscr{H}$:

$$\mathscr{H}([\mathscr{N}, \mathbf{v}], \lambda) = \mathscr{L}_1(\mathscr{N}, \mathbf{v}) + \int_\Omega \lambda_{ij}(e_{ij}(\mathbf{v}) - \mathscr{N}_{ij}) \,\mathrm{d}\mathbf{x} .$$

It is readily seen that

$$\sup_{\lambda \in S} \int_\Omega \lambda_{ij}(e_{ij}(\mathbf{v}) - \mathscr{N}_{ij}) \,\mathrm{d}\mathbf{x} = \begin{cases} 0 & \text{if } \mathscr{N}_{ij} = e_{ij}(\mathbf{v}) \quad \forall i,j \\ +\infty & \text{if not} . \end{cases}$$

Consequently, any solution $\mathbf{u}$ of the primary problem (1.2) satisfies the relation

$$(1.4) \qquad \mathscr{L}(\mathbf{u}) = \inf_{\mathbf{v}\in K} \mathscr{L}(\mathbf{v}) = \inf_{\substack{\mathbf{v}\in K \\ \mathcal{N}\in S}} \sup_{\lambda\in S} \mathscr{H}([\mathcal{N}, \mathbf{v}], \lambda) .$$

The following variational problem

$$(1.5) \qquad \sup_{\lambda\in S} \inf_{[\mathcal{N},\mathbf{v}]\in S\times K} \mathscr{H}([\mathcal{N}, \mathbf{v}], \lambda)$$

is called *dual* to the (primary) problem (1.4). It is desirable to investigate the relation between the values of (1.4) and (1.5). To this end we apply lemma 1.1. First we prove

Lemma 1.2. *Let $\{[\mathcal{N}^*, \mathbf{v}^*], \lambda^*\}$ be a saddle point of $\mathscr{H}$ on $\mathscr{W} \times S$, where $\mathscr{W} = S \times K$. Then a solution $\mathbf{u}$ of the primary problem exists and*

$$\mathcal{N}^* = e(\mathbf{u}) , \quad \mathbf{v}^* = \mathbf{u} , \quad \lambda^* = \tau(\mathbf{u}) ,$$

where $e(\mathbf{u})$ and $\tau(\mathbf{u})$ are the corresponding strain and stress tensor, respectively.

Proof. From (1.1) we deduce:

$$(1.6) \qquad \delta_\lambda \mathscr{H}([\mathcal{N}^*, \mathbf{v}^*], \lambda^*) = 0 \Leftrightarrow \mathcal{N}^*_{ij} = e_{ij}(\mathbf{v}^*) ,$$

$$(1.7) \qquad \delta_{\mathcal{N}} \mathscr{H}([\mathcal{N}^*, \mathbf{v}^*], \lambda^*) = 0 \Leftrightarrow \lambda^*_{ij} = c_{ijkl}\mathcal{N}^*_{kl} ,$$

$$(1.8) \qquad \delta_v \mathscr{H}([\mathcal{N}^*, \mathbf{v}^*], \lambda^*) (\mathbf{v} - \mathbf{v}^*) \geqq 0 \quad \forall \mathbf{v} \in K$$

(where e.g. $\delta_\lambda \mathscr{H}$ denotes the partial Gâteaux derivative of $\mathscr{H}$ with respect to λ). The inequality (1.8) together with (1.6) and (1.7) implies

$$A(\mathbf{v}^*, \mathbf{v} - \mathbf{v}^*) \geqq L(\mathbf{v} - \mathbf{v}^*) \quad \forall \mathbf{v} \in K .$$

Consequently, $\mathbf{v}^*$ is a solution of the problem (1.2).

Lemma 1.3. *Let $\mathbf{u}$ be a solution of the primary problem. Then $\{[e(\mathbf{u}), \mathbf{u}], \tau(\mathbf{u})\}$ is a saddle point of $\mathscr{H}$ on $\mathscr{W} \times S$.*

Proof. We have to verify that

$$\mathscr{H}([e(\mathbf{u}), \mathbf{u}], \mu) \leqq \mathscr{H}([e(\mathbf{u}), \mathbf{u}], \tau(\mathbf{u})) \leqq \mathscr{H}([\mathcal{N}, \mathbf{v}], \tau(\mathbf{u}))$$

holds for any $\mu \in S$ and $[\mathcal{N}, \mathbf{v}] \in \mathscr{W}$.

The left-hand inequality turns to an equality due to the definition of $\mathscr{H}$. The right-hand inequality can be written in the following form

$$\frac{1}{2}\int_\Omega c_{ijkl}\, e_{ij}(\mathbf{u})\, e_{kl}(\mathbf{u})\, \mathrm{d}\mathbf{x} - L(\mathbf{u}) \leqq$$

$$\leqq \frac{1}{2}\int_\Omega c_{ijkl}\mathcal{N}_{ij}\mathcal{N}_{kl}\, \mathrm{d}\mathbf{x} - L(\mathbf{v}) + \int_\Omega c_{ijkl}\, e_{kl}(\mathbf{u}) \left(e_{ij}(\mathbf{v}) - \mathcal{N}_{ij}\right) \mathrm{d}\mathbf{x} ,$$

which is equivalent to

$$\frac{1}{2}\int_\Omega c_{ijkl}(\mathscr{N}_{ij} - e_{ij}(\mathbf{u}))\,(\mathscr{N}_{kl} - e_{kl}(\mathbf{u}))\,\mathrm{d}\mathbf{x} +$$

$$+\int_\Omega c_{ijkl}\, e_{kl}(\mathbf{u})\, e_{ij}(\mathbf{v} - \mathbf{u})\,\mathrm{d}\mathbf{x} \geqq L(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in K\,, \quad \forall \mathscr{N} \in S\,.$$

The latter inequality, however, is an easy consequence of the ellipticity of the coefficients c_{ijkl} (cf. (1.4) in [1] – I) and the definition of $\mathbf{u}$.

Lemma 1.4. *Let a solution* $\mathbf{u}$ *of the primary problem* (1.2) *exist. Then*

$$\mathscr{L}(\mathbf{u}) = \inf_{\substack{\mathscr{N}\in S\\ \mathbf{v}\in K}} \sup_{\lambda\in S} \mathscr{H}([\mathscr{N}, \mathbf{v}], \lambda) = \sup_{\lambda\in S} \inf_{\substack{\mathscr{N}\in S\\ \mathbf{v}\in K}} \mathscr{H}([\mathscr{N}, \mathbf{v}], \lambda)\,. \tag{1.9}$$

Proof. Using (1.4), Lemma 1.3 and Lemma 1.1, the assertion follows.

Next let us simplify the dual problem (1.5) by an elimination of the variables $[\mathscr{N}, \mathbf{v}]$. Consider the „inner" problem

$$\inf_{[\mathscr{N},\mathbf{v}]\in\mathscr{W}} \mathscr{H}([\mathscr{N}, \mathbf{v}], \lambda)\,,$$

where $\lambda \in S$ is a fixed element. Obviously, we have

$$\inf_{\mathscr{W}} \mathscr{H} = \inf_{\mathscr{N}\in S} \mathscr{H}_1(\mathscr{N}, \lambda) + \inf_{\mathbf{v}\in K} \mathscr{H}_2(\mathbf{v}, \lambda) \tag{1.10}$$

where

$$\mathscr{H}_1(\mathscr{N}, \lambda) = \frac{1}{2}\int_\Omega c_{ijkl}\mathscr{N}_{ij}\mathscr{N}_{kl}\,\mathrm{d}\mathbf{x} - \int_\Omega \lambda_{ij}\mathscr{N}_{ij}\,\mathrm{d}\mathbf{x}\,,$$

$$\mathscr{H}_2(\mathbf{v}, \lambda) = \int_\Omega \lambda_{ij}\, e_{ij}(\mathbf{v})\,\mathrm{d}\mathbf{x} - L(\mathbf{v})\,. \tag{1.11}$$

We can find easily that

$$\inf_{\mathscr{N}\in S} \mathscr{H}_1(\mathscr{N}, \lambda) = \mathscr{H}_1(\mathscr{N}^*, \lambda) = -\frac{1}{2}\int_\Omega a_{ijkl}\lambda_{ij}\lambda_{kl}\,\mathrm{d}\mathbf{x}\,, \tag{1.12}$$

where a_{ijkl} are coefficients of the inverse generalized Hooke's law and $\mathscr{N}^*_{ij} = a_{ijkl}\lambda_{kl}$.

Let there exist a $\mathbf{v}_0 \in K$ such that $\mathscr{H}_2(\mathbf{v}_0, \lambda) < 0$. Since K is a convex cone, $t\mathbf{v}_0 \in K$ and

$$\mathscr{H}_2(t\mathbf{v}_0, \lambda) \to -\infty \quad \text{for} \quad t \to +\infty\,.$$

If $\mathscr{H}_2(\mathbf{v}, \lambda) \geqq 0$ for all $\mathbf{v} \in K$, then

$$\inf_{\mathbf{v}\in K} \mathscr{H}_2(\mathbf{v}, \lambda) = \mathscr{H}_2(0, \lambda) = 0\,.$$

Denote

$$K^+_{F,P} = \{\lambda \in S \mid \mathscr{H}_2(\mathbf{v}, \lambda) \geqq 0 \ \forall \mathbf{v} \in K\}\,. \tag{1.13}$$

We thus obtained that

$$(1.14) \qquad \inf_{\mathbf{v}\in K} \mathscr{H}_2(\mathbf{v}, \lambda) = \begin{cases} 0 & \text{if } \lambda \in K^+_{F,P}, \\ -\infty & \text{if } \lambda \notin K^+_{F,P}. \end{cases}$$

Lemma 1.5. *Let a solution $\mathbf{u}$ of the primary problem exist. Then $K^+_{F,P}$ is a non-empty, closed and convex subset of S.*

Proof. We show that $\tau(\mathbf{u})$ belongs to $K^+_{F,P}$. In fact,

$$\int_\Omega \tau_{ij}(\mathbf{u})\, e_{ij}(\mathbf{v} - \mathbf{u})\, \mathrm{d}\mathbf{x} \geqq L(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in K$$

holds and substituting $\mathbf{v} = \mathbf{u} + \mathbf{w}$, where $\mathbf{w}$ is any element of K, we obtain

$$\mathscr{H}_2(\mathbf{w}, \tau(\mathbf{u})) = \int_\Omega \tau_{ij}(\mathbf{u})\, e_{ij}(\mathbf{w}) - L(\mathbf{w}) \geqq 0 \quad \forall \mathbf{w} \in K.$$

The convexity and closedness of $K^+_{F,P}$ is easy to see.

Q.E.D.

From (1.10), (1.12) and (1.14) it follows that

$$\inf_{[\mathscr{N}, \mathbf{v}]\in\mathscr{W}} \mathscr{H}([\mathscr{N}, \mathbf{v}], \lambda) = \begin{cases} -\mathscr{S}(\lambda) & \text{if } \lambda \in K^+_{F,P} \\ -\infty & \text{if } \lambda \notin K^+_{F,P} \end{cases}$$

where

$$\mathscr{S}(\lambda) = \frac{1}{2} \int_\Omega a_{ijkl} \lambda_{ij} \lambda_{kl}\, \mathrm{d}\mathbf{x}.$$

Consequently, we have

$$\sup_{\lambda\in S} \inf_{[\mathscr{N}, \mathbf{v}]\in\mathscr{W}} \mathscr{H}([\mathscr{N}, \mathbf{v}], \lambda) = \sup_{\lambda\in K_{F,P}{}^+} (-\mathscr{S}(\lambda)) = -\inf_{\lambda\in K_{F,P}{}^+} \mathscr{S}(\lambda).$$

Moreover, if a saddle point of $\mathscr{H}$ on $\mathscr{W} \times S$ exists, then

$$(1.15) \qquad -\inf_{\lambda\in K_{F,P}{}^+} \mathscr{S}(\lambda) = -\mathscr{S}(\tau(\mathbf{u})) = \mathscr{L}(\mathbf{u}),$$

where $\mathbf{u}$ is a solution to the primary problem. In fact, the assertion follows from Lemmas 1.2, 1.1 and 1.4.

On the other hand, Lemma 1.3 guarantees the existence of a saddle point, provided a solution of the primary problem exists.

In [1] – I, Section 2, we studied the existence and uniqueness of weak solutions of the primary problem $\mathscr{P}_1$, i.e., (1.2). Under the assumptions of Theorem 2.2 there, i.e. if

$$(1.16) \qquad L(y) \leqq 0 \quad \forall \mathbf{y} \in K \cap \mathscr{R},$$

$$L(y) < 0 \quad \forall \mathbf{y} \in K \cap \mathscr{R} \dot{-} \mathscr{R}^*,$$

a solution $\boldsymbol{u}$ of the primary problem exists. Since the difference of any two solutions belongs to the subspace $\mathscr{R}$ of rigid displacements, both $e(\boldsymbol{u})$ and $\tau(\boldsymbol{u})$ are determined uniquely. Consequently, if (1.16) holds, the saddle point of $\mathscr{H}$ exists, its last component is uniquely determined and (1.15) is true. In other words, the *dual problem* to find $\lambda \in K^{+}_{F,P}$ such that

$$(1.17) \qquad \mathscr{S}(\lambda) \leqq \mathscr{S}(\mu) \quad \forall \mu \in K^{+}_{F,P}\,,$$

has a *unique* solution $\lambda = \tau(\boldsymbol{u})$, where $\boldsymbol{u}$ is any solution of the primary problem.

Thus we obtained a *uniquely* solvable formulation of a general class of contact problems in contradistinction to the primary variational formulation (see Section 2 of [1] – I, where only the cases of one-dimensional spaces of rigid virtual displacements have been considered).

Remark 1.1. The existence and uniqueness of the solution of (1.17) can be proved directly, using Lemma 1.5 and the strict convexity of the functional $\mathscr{S}$.

Remark 1.2. Let us ephasize that the dual problem is uniquely solvable, whenever the primary problem possesses a solution. The existence (and uniqueness) of the solution for the dual problem, however, follows directly, if the set $K^{+}_{F,P}$ is non-empty. Thus a possibility arises that the dual problem may have a solution even in some cases when the primary problem has not. We shall not investigate the latter question, but remark only that the set $K^{+}_{F,P}$ is non-empty only if the condition $(1.16)_1$ is satisfied. Consequently, $(1.16)_1$ is a necessary condition for the existence of a solution for both the primary and the dual problem.

Interpretation of the set $K^{+}_{F,P}$

For the purposes of an approximation, it is useful to study the set $K^{+}_{F,P}$ of admissible stress fields more closely.

Lemma 1.6. 1° *Let $\lambda \in K^{+}_{F,P}$ be sufficiently smooth. Then λ satisfies the following conditions*:

$$(1.18) \qquad \frac{\partial \lambda_{ij}}{\partial x_j} + F_i = 0 \quad in \quad \Omega = \Omega' \cup \Omega''\,, \quad i = 1, 2,,$$

$$(1.19) \qquad \lambda_{ij} n_j = P_i \quad on \quad \Gamma_\tau = \Gamma'_\tau \cup \Gamma''_\tau\,, \quad i = 1, 2\,,$$

$$(1.20) \qquad T_t(\lambda) = 0 \quad on \quad \Gamma_0\,,$$

$$(1.21) \qquad T_t(\lambda') = T_t(\lambda'') = 0 \quad on \quad \Gamma_K\,,$$

$$(1.22) \qquad T_n(\lambda') = T_n(\lambda'') \leqq 0 \quad on \quad \Gamma_K\,.$$

2° *Conversely, let $\lambda \in S$ be sufficiently smooth and let it satisfy* (1.18)–(1.22). *Then $\lambda \in K^{+}_{F,P}$.*

Proof. 1° Integrating by parts we obtain for any $\mathbf{v} \in K$

$$\int_\Omega \lambda_{ij}\, e_{ij}(\mathbf{v})\,\mathrm{d}\mathbf{x} = -\int_\Omega v_i \frac{\partial \lambda_{ij}}{\partial x_j} + \int_{\partial\Omega' \cup \partial\Omega''} (T_n(\lambda)\, v_n + T_t(\lambda)\, v_t)\,\mathrm{d}s \geqq \int_\Omega F_i v_i\,\mathrm{d}\mathbf{x} + \int_{\Gamma_\tau} P_i v_i\,\mathrm{d}s.$$

Inserting $v_i^M = \pm\varphi_i \in C_0^\infty(\Omega^M)$, $M = {}', {}''$, we are led to (1.18). Hence we obtain

$$\int_{\partial\Omega' \cup \partial\Omega''} T_i(\lambda)\, v_i\,\mathrm{d}s \geqq \int_{\Gamma_\tau} P_i v_i\,\mathrm{d}s \quad \forall \mathbf{v} \in K\,.$$

Choosing $v_i = \pm\psi_i$ such that the traces of ψ have their support in Γ_τ, we obtain (1.19).

Let us choose $\mathbf{v}' = 0$, $\mathbf{v}''$ with $v_n'' = 0$, $v_t'' = \pm\psi$ on Γ_0, where the support of ψ is in Γ_0. Consequently, (1.20) follows.

It remains to analyze the inequality

$$(1.23) \qquad \int_{\Gamma_K} (T_n(\lambda')\, v_n' + T_t(\lambda')\, v_t' + T_n(\lambda'')\, v_n'' + T_t(\lambda'')\, v_t'')\mathrm{d}s \geqq 0 \quad \forall \mathbf{v} \in K\,.$$

Choosing $\mathbf{v} \in V$ such that $v_t' = v_t'' = 0$ and $v_n' = -v_n'' = \pm\varphi$ on Γ_K, $\varphi \in C_0^\infty(\Gamma_K)$, we obtain

$$\int_{\Gamma_K} (T_n(\lambda') - T_n(\lambda''))\, \varphi\,\mathrm{d}s = 0\,.$$

Hence

$$T_n(\lambda') = T_n(\lambda'') \quad \text{on} \quad \Gamma_K\,.$$

Let us choose $\mathbf{v} \in V$ such that $v_n' = v_n'' = 0$, $v_t'' = 0$ $v_t' = \pm\varphi$ on Γ_K. Then (1.23) yields $T_t(\lambda') = 0$ on Γ_K. The condition $T_t(\lambda'') = 0$ can be derived in a similar way.

It remains

$$\int_{\Gamma_K} T_n(\lambda)\,(v_n' + v_n'')\,\mathrm{d}s \geqq 0 \quad \forall \mathbf{v} \in K$$

and $T_n(\lambda) \leqq 0$ follows, using the condition $v_n' + v_n'' \leqq 0$ on Γ_K.

2° Let us multiply (1.18) by a function v_i, where $\mathbf{v} \in K$, and integrate over Ω by parts. We obtain

$$0 = -\int_\Omega \lambda_{ij} \frac{\partial v_i}{\partial x_j}\,\mathrm{d}\mathbf{x} + \int_\Omega F_i v_i\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega' \cup \partial\Omega''} \lambda_{ij} n_j v_i\,\mathrm{d}s =$$

$$= -\int_\Omega \lambda_{ij}\, e_{ij}(\mathbf{v})\,\mathrm{d}\mathbf{x} + L(\mathbf{v}) + \int_{\Gamma_K} T_n(\lambda)\,(v_n' + v_n'')\,\mathrm{d}s\,.$$

From the definition of K and (1.22) we conclude that the last integral is non-negative and consequently, $\lambda \in K_{F,P}^+$.

2. APPROXIMATIONS OF THE DUAL PROBLEM WITH A BOUNDED CONTACT ZONE

As usually, finite-dimensional approximations of the set $K^+_{F,P}$ of admissible stress-functions are needed. To this end, it is suitable first to find a „particular" solution $\bar{\lambda}$ of the non-homogeneous conditions (1.18), (1.19) and then to write

$$\lambda = \bar{\lambda} + \tau ,$$

where τ are self-equilibriated stress fields.

Since the system of forces F_i, P_i acting on Ω'' has a non-zero resultant, it is necessary to introduce reaction forces (normal vector loading) on Γ_K, with respect to both Ω'' and Ω'.

Lemma 2.1. *If* $\bar{\lambda} \in S$ *satisfies* (1.18), (1.19), *then*

$$\int_{\Gamma_K} T_i(\bar{\lambda}'') \, \mathrm{d}s = -\left(\int_{\Omega''} F_i \, \mathrm{d}\mathbf{x} + \int_{\Gamma_\tau''} P_i \, \mathrm{d}s + \int_{\Gamma_0} T_i(\bar{\lambda}'') \, \mathrm{d}s\right), \quad i = 1, 2 . \tag{2.1}$$

Proof. Using (1.18) and (1.19), we may write

$$-\int_{\Omega''} F_i \, \mathrm{d}\mathbf{x} = \int_{\Omega''} \frac{\partial \bar{\lambda}''_{ij}}{\partial x_j} \, \mathrm{d}\mathbf{x} = \int_{\partial\Omega''} \bar{\lambda}''_{ij} n''_j \, \mathrm{d}s = \int_{\Gamma_\tau''} P_i \, \mathrm{d}s + \int_{\Gamma_0} T_i(\bar{\lambda}'') \, \mathrm{d}s + \int_{\Gamma_K} T_i(\bar{\lambda}'') \, \mathrm{d}s ,$$

and (2.1) follows. Q.E.D.

Observing Lemma 2.1, we can choose a simplest distribution of the reaction forces $T_n(\bar{\lambda})$ on Γ_K.

Example 1. Let Γ_0 consist of straight segments parallel with x_1-axis and Γ_K be a straight segment such that $n''_1 > 0$ on Γ_K. We can choose $\bar{\lambda} \in K^+_{F,P}$ such that

$$T_n(\bar{\lambda}') = T_n(\bar{\lambda}'') = g \quad \text{on} \quad \Gamma_K , \tag{2.2}$$

where

$$g = -\left(\int_{\Omega''} F_1 \, \mathrm{d}x + \int_{\Gamma_\tau''} P_1 \, \mathrm{d}s\right) \Big/ \int_{\Gamma_K} \mathrm{d}x_2 = \text{const} . \tag{2.3}$$

In fact, using $T_1(\bar{\lambda}) = T_t(\bar{\lambda}) = 0$ on Γ_0 and $\mathrm{d}x_2 = n''_1 \, \mathrm{d}s$, $T_t(\bar{\lambda}) = 0$ on Γ_K, we realize that the choice (2.2), (2.3) satisfies the necessary equilibrium conditions (2.1). Moreover, the necessary condition for the existence of a solution is

$$V_1 = \int_{\Omega''} F_1 \, \mathrm{d}\mathbf{x} + \int_{\Gamma_\tau''} P_1 \, \mathrm{d}s \geqq 0 \tag{2.4}$$

(cf. Remark 1.2 and $(1.16)_1$). Hence $g \leqq 0$ and (1.22) is also true.

Example 2. For the same Γ_0 and Γ_K with $n_2'' > 0$ on Γ_K, we choose (2.2) with $g = 0$ and

$$(2.5) \qquad T_t(\bar{\lambda}') = T_t(\bar{\lambda}'') = g_1 = -V_1 \Big/ \int_{\Gamma_K} \mathrm{d}x_1 \quad \text{on} \quad \Gamma_K .$$

Obviously, $\bar{\lambda} \notin K_{F,P}^+$, unless $V_1 = 0$.

Example 3. Let $\Gamma_0 = \emptyset$ and Γ_K be a straight segment parallel with x_1-axis. We can choose $\bar{\lambda}$ such that

$$(2.6) \qquad T_n(\bar{\lambda}'') = -T_2(\bar{\lambda}'') = V_2 \Big/ \int_{\Gamma_K} \mathrm{d}s = g \quad \text{on} \quad \Gamma_K ,$$

where $V_2 = \int_{\Omega''} F_2 \,\mathrm{d}\mathbf{x} + \int_{\Gamma_\tau''} P_2 \,\mathrm{d}s$. Since $V_2 \leqq 0$ is necessary for the existence of a solution, we have $T_n(\bar{\lambda}'') \leqq 0$ on Γ_K and (1.22) can also be satisfied.

Remark 2.1. It is not essential for $\bar{\lambda}$ to belong to $K_{F,P}^+$. From the practical point of view, however, $\bar{\lambda}$ should be such that $T_i(\bar{\lambda})$ are constant on Γ_K. Then we can construct internal approximations of $K_{F,P}^+$, which is advantageous.

Remark 2.2. An algorithm for finding $\bar{\lambda}$ will be considered in Section 3.

Henceforth *we choose the configuration of Example* 1 to show the approximations of $K_{F,P}^+$ in detail. Let the stress field $\bar{\lambda}$ satisfy the following condition

$$(2.7) \qquad \int_\Omega \bar{\lambda}_{ij}\, e_{ij}(\mathbf{v}) \,\mathrm{d}\mathbf{x} = L(\mathbf{v}) + \int_{\Gamma_K} g(v_n' + v_n'') \,\mathrm{d}s \quad \forall \mathbf{v} \in V .$$

Obviously, it holds

$$\lambda \in K_{F,P}^+ \Leftrightarrow \lambda - \bar{\lambda} \equiv \tau \in \mathscr{U}_0 ,$$

where

$$\mathscr{U}_0 = \left\{ \tau \in S \mid \int_\Omega \tau_{ij}\, e_{ij}(\mathbf{v}) \,\mathrm{d}\mathbf{x} \geqq -g \int_{\Gamma_K} (v_n' + v_n'') \,\mathrm{d}s \ \forall \mathbf{v} \in K \right\}.$$

Lemma 2.2. 1° *Let* $\tau \in \mathscr{U}_0$ *be sufficiently smooth. Then* τ *satisfies homogeneous equations* (1.18), (1.19), (1.20), (1.21) *and*

$$(2.8) \qquad T_n(\tau') = T_n(\tau'') \leqq -g \quad \text{on} \quad \Gamma_K .$$

2° *Let* Ω *be divided into a finite number of closed subdomains* K_r,

$$\Omega = \bigcup_r K_r , \quad \mathring{K}_r \cap \mathring{K}_s = \emptyset \quad \text{if} \quad r \neq s .$$

Let $\tau \in S$ *satisfy the homogeneous equations* (1.18) *in every* K_r, *homogeneous boundary conditions* (1.19), (1.20), (1.21) *and* (2.8). *Moreover, let the stress vector* $\mathbf{T}(\tau)$ *be continuous across any common boundary of two adjoint subdomains, i.e.*

$$\mathbf{T}(\tau)|_{K_r} + \mathbf{T}(\tau)|_{K_s} = 0 \quad \text{on} \quad K_r \cap K_s \quad \forall r \neq s .$$

Then $\tau \in \mathscr{U}_0$.

Proof is analogous to that of Lemma 1.6.

Remark 2.3. If the condition (2.4) holds, $\mathscr{U}_0$ is non-empty, since it contains the zero element. Moreover, $\mathscr{U}_0$ is convex and closed subset of S.

Substituting $\lambda = \bar{\lambda} + \tau$ into the definition (1.17) of the dual problem, we obtain an *equivalent dual problem*: to find $\tau^0 \in \mathscr{U}_0$ such that

$$J(\tau^0) \leqq J(\tau) \quad \forall \tau \in \mathscr{U}_0 , \tag{2.9}$$

where

$$J(\tau) = \frac{1}{2} \int_\Omega a_{ijkl}\, \tau_{ij}(\tau_{kl} + 2\bar{\lambda}_{kl})\, \mathrm{d}\mathbf{x} .$$

2.1. AN EQUILIBRIUM FINITE ELEMENT MODEL

Since the admissible stress functions in the equivalent dual problem (2.9) have to satisfy the homogeneous equations (1.18) of equilibrium, we need some finite-dimensional subspaces with the same property. To this end, we can employ an equilibrium stress field model proposed by Watwood and Hartz [3].

The model consists of triangular block-elements, each of them being generated by connecting the vertices of a general triangle with its centre of gravity. On each subtriangle 3 linear functions – components of a self-equilibriated stress field are defined. The stress vector has to be continuous when crossing any common boundary of two subtriangles.

For an arbitrary triangle K let us define the space of self-equilibriated linear stress fields as follows

$$M(K) = \{\tau_{11} = \beta_1 + \beta_2 x_1 + \beta_3 x_2,\ \tau_{22} = \beta_4 + \beta_5 x_1 + \beta_6 x_2,$$
$$\tau_{12} = \tau_{21} = \beta_7 - \beta_6 x_1 - \beta_2 x_2\} ,$$

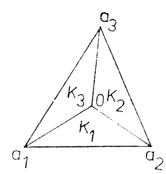

Fig. 1

where $\beta \in R^7$ is an arbitrary vector. It is readily seen that $\partial\tau_{ij}/\partial x_j = 0$ in K for any $\tau \in M(K)$, $i = 1, 2$. Next let us consider a triangular block-element $K = \bigcup_{i=1}^{3} K_i$ (see Fig. 1) and define

$$N(K) = \{\tau = (\tau^1, \tau^2, \tau^3) |\ \tau^i = \tau|_{K_i} \in M(K_i) ,$$
$$\mathbf{T}(\tau^i) + \mathbf{T}(\tau^{i-1}) = 0 \text{ on } 0a_i,\ i = 1, 2, 3, \text{ modulo } 3\} .$$

The last equations mean that the stress vectors are continuous across any side $0a_i$.

Theorem 2.1. *Let an arbitrary external loading $\mathbf{T}^0$ of the triangle K be given,* *such that* (i) *it varies linearly along the sides of* K, *and* (ii) *it satisfies the global equilibrium conditions.*

Then a unique stress field $\tau \in N(K)$ *exists such that*

$$\mathbf{T}(\tau) = \mathbf{T}^0 \quad on \quad \partial K .$$

Moreover the following estimate holds

$$\max_{i=1,2,3} \|\tau^i\|_{C(K_i)} \leqq c(\alpha) \max_{\substack{s \in \partial K \\ k=1,2}} |T_k^0(s)| , \tag{2.10}$$

where the constant $c(\alpha)$ *depends on the minimal angle* α *of* K *only.*

For the proof – see [4] – Theorem 2.1.

Let G be a bounded polygonal domain, $h \in (0, 1\rangle$ a parameter, $\mathcal{T}_h$ a triangulation of G. Define

$$h = \max_{K \in \mathcal{T}_h} \operatorname{diam} K ,$$

$$N_h(G) = \{\tau \in S |\, \tau|_K \in N(K)\ \forall K \in \mathcal{T}_h,\ \mathbf{T}(\tau)|_K + \mathbf{T}(\tau)|_{K'} = 0 \text{ on any } \bar{K} \cap \bar{K}'\} ,$$

$$E(G) = \{\tau \in [H^1(G)]^4 |\, \tau_{12} = \tau_{21},\ \partial\tau_{ij}/\partial x_j = 0 \text{ in } G,\ i = 1, 2\} .$$

Theorem 2.2. *There exists a linear continuous mapping* $r_h : E(G) \to N_h(G)$ *such that for any* $\tau \in E(G) \cap [H^2(G)]^4$

$$\|\tau - r_h\tau\|_{[L_2(G)]^4} \leqq C\, h^2 \|\tau\|_{[H^2(G)]^4} \tag{2.11}$$

holds, where C *does not depend on* h *and* τ, *provided the family* $\{\mathcal{T}_h\}$ *of triangulations is regular* (cf. [1] – II, Section 2.1).

For the proof – see [4] – Theorem 2.5.

Remark 2.4. The estimate (2.11) follows also from some results of Johnson and Mercier [5].

The mapping r_h from Theorem 2.2 is defined "locally" on every triangular block-element $K \in \mathcal{T}_h$ as follows:

The stress vector components $T_k(\tau)$ on any side S_j of K are projected in $L_2(S_j)$ onto the space $P_1(S_j)$ of linear functions. By these projections $\mathbf{T}^0$, the stress field $r_h\tau|_K \in N(K)$ is determined uniquely, by virtue of Theorem 2.1.

Remark 2.5. Any stress field $\tau \in N_h(G)$ satisfies the equilibrium equations $\partial\tau_{ij}/\partial x_j = 0$, $i = 1, 2$ in the sense of distributions.

2.2. APPROXIMATIONS OF THE DUAL PROBLEM BY AN EQUILIBRIUM FINITE ELEMENT MODEL

Assume that both Ω' and Ω'' are bounded polygonal domains. We define an approximation of $\mathcal{U}_0$ as follows:

$$\mathcal{U}_{0h} = \mathcal{U}_0 \cap N_h(\Omega) ,$$

where

$$N_h(\Omega) = \{(\tau', \tau'') | \tau^M \in N_h(\Omega^M),\ M = ', ''\} .$$

We say that $\tau^h \in \mathcal{U}_{0h}$ is an approximation of the dual problem, if

$$J(\tau^h) \leqq J(\tau) \quad \forall \tau \in \mathcal{U}_{0h} . \tag{2.12}$$

Lemma 2.3. *If* (2.4) *holds, there exists a unique solution of the problem* (2.12).

Proof. Obviously, $N_h(\Omega) \subset S$ is a linear and finite-dimensional subset, therefore it is closed and convex. Using Remark 2.3, we conclude that also $\mathcal{U}_{0h}$ is closed, convex and non-empty. Since the functional J is differentiable and strictly convex, the existence and uniqueness of τ^h follows easily.

An algorithm for finding τ^h will be presented in the next Section. Here we shall try to estimate the error

$$\|\lambda - \lambda^h\|_{0,\Omega} = \|\tau^0 - \tau^h\|_{0,\Omega}$$

where

$$\lambda = \bar{\lambda} + \tau^0 , \quad \lambda^h = \bar{\lambda} + \tau^h , \quad \|\cdot\|_{0,\Omega} = \|\cdot\|_{[L_2(\Omega)]^4} .$$

To this end we employ a lemma of Mosco and Strang [6].

Lemma 2.4. *Let an element* $W^h \in \mathcal{U}_{0h}$ *exist such that*

$$2\tau^0 - W^h \in \mathcal{U}_0 . \tag{2.13}$$

Then it holds

$$\|\tau^0 - \tau^h\|_{0,\Omega} \leqq C\|\tau^0 - W^h\|_{0,\Omega} . \tag{2.14}$$

For the proof – see [6] or [7] – Lemma 2.1.

Hence it remains to construct a $W^h \in \mathcal{U}_{0h}$, satisfying (2.13) and sufficiently close to τ^0.

Theorem 2.3. *Let* Γ_0 *consist of straight segments parallel with* x_1*-axis and* Γ_K *be a straight segment such that* $n_1'' > 0$ *on* Γ_K.

Assume that $\tau^0|_{\Omega^M} \in [H^2(\Omega^M)]^2$, $M = {}^{\mathrm{I}}, {}^{\mathrm{II}}$, *and* $T_n(\tau^0) \in H^2(\Gamma_K)$. *Let the family of triangulations* $\{\mathcal{T}_h\}$, $0 < h \leqq 1$, *satisfy the following conditions*:

(A1) *it is regular* (cf. [1] – II, *Section* 2.1);

(A2) *a positive number* β *exists, independent of* h *and such that the ratio of any two sides in* $\mathcal{T}_h$ *is less than* β;

(A3) *between* Γ_K *and* Γ_0 *in* Ω'' *and between* Γ_K *and* Γ_u *in* Ω' *the triangulation* $\mathcal{T}_h$ *is inscribed into smooth "vaulted strips" with bounded curvature and bounded slope* $|\vartheta| \leqq \Theta < \pi/2$, ($\Theta$ independent of h), *which are perpendicular to* Γ_0 *and* Γ_K – *see Fig.* 2.

Then

$$\|\tau^0 - \tau^h\|_{0,\Omega} \leqq C(\tau^0)\, h^{3/2}\,,$$

where $C(\tau^0)$ does not depend on h.

Proof is based on Lemma 2.4 and the two following lemmas.

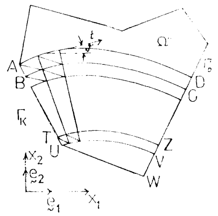

Fig. 2

Lemma 2.5. *Assume that $\tau^0 \in H^2(\Omega^M)$, $M = {}^{\mathrm{I}}, {}^{\mathrm{II}}$, $T_n(\tau^0) \in H^2(\Gamma_K)$. Then a piecewise linear function ψ^h on Γ_K exists with the nodes determined by the triangulation $\mathcal{T}_h$ (discontinuous, in general) and such that*

$$\int_{\Gamma_K} \psi^h \,\mathrm{d}s = \int_{\Gamma_K} T_n(r_h \tau^0)\,\mathrm{d}s\,,$$

$$2\,T_n(\tau^0) + g \leqq \psi^h \leqq -g \quad \text{on} \quad \Gamma_K\,,$$

$$\|T_n(r_h\tau^0) - \psi^h\|_{0,\Gamma_K} \leqq C\,h^2 \left\| \frac{\mathrm{d}^2}{\mathrm{d}s^2}\, T_n(\tau^0) \right\|_{0,\Gamma_K}\,,$$

where r_h is the mapping from Theorem 2.2.

Proof is analogous to that of Lemma 4.2 in [8].

Lemma 2.6. *Let the triangulations $\mathcal{T}_h$ satisfy conditions (A1), (A2), (A3). Given a piecewise linear function φ^h on Γ_K, with the nodes determined by $\mathcal{T}_h$ and such that*

$$\int_{\Gamma_K} \varphi^h \,\mathrm{d}s = 0\,. \tag{2.15}$$

Then there exists a function $w^h \in N_h(\Omega)$ such that

$$\mathbf{T}(w^h) = 0 \quad on \quad \Gamma_\tau\,, \tag{2.16}$$

$$T_t(w^h) = 0 \quad on \quad \Gamma_0\,, \tag{2.17}$$

$$T_t(w^{h\mathrm{I}}) = T_t(w^{h\mathrm{II}}) = 0 \quad on \quad \Gamma_K\,, \tag{2.18}$$

$$T_n(w^{h\mathrm{I}}) = T_n(w^{h\mathrm{II}}) = \varphi^h \quad on \quad \Gamma_K\,, \tag{2.19}$$

$$\|w^h\|_{0,\Omega} \leqq C\,h^{-1/2} \|\varphi^h\|_{0,\Gamma_K}\,. \tag{2.20}$$

Proof. Consider e.g. the domain Ω'' and the triangulation $\mathcal{T}_h$, satisfying (A1)–(A3) – see Fig. 2, where $\Gamma_K = AU$, $\Gamma_0 \supset DV$.

1° Let $ABCD$ be the upper "vaulted strip". On the edge AB we have a force

$$\mathbf{P}_1 = \int_A^B \varphi \, ds$$

(henceforth the superscript h will be omitted), perpendicular to AB, acting in AB and a moment M_1, where

$$|M_1| = \tfrac{1}{3} l_{AB}^2 \frac{|\varphi(A)|^2}{|\varphi(A)| + |\varphi(B)|} \tag{2.21}$$

holds if $\varphi(A) < 0$, $\varphi(B) > 0$, $|\varphi(A)| \leqq |\varphi(B)|$ or $M_1 = 0$ if $\varphi(A)\,\varphi(B) \geqq 0$.

The stress field response to the loading $\mathbf{P}_1$ and M_1 will be estimated separately.

2° *The loading by the force* $\mathbf{P}_1$. Let the stress vectors on the upper edge $AE_1F_1 \ldots$ of the strip be zero (Fig. 3). On the lower edge $BEF \ldots$, however, let a piecewise constant loading q_i acts, parallel to the x_2-axis, and such that the resultants $\mathbf{V}_j = \mathbf{P}_1 - \sum_{i=1}^{j} q_i l_i \mathbf{e}_2$, $j = 1, 2, \ldots$, are parallel to the tangent $\mathbf{t}$ of the arc AD at the points $E_1, F_1, \ldots$, and act inside the sides E_1E, F_1F, ...

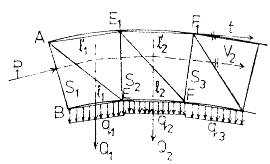

Fig. 3

We can show that *all stress vector components* in the strip $ABCD$ *are bounded above by* the number

$$C \max_{\Gamma_K} |\varphi| ,$$

where C is independent of h.

In fact, we may write (see Fig. 4)

$$\begin{aligned} |P_1| &= \tfrac{1}{2} s_1 |\varphi(A) + \varphi(B)| \leqq s_1 \max_{AB} |\varphi| , \\ |V_1| &= |P_1| \cos \vartheta_0 \cos^{-1} \vartheta_1 \leqq |P_1| \,.\, \cos^{-1} \Theta \\ |Q_1| &= |q_1 l_1| = |P_1| \cos \vartheta_0 (\operatorname{tg} \vartheta_0 - \operatorname{tg} \vartheta_1) = \\ &= |P_1| \cos^{-1} \vartheta_1 \,.\, \sin (\vartheta_0 - \vartheta_1) \leqq \\ &\leqq |P_1| \cos^{-1} \Theta \,.\, l_1' \varrho^{-1} \leqq C \beta |P_1| \,.\, l_1 , \end{aligned} \tag{2.22}$$

where ϱ denotes the radius of the curvature at the point E_1.

Consequently, we have

$$|q_1| \leqq C\,\beta|P_1| = C_1 h \max_{AB} |\varphi|\,. \tag{2.23}$$

The values of V_i, q_i, $i = 2, 3, \ldots$, can be estimated in a similar way.

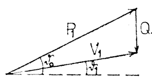

Fig. 4

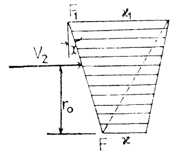

Fig. 5

Next let us consider the stress vectors on the side FF_1. They are equivalent with the resultant V_2, which acts at the point F_2, $FF_2 = z_0$, $FF_1 = s_3$ (see Fig. 5). It holds

$$\begin{aligned} V_2 \quad &= \tfrac{1}{2}\, s_3(\varkappa_1 + \varkappa) \\ V_2 r_0 &= \tfrac{1}{2}\, s_3(\varkappa_1 r_1 + \varkappa r) \end{aligned} \tag{2.24}$$

where

$$r_1 = \tfrac{2}{3}s_3 \cos\chi\,, \quad r = \tfrac{1}{3}s_3 \cos\chi\,,$$

$$r_0 = z_0 \cos\chi\,, \quad z_0 < s_3\,.$$

The solution of (2.24) with respect to $\varkappa_1$, $\varkappa$ yields that

$$\varkappa_1 = \frac{2V_2(r - r_0)}{s_3(r - r_1)}\,.$$

One easily derives the following inequalities

$$\left|\frac{r - r_0}{r - r_1}\right| = \left|\frac{\frac{1}{3}s_3 - z_0}{\frac{1}{3}s_3}\right| = \left|1 - 3\,\frac{z_0}{s_3}\right| \leqq 2\,,$$

$$|\varkappa_1| \leqq \frac{4}{s_3}\,|V_2| \leqq C\,\frac{s_1}{s_3} \max_{AB} |\varphi| \leqq C\beta \max_{AB} |\varphi|\,,$$

where (2.22) has been employed.

A similar estimate can be deduced for $\varkappa$. The stress vectors on the side F_1G can be estimated by an analogous way.

3° *The loading by the moment* M_1. From (2.21) it follows

$$|M_1| \leqq \tfrac{1}{6}s_1^2 \max_{AB} |\varphi|\,.$$

Let both the upper and lower edge of the strip $ABCD$ be without loading. Consider any side connecting the upper with the lower edge, e.g. FF_1. The moment M_1 will be equivalent with a linear loading, perpendicular to FF_1, the maximal value of which can be estimated as follows

$$|m| = \frac{6}{s_3^2}|M_1| \leqq \left(\frac{s_1}{s_3}\right)^2 \max_{AB}|\varphi| = \beta^2 \max_{AB}|\varphi| . \tag{2.25}$$

4° Consider an arbitrary "interior vaulted strip", between the strips $ABCD$ and $TUVZ$ – see Fig. 2. We shall construct the stress vectors, bounded again by $C \max_{\Gamma_K}|\varphi|$. Let the upper edge be loaded by a piecewise constant load $-q_i^H$, acting "upwards" and the lower edge by a piecewise constant load q_i^D "downwards". The differences $\Delta q_i = q_i^D - q_i^H$ are such that the resultants

$$\mathbf{V}_j^{(n)} = \mathbf{P}_n - \sum_{i=1}^{j} \Delta q_i l_i \mathbf{e}_2 , \quad j = 1, 2, \ldots$$

where

$$\mathbf{P}_n = \int_{s_1(n)} \varphi \, \mathrm{d}s ,$$

are parallel to the corresponding tangent of the upper arc, as previously.

Besides, q_i^H equals to q_i^D of the neighbouring strip above, of course.

The resultants $V_i^{(n)}$ can be estimated as in the strip $ABCD$. By an analogy to (2.23), we may write

$$|q_i^D| \leqq \sum_{j=1}^{n} |\Delta_i^{(j)}| \leqq \sum_{j=1}^{n} Ch \max_{\Gamma_K}|\varphi| \leqq C_1 \max_{\Gamma_K}|\varphi| ,$$

using the inequalities

$$nh \leqq h\beta \min_{j=1,2,\ldots,n} s_1^{(n)} \leqq \beta \sum_j s_1^{(n)} \leqq \beta \operatorname{mes} \Gamma_K .$$

5° The loading by the moment M_n, acting on the interior vaulted strip, can be estimated as that by M_1 in the upper strip.

6° Finally, let us show that the loading q_i^D of the lower strip $TUVZ$ equals to zero if the resultants $\mathbf{V}_i^{(N)}$ are parallel to the same tangent $\mathbf{t}$ at the corresponding vertex of the edge $AE_1 \ldots D$.

Let $AU = \Gamma_K$ consist of N sides $S_1^{(1)}, \ldots, S_1^{(N)}$. The following equations hold for the quadrangles adjoint to Γ_K:

$$\begin{aligned} \mathbf{P}_1 - \quad Q_1^{(1)}\mathbf{e}_2 &= \mathbf{V}_1^{(1)} , \\ \mathbf{P}_2 - \Delta Q_1^{(2)}\mathbf{e}_2 &= \mathbf{V}_1^{(2)} , \\ &\vdots \\ \mathbf{P}_N - \Delta Q_1^{(N)}\mathbf{e}_2 &= \mathbf{V}_1^{(N)} , \end{aligned}$$

where $\Delta Q_1^{(j)} = Q_1^{(j)} - Q_1^{(j-1)}$, $Q_1^{(j)} = l_1^{(j)} q_1^{D(j)}$, $j = 1, 2, \ldots, N$.

Since (2.15) implies that

$$\sum_{j=1}^{N} \mathbf{P}_j = \int_{\Gamma_K} \varphi \, \mathrm{d}s = 0 ,$$

by adding we obtain

$$- Q_1^{(N)} \mathbf{e}_2 = \sum_{j=1}^{N} \mathbf{V}_1^{(j)} .$$

The vectors on both sides have different directions and consequently $\mathbf{Q}_1^{(N)} = 0$, $\sum_{j=1}^{N} \mathbf{V}_1^{(j)} = 0$.

Considering the second column of quadrangles, we prove that $\mathbf{Q}_2^{(N)} = 0$, changing only $\mathbf{P}_j$ for $\mathbf{V}_1^{(j)}$, $j = 1, 2, \ldots, N$, a.s.o.

7° The triangulation of Ω'' outside the vaulted strip $AUVD$ can be chosen arbitrarily, it must fit in the division of the arcs $\widehat{AD}$ and $\widehat{UV}$, of course. We choose zero stress field everywhere outside $AUVD$.

8° On the domain Ω' a parallel approach can be used.

9° The above method of construction of the stress vectors guarantees that each triangular element K is loaded by self-equilibriated linearly distributed external forces. Theorem 2.1 can be applied to obtain a uniquely determined stress field $w^h|_K \in N(K)$ and the estimate (2.10) holds. Since the condition of continuity of stress vector is also satisfied, we arrive at a stress field $w^h \in N_h(\Omega)$. From (2.10) and the above estimates it follows

$$\max_{i=1,2,3} \|w^{hi}\|_{C(\bar{K}_i)} \leqq c \max_{\Gamma_K} |\varphi^h| \quad \forall K \in \mathcal{T}_h ,$$

and the same bound is true for $\|w^h\|_{0,\Omega}$.

It holds

$$\max_{\Gamma_K} |\varphi^h| \leqq C\, h^{-1/2} \|\varphi^h\|_{L_2(\Gamma_K)} . \tag{2.26}$$

In fact, let AB be an arbitrary side of $\mathcal{T}_h$ on Γ_K. Then

$$\int_A^B \varphi^2 \, \mathrm{d}s = \tfrac{1}{6} l_{AB}[\varphi(A)^2 + \varphi(B)^2 + (\varphi(A) + \varphi(B))^2] \geqq \tfrac{1}{6} l_{AB}[\varphi(A)^2 + \varphi(B)^2],$$

consequently

$$|\varphi(A)|^2 + |\varphi(B)|^2 \leqq 6 l_{AB}^{-1} \|\varphi\|^2_{L_2(AB)} \leqq 6\beta\, h^{-1} \|\varphi\|^2_{L_2(\Gamma_K)}$$

and (2.26) follows.

Thus we obtain the inequality (2.20). By the construction of w^h (especially of the stress vector $\mathbf{T}(w^h)$) it is easy to verify the boundary conditions (2.16)–(2.19).

Q.E.D.

Proof of Theorem 2.3. Let ψ^h be the function from Lemma 2.5. If we set

$$\varphi^h = T_n(r_h \tau^0) - \psi^h ,$$

then Lemma 2.6 implies that a $w^h \in N_h(\Omega)$ exists, satisfying the boundary conditions (2.16)–(2.19) and the estimate (2.20). Defining $W^h = r_h\tau^0 - w^h$, we obtain $W^h \in \mathscr{U}_{0h}$. In fact, $W^h \in N_h(\Omega)$ and, by virtue of the definition of r_h (cf. Remark 2.4)

$$\mathbf{T}(W^h) = 0 \quad \text{on} \quad \Gamma_\tau ,$$

$$T_t(W^h) = 0 \quad \text{on} \quad \Gamma_0 ,$$

$$T_t(W^{hI}) = T_t(W^{hII}) = 0 \quad \text{on} \quad \Gamma_K ,$$

$$T_n(W^{hI}) = T_n(W^{hII}) = T_n(r_h\tau^0) - \varphi^h = \psi^h \leqq -g \quad \text{on} \quad \Gamma_K .$$

Hence (see also Remark 2.5) $W^h \in \mathscr{U}_0$ follows.

Moreover, we show that

$$(2.27) \qquad 2\tau^0 - W^h \in \mathscr{U}_0 .$$

In fact, both τ^0 and W^h satisfy homogeneous boundary conditions on Γ_τ for T and on Γ_0, Γ_K for T_t, respectively. Moreover,

$$T_n(2\tau^0 - W^h) = 2\, T_n(\tau^0) - \psi^h \leqq -g \quad \text{on} \quad \Gamma_K .$$

Consequently, (2.27) follows.

Using Theorem 2.2, we write

$$\|\tau^0 - W^h\|_{0,\Omega} \leqq \|\tau^0 - r_h\tau^0\|_{0,\Omega} + \|r_h\tau^0 - W^h\|_{0,\Omega} \leqq$$
$$\leqq Ch^2\|\tau^0\|_{2,\Omega} + \|w^h\|_{0,\Omega} ,$$

and for the last term Lemmas 2.6 and 2.5 imply

$$\|w^h\|_{0,\Omega} \leqq C\, h^{-1/2}\|\varphi^h\|_{0,\Gamma_K} \leqq C_1\, h^{3/2} \left\| \frac{d^2}{ds^2}\, T_n(\tau^0) \right\|_{0,\Gamma_K} .$$

Finally, from Lemma 2.4 the error estimate follows. Q.E.D.

Remark 2.4. Let us discuss also the configuration of Example 3 in short, i.e., let $\Gamma_0 = \emptyset$ and Γ_K be a straight segment, parallel with the x_1-axis.

We construct a $\bar{\lambda}$ satisfying (2.7), define $\mathscr{U}_0$, the equivalent dual problem (2.9), $\mathscr{U}_{0h}$ and the approximations (2.12). If $V_2 \leqq 0$, $\mathscr{U}_0$ contains the zero element and there exists a unique approximation τ^h.

We can prove an analogue of Theorem 2.3, where the same regularity of τ^0 is required, assumptions (A1), (A2) are preserved and (A3) is replaced by the following (A3'): in Ω'' the triangulations $\mathscr{T}_h$ contain a fixed rectangle $AUBC$, independent of h, with $AU = \Gamma_K$, which is divided into small rectangular elements. In Ω' the triangulations $\mathscr{T}_h$ satisfy the same conditions as in (A3).

The proof is based on a modified Lemma 2.5, where a moment equilibrium condition

$$\int_{\Gamma_K} \psi^h s \, ds = \int_{\Gamma_K} T_n(r_h\tau^0)\, s \, ds$$

is added, and on an analogue of Lemma 2.6. To prove the latter lemma, we employ the idea of stress fields in a "beam" $AUBC$. Thus we obtain upper bounds $C \max_{\Gamma_K} |\varphi|$ for the linear stress vectors components on every side, as previously.

3. ALGORITHM FOR APPROXIMATIONS OF THE DUAL PROBLEM

In the theoretical analysis we used the stress vector components to determine the stress fields $\tau \in N_h(\Omega)$ – cf. Theorem 2.1. For practice, however, the stress tensor parameters $\beta_1^i, \ldots, \beta_7^i$ $(i = 1, 2, 3)$ (cf. Section 2.1) are more suitable. First we present a survey of some results of Watwood and Hartz [3], which can be employed immediately to construct an algorithm for the solution of the problem (2.12).

On every subtriangle K_i the stress field is given as follows

$$(3.1)\qquad \tau = \begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \end{bmatrix} = \frac{E}{\sqrt{(AE_0)}} \begin{bmatrix} \sqrt{A} & 0 & 0 & 0 & x_1 & 0 & x_2 \\ 0 & \sqrt{A} & 0 & x_1 & 0 & x_2 & 0 \\ 0 & \sqrt{0} & \sqrt{A} & 0 & -x_2 & -x_1 & 0 \end{bmatrix} S = MS\,,$$

where $S \in \mathbb{R}^7$ is a vector of coefficients, E Young's modulus, E_0 a dimensionless quantity, being equal to the reference modulus, A the area of K_i.

Let the origin of coordinates (x_1, x_2) be at the center of gravity of K_i, the material be homogeneous and isotropic in K_i, with a Poisson's constant σ. Then we have

$$(3.2)\qquad \int_{K_i} a_{mjkl} \tau_{mj} \tau_{kl} \, \mathrm{d}\mathbf{x} = S^{\mathsf{T}} f S\,,$$

where (t is the thickness of the element K_i):

$$f = \frac{tE}{E_0} \begin{bmatrix} \alpha A, & & & & & & \\ \beta A, & \alpha A, & & & & & \\ 0, & 0, & \gamma A, & & \text{symmetry} & & \\ 0, & 0, & 0, & \alpha\delta_1, & & & \\ 0, & 0. & 0, & \beta\delta_1, & \alpha\delta_1 + \gamma\delta_2, & & \\ 0, & 0, & 0, & \alpha\delta_{12}, & (\beta + \gamma)\,\delta_{12}, & \alpha\delta_2 + \gamma\delta_1, & \\ 0, & 0, & 0, & \beta\delta_{12}, & \alpha\delta_{12}, & \beta\delta_2, & \alpha\delta_2 \end{bmatrix}$$

$$\delta_j = \frac{1}{A} \int_{K_i} (x_j)^2 \, \mathrm{d}x_1 \, \mathrm{d}x_2\,, \quad j = 1, 2\,,$$

$$\delta_{12} = \frac{1}{A} \int_{K_i} x_1 x_2 \, \mathrm{d}x_1 \, \mathrm{d}x_2;$$

α, β, γ are constants, specified for plane stress as

$$\alpha = 1\,, \quad \beta = -\sigma\,, \quad \gamma = 2(1 + \sigma)$$

and for plane strain as

$$\alpha = 1 - \sigma^2, \quad \beta = -\sigma(1 + \sigma), \quad \gamma = 2(1 + \sigma).$$

Likewise we could derive

$$\int_{K_i} a_{mjkl} \tau_{mj} \bar{\lambda}_{kl} \, dx = \left(\int_{K_i} \bar{\lambda} B^{-1} M \, dx \right) S, \tag{3.3}$$

where $\bar{\lambda} = (\bar{\lambda}_{11}, \bar{\lambda}_{22}, \bar{\lambda}_{12})^{\mathrm{T}}$, M is the matrix from (3.1) and B^{-1} is a (3×3) matrix of the corresponding inverse Hooke's law.

Each of the stress vector components on a side $a_i a_{i+1}$ can be expressed in terms of external parameters $S^* \in \mathbb{R}^4$ and a parameter $p \in \langle -1, 1 \rangle$, as follows:

$$T_1(p) = S_1^* + S_2^* p,$$

$$T_2(p) = S_3^* + S_4^* p.$$

For example, consider the side $a_2 a_3$. Then $p = -1$ at a_2, $p = 1$ at a_3,

$$S^* = \frac{1}{2 l_a \sqrt{A}} \overset{(a)}{C} S,$$

where l_a is the length of $a_2 a_3$ and $\overset{(a)}{C}$ the following (4×7) matrix:

$$\overset{(a)}{C} = \begin{bmatrix} -2\sqrt{(A)}(Y_2 - Y_3), & 0, & 2\sqrt{(A)}(X_2 - X_3) & 0, \\ 0, & 0, & 0, & 0, \\ 0, & 2\sqrt{(A)}(X_2 - X_3), & -2\sqrt{(A)}(Y_2 - Y_3), & X_2^2 - X_3^2, \\ 0, & 0, & 0, & -(X_2 - X_3)^2, \end{bmatrix.$$

$$\begin{matrix} -2(X_2 Y_2 - X_3 Y_3), & -(X_2^2 - X_3^2), & -(Y_2^2 - Y_3^2) \\ -2(X_2 - X_3)(Y_2 - Y_3), & (X_2 - X_3)^2, & (Y_2 - Y_3)^2 \\ Y_2^2 - Y_3^2, & 2(X_2 Y_2 - X_3 Y_3), & 0 \\ -(Y_2 - Y_3)^2, & -2(X_2 - X_3)(Y_2 - Y_3), & 0 \end{matrix} \Bigg]$$

By a cyclic permutation of indices, the matrices $\overset{(b)}{C}$ and $\overset{(c)}{C}$ can be obtained. Denote the external parameters on the side b $S_5^*, S_6^*, S_7^*, S_8^*$ and on c $S_9^*, S_{10}^*, S_{11}^*, S_{12}^*$. For the total vector $S^* \in \mathbb{R}^{12}$ it holds

$$S^* = CS, \tag{3.4}$$

where the $(12 \times .7)$ matrix C is composed of $\overset{(a)}{C}, \overset{(b)}{C}, \overset{(c)}{C}$:

$$C = \frac{1}{2\sqrt{A}} \begin{bmatrix} l_a^{-1} & \overset{(a)}{C} \\ l_b^{-1} & \overset{(b)}{C} \\ l_c^{-1} & \overset{(c)}{C} \end{bmatrix}.$$

The conditions of continuity for the stress vector across any common side of adjoint triangles take the form

$$S_i^* + S_j^* = 0\,, \tag{3.5}$$

(where the indices i, j correspond with the same basis functions but with different triangles).

It is readily seen from the definition of $\mathcal{U}_{0h} = \mathcal{U}_0 \cap N_h(\Omega)$, that $\tau \in \mathcal{U}_{0h}$ if and only if:

all constraints of the type (3.5) hold,

$$S_j^* = 0 \quad \text{on any side} \quad a_i a_{i+1} \subset \bar{\Gamma}_\tau\,, \tag{3.6}$$

$$S_j^* t_1 + S_{j+2}^* t_2 = 0 \quad \text{on} \quad \bar{\Gamma}_0 \cup \bar{\Gamma}_K\,, \tag{3.7}$$

(where t_k denote the tangential vector components and (3.7) hold independently for $\tau^M \in N_h(\Omega^M)$, $M = {}^{\text{I}}, {}^{\text{II}}$),

$$\begin{aligned} &[S_j^* n_1' + S_{j+2}^* n_2' - (S_{j+1}^* n_1' + S_{j+3}^* n_2')]_{\Omega'} \leqq -g \quad \text{on} \quad \bar{\Gamma}_K\,, \\ &[S_j^* n_1' + S_{j+2}^* n_2' + (S_{j+1}^* n_1' + S_{j+3}^* n_2')]_{\Omega'} \leqq -g \quad \text{on} \quad \bar{\Gamma}_K \end{aligned} \tag{3.8}$$

and conditions (3.5) for any common side of two triangles, which belongs to $\bar{\Gamma}_K$.

The following *compensation of parameters* is recommended. In each triangular block-element we exclude the internal degrees of freedom. Let the conditions of continuity at the segment $0a_i$ (see Fig. 1) be written as

$$A_u S = 0 \tag{3.9}$$

where A_u is a (12×21) matrix and S a (21×1) matrix. There exists a regular (21×21) matrix Q such that

$$A_u Q = [I \mid 0]\,, \tag{3.10}$$

where I is the unit matrix. The matrix Q is not determined uniquely. It suffices even to replace I in (3.10) by any regular (12×12) matrix. Moreover, only the last nine columns of Q, i.e. a matrix Q_1, is needed.

Let us transform S as follows

$$S = Q\hat{S} = [Q_0 \mid Q_1] \begin{bmatrix} \hat{S}_u \\ \hat{S}_l \end{bmatrix}, \tag{3.11}$$

where Q is divided between 12th and 13th column and $\hat{S}$ accordingly. Substituting into (3.9), we obtain

$$A_u Q \hat{S} = [I \mid 0]\, \hat{S} = 0 \Rightarrow \hat{S}_u = 0$$

and the transformation (3.11) is reduced to

$$S = Q_1 \hat{S}_l\,, \tag{3.12}$$

where Q_1 is a (21×9) matrix and $\hat{S}_l$ a (9×1) matrix. The parameters $\hat{S}_l$ are independent degrees of freedom of the block-element.

The functional $J(\tau)$ of the equivalent dual problem can be evaluated in terms of $\hat{S}_l$:

$$J(\tau) = \tfrac{1}{2} \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{3} \int_{K_i} a_{mjkl} \tau_{mj} (\tau_{kl} + 2\bar{\lambda}_{kl}) \, d\mathbf{x} =$$

$$= \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{3} (\tfrac{1}{2} S^\mathsf{T} f S + B_0^\mathsf{T} S) = \sum_{K \in \mathcal{T}_h} (\tfrac{1}{2} S^\mathsf{T} F S + b_0^\mathsf{T} S) =$$

$$= \sum_{K \in \mathcal{T}_h} (\tfrac{1}{2} \hat{S}_l^\mathsf{T} Q_1^\mathsf{T} F Q_1 \hat{S}_l + b_0^\mathsf{T} Q_1 \hat{S}_l) = \tfrac{1}{2} \hat{S}_l^\mathsf{T} A \hat{S}_l + b^\mathsf{T} \hat{S}_l = J(\hat{S}_l),$$

(where the vectors $S, \hat{S}_l$ correspond with a subtriangle, block-element and the whole triangulation, respectively). The $(N \times N)$ matrix A is positive definite.

Inserting (3.4) and (3.12) into the conditions of the type (3.5)–(3.8), we obtain the constraints

$$D\hat{S}_l = 0\,, \tag{3.13}$$

$$E\hat{S}_l \leqq -\mathbf{g}\,, \tag{3.14}$$

where D and E are $(p_1 \times N)$ and $(p_2 \times N)$ matrices, respectively, $\mathbf{g}$ a vector with identical components, equal to g.

Let us define the set

$$\mathcal{B} = \{\hat{S}_l \in \mathbb{R}^N \mid (3.13) \text{ and } (3.14) \text{ are satisfied}\}\,.$$

Thus we arrived at the problem to find $\sigma \in \mathcal{B}$ such that

$$J(\sigma) \leqq J(\hat{S}_l) \quad \forall \hat{S}_l \in \mathcal{B}\,. \tag{3.15}$$

One can apply e.g. the *algorithm of Uzawa* for solving (3.15) (cf. [2] – chpt. 5). Denote $p = p_1 + p_2$ and

$$B = \begin{bmatrix} D \\ E \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ \mathbf{g} \end{bmatrix}$$

the $(p \times N)$ and $(p \times 1)$ matrix, respectively. Introduce the set of admissible Lagrange multipliers

$$\Lambda = \{z \in \mathbb{R}^p \mid z_j \geqq 0 \text{ for } j = p_1 + 1, \ldots, p\}\,.$$

Choosing a $z^0 \in \Lambda$, we calculate $s^0 \in \mathbb{R}^N$ from the system

$$As^0 = -b - B^\mathsf{T} z^0\,.$$

Having z^n, s^n, the values of z^{n+1}, s^{n+1} are determined as follows

$$z^{n+1} = P_\Lambda[z^n + \varrho(Bs^n + G)]\,,$$

$$As^{n+1} = -b - B^\mathsf{T} z^{n+1}\,,$$

where P_Λ denotes the projection onto the set Λ, i.e.,

$$y = P_\Lambda v \Leftrightarrow \begin{cases} y_j = v_j\,, & j = 1, \ldots, p_1 \\ y_j = \max\{0, v_j\}\,, & j = p_1 + 1, \ldots, p\,, \end{cases}$$

and ϱ is a sufficiently small parameter.

It can be proven that $s^n \to \sigma$ in $\mathbb{R}^N$ for $n \to \infty$, where σ is the (unique — see Lemma 2.3) solution of (3.15), provided that rang $B = p_1 + p_2$.

Finally, let us discuss again a possible *construction of the auxiliary stress field* $\bar{\lambda}$ (cf. Section 2). Again, we consider the configuration of Example 1 and choose the boundary condition (2.2) on Γ_K. According to the interpretation of the set $K^+_{F,P}$ and (2.7) (see also Lemma 1.6 and Lemma 2.2), we may proceed as follows.

Choose $\lambda^{(1)} \in S$, satisfying the equations (1.18) in $\Omega = \Omega' \cup \Omega''$ (by simple integration with respect to x_1 or x_2).

Assume that $\boldsymbol{F}$ is a constant vector field (zero or gravitational forces in most cases) and $\boldsymbol{P}$ is a piecewise linear vector field. Then $\lambda^{(1)}_{ij}$ are linear polynomials over Ω and $T_i(\lambda^{(1)})$ are linear on every side of the polygonal boundaries $\partial\Omega' \cup \partial\Omega''$.

Setting $\bar{\lambda} = \lambda^{(1)} + \lambda^{(2)}$ we may look for $\lambda^{(2)}$ in the space $N_h(\Omega)$, satisfying moreover the following boundary conditions

$$\boldsymbol{T}(\lambda^{(2)}) = \boldsymbol{P} - \boldsymbol{T}(\lambda^{(1)}) \quad \text{on} \quad \Gamma_\tau\,, \tag{3.16}$$

$$T_t(\lambda^{(2)}) = -T_t(\lambda^{(1)}) \quad \text{on} \quad \Gamma_0\,, \tag{3.17}$$

$$T_t(\lambda^{(2)\mathrm{I}}) = T_t(\lambda^{(2)\mathrm{II}}) = -T_t(\lambda^{(1)}) \quad \text{on} \quad \Gamma_K\,, \tag{3.18}$$

$$T_n(\lambda^{(2)\mathrm{I}}) = T_n(\lambda^{(2)\mathrm{II}}) = g - T_n(\lambda^{(1)}) \quad \text{on} \quad \Gamma_K\,. \tag{3.19}$$

Since the right-hand sides in (3.16)–(3.19) are linear or piecewise linear functions, the stress field $\lambda^{(2)}$ exists. It can be constructed using the procedure described above. We employ the parameters $\hat{S}_l$, and the formulas (3.1), (3.12), (3.4), write the continuity conditions of the type (3.5) and express the boundary conditions (3.16)–(3.19) in terms of $\hat{S}_l$ via (3.4) and (3.12). The undetermined values of $T_n(\lambda^{(2)})$ on Γ_0 and $T_i(\lambda^{(2)})$ on Γ_u, $i = 1, 2$, can be chosen in such a way that the resulting system of linear equations is solvable. The choice corresponds with the global equilibrium of the body Ω'' and Ω', respectively.

Remark 3.1. The dual analysis enables us to find *a posteriori error estimates* and two-sided bounds of the exact energy. The derivation may follow e.g. the approach of [8] — Section 5.

References

[1] *Haslinger, J., Hlaváček, I.:* Contact between elastic bodies. I. Continuous problems. Apl. mat. 25 (1980), 324—347. II. Finite element analysis. Apl. mat. 26. (1981), 263—290.

[2] *Céa, J.:* Optimisation, théorie et algorithmes. Dunod, Paris 1971.
[3] *Watwood, V. B., Hartz, B. J.:* An equilibrium stress field model for finite element solution of two-dimensional elastostatic problems. Int. J. Solids Structures 4 (1968), 857—873.
[4] *Hlaváček, I.:* Convergence of an equilibrium finite element model for plane elastostatics. Apl. mat. 24 (1979), 427—457.
[5] *Johnson, C., Mercier, B.:* Some equilibrium finite element methods for two-dimensional elasticity problems. Numer. Math. 30, (1978), 103—116.
[6] *Mosco, U., Strang, G.:* One-sided approximations and variational inequalities. Bull. Am. Math. Soc. 80 (1974), 308—312.
[7] *Hlaváček, I.:* Dual finite element analysis for unilateral boundary value problems. Apl. mat. 22 (1977), 14—51.
[8] *Hlaváček, I.:* Dual finite element analysis for semi-coercive unilateral boundary value problems. Apl. mat. 23 (1978), 52—71.

Souhrn

KONTAKT PRUŽNÝCH TĚLES — III. DUÁLNÍ ANALÝZA METODOU KONEČNÝCH PRVKŮ

JAROSLAV HASLINGER, IVAN HLAVÁČEK

Jednostranná kontaktní úloha dvou pružných těles s omezeným rozsahem kontaktu je formulována v napětích pomocí principu minima doplňkové energie. Definují se aproximace řešení, které se skládají z trojúhelníkových blokových rovnovážných prvků. V případě, že přesné řešení je dostatečně regulární, odvozuje se odhad chyby v normě L^2.

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