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ON CONVERGENCE OF HOMOGENEOUS MARKOV CHAINS

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In the paper we study the validity of an inequality, which may be useful in investigating the character of convergence of distributions in Markov chains.

Let $\mathbf{P} = (p_{ij})$ be a finite stochastic matrix, $\sum_j p_{ij} = 1$, and let $\mathbf{p}_t, \mathbf{p}_{t+1} = \mathbf{p}_t\mathbf{P}$ and $\mathbf{p}_{t+2} = \mathbf{p}_{t+1}\mathbf{P}$ be row vectors of distributions of probabilities in the corresponding Markov chain. We denote the matrix-transposition by a prime and the norm of a vector $\mathbf{x} = (x_1, x_2, \ldots)$ by $\|\mathbf{x}\| = \sum_i |x_i|$. The corresponding norm of the matrix $\mathbf{P}$ is $\|\mathbf{P}\| = \max_i \sum_j p_{ij} = 1$, therefore

$$\|\mathbf{p}_{t+2} - \mathbf{p}_{t+1}\| = \|(\mathbf{p}_{t+1} - \mathbf{p}_t)\,\mathbf{P}\| \leqq \|\mathbf{p}_{t+1} - \mathbf{p}_t\| \,. \tag{1}$$

With the help of simple calculations it is easy to prove that even the strict inequality holds for two-state Markov chains in (1) in nontrivial cases. Professor Alladi Ramakrishnan*) has conjuctured that the strict inequality holds for every irreducible aperiodic homogeneous Markov chain**). However, the conjecture turns out not to be true in general. We give a necessary and sufficient condition for its validity in the following

Theorem *Let X_t, $t = 1, 2, \ldots$, be an irreducible aperiodic homogeneous Markov chain with a finite state space $S = \{s_1, s_2, \ldots, s_k\}$. Denote the absolute distributions by $\mathbf{p}_t(i) = P(X_t = s_i)$, $s_i \in S$, and the row vector by $\mathbf{p}_t = (p_t(1), p_t(2), \ldots, p_t(k))$ at a time t.*

Then the strict inequality

$$\sum_{i=1}^{k} |p_{t+2}(i) - p_{t+1}(i)| < \sum_{i=1}^{k} |p_{t+1}(i) - p_t(i)| \tag{2}$$

*) Director of The Institute of Mathematical Sciences, Madras.

**) Private communication by F. Zítek.

holds for each nonstationary $\mathbf{p}_t$ *if and only if the product* $\mathbf{PP}'$ *is a positive matrix, i.e. if and only if for each pair of distinct states* $s_i, s_j \in S$, $i \neq j$, *there is a state* $s_m \in S$ *such that there are transitions to the state* s_m, $p_{im} > 0$, $p_{jm} > 0$.

Remark. In the case of a two-state Markov chain, the assumptions of Theorem imply positivity of the matrix $\mathbf{P}$, therefore (2) is satisfied as we have mentioned.

Introduce a set of vectors

$$Z = \{\mathbf{x};\ \mathbf{x} = (x_1, x_2, \ldots, x_k),\quad \sum_{i=1}^{k} x_i = 0 \quad \text{and} \quad \sum_{i=1}^{k} |x_i| > 0\}\,.$$

In the proof of Theorem, we shall use the following

Lemma. *Under the suppositions of Theorem, the inequality (2) holds for each nonstationary* $\mathbf{p}_t$ *if and only if*

$$\sum_{i=1}^{k} \Big| \sum_{j=1}^{k} x_j p_{ji} \Big| < \sum_{i=1}^{k} |x_i| \quad \textit{for each} \quad \mathbf{x} \in Z\,. \tag{3}$$

Proof of Lemma. Sufficiency of the condition (3). Put $x_i = p_{t+1}(i) - p_t(i)$. Then $\mathbf{x} \in Z$ and (3) implies (2) immediately.

Necessity of (3). Let the relations (2) be not true, i.e. let there exist $\mathbf{b} \in Z$ such that

$$\sum_{i=1}^{k} \Big| \sum_{m=1}^{k} b_m p_{mi} \Big| = \sum_{i=1}^{k} |b_i|\,.$$

(Notice that the left hand side cannot be greater than the right hand side:

$$\sum_{i=1}^{k} \Big| \sum_{m=1}^{k} b_m p_{mi} \Big| \leqq \sum_{i=1}^{k} \sum_{m=1}^{k} |b_m|\, p_{mi} = \sum_{m=1}^{k} |b_m| \sum_{i=1}^{k} p_{mi}\,.)$$

Denote by $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ a stationary distribution of the chain under consideration. The irreducibility implies $0 < \pi_i < 1$ for each $i = 1, 2, \ldots, k$. Since (1) is is a simple characteristic root of $\mathbf{P}$, the rank of the matrix of the system

$$\sum_{j=1}^{k} z_j p_{jm} - z_m = b_m\,, \quad m = 1, 2, \ldots, k \tag{4}$$

of linear equations is equal to $k - 1$. Hence, $\sum_{m=1}^{k} \big(\sum_{j=1}^{k} z_j p_{jm} - z_m \big) = 0 = \sum_{m=1}^{k} b_m$ implies that the system (4) possesses a nonzero solution $\mathbf{z} = (z_1, z_2, \ldots, z_k)$. The vector $\mathbf{z}$ cannot be proportional to π, for π is a solution of the corresponding homogeneous system. There is a sufficiently small positive constant c such that $x_m = \pi_m + c z_m > 0$ for all $m = 1, 2, \ldots, k$. Denote $d = \sum_{m=1}^{k} x_m$ and $p_t(m) = x_m/d$. The vector $\mathbf{p}_t = (p_t(1), p_t(2), \ldots, p_t(k))$ is not proportional to π, therefore it is a nonstationary distribution and it is a solution of a system analogous to (4) with the right hand sides replaced by $c b_m/d$, $m = 1, 2, \ldots, k$. We get

$$\sum_{i=1}^{k} |p_{t+2}(i) - p_{t+1}(i)| =$$

$$= \sum_{i=1}^{k} \Big| \sum_{m=1}^{k} \sum_{j=1}^{k} p_t(j)\, p_{jm} p_{mi} - \sum_{m=1}^{k} p_t(m)\, p_{mi} \Big| =$$

$$= (c/d) \sum_{i=1}^{k} \Big| \sum_{m=1}^{k} p_{mi} \Big(\sum_{j=1}^{k} z_j p_{jm} - z_m \Big) \Big| = (c/d) \sum_{i=1}^{k} \Big| \sum_{m=1}^{k} p_{mi} b_m \Big| =$$

$$= (c/d) \sum_{i=1}^{k} |b_i| = (c/d) \sum_{i=1}^{k} \Big| \sum_{j=1}^{k} z_j p_{ji} - z_i \Big| = \sum_{i=1}^{k} |p_{t+1}(i) - p_t(i)| ,$$

which means that (2) is not true.

Proof of Theorem. Necessity of the condition. Suppose that $\boldsymbol{PP'}$ is not positive, i.e. there are s and u such that $\sum_{j=1}^{k} p_{sj} p_{uj} = 0$. Put $x_s = 1, x_u = -1$ and $x_i = 0$ for $s \neq i \neq u$. Then $\sum_{j=1}^{k} x_j = 0$ and the vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_k)$ belongs to Z. However, $\sum_{i=1}^{k} \big| \sum_{j=1}^{k} x_j p_{ji} \big| = \sum_{i=1}^{k} |p_{si} - p_{ui}| = \sum_{i=1}^{k} (p_{si} + |-p_{ui}|) = 2$ as p_{si} or p_{ui} equals zero for all i. The identity $\sum_{j=1}^{k} |x_j| = 2$ means that (3) is not true. According to Lemma, (2) is not satisfied, either.
Sufficiency of the condition. Denote $M = \{i;\, x_i \geqq 0\}$ and $L = \{i;\, x_i < 0\}$. Denote for brevity $c = \sum_{i=1}^{k} |x_i|$. We get

$$\sum_{j \in M} x_j = - \sum_{j \in L} x_j = c/2 . \tag{5}$$

For each $i = 1, 2, \ldots, k$, denote $r_i = \sum_{j \in M} x_j p_{ji}$, $s_i = -\sum_{j \in L} x_j p_{ji}$, $A = \{i;\, r_i \geqq s_i\}$, $B = \{i;\, r_i < s_i\}$, $r = \sum_{i \in A} r_i$, $s = \sum_{i \in A} s_i$. The identities (5) imply $\sum_{i=1}^{k} r_i = \sum_{i=1}^{k} s_i = c/2$, and of course, $\sum_{i \in B} r_i = c/2 - r$, $\sum_{i \in B} s_i = c/2 - s$. We get $\sum_{i=1}^{k} \big| \sum_{j=1}^{k} x_j p_{ji} \big| = \sum_{i=1}^{k} |r_i - s_i| = \sum_{i \in A} (r_i - s_i) + \sum_{i \in B} (s_i - r_i) = (r - s) + (r - s) = 2(r - s)$. Since both the numbers r and s are in the square $0 \leqq r \leqq c/2$, $0 \leqq s \leqq c/2$, the inequality $2(r - c) \leqq c$ is true.

Now, if (2) is not satisfied, then the equivalent condition (3) is not satisfied either, which means $2(r - s) = c$. Moreover, this identity holds if and only if $r = c/2$ and $s = 0$. i.e., if $0 = c/2 - r = \sum_{i \in B} r_i = \sum_{i \in B} \sum_{j \in M} x_j p_{ji}$, $0 = s = \sum_{i \in A} s_i = -\sum_{i \in A} \sum_{j \in L} x_j p_{ji}$. The sum of the components of the nonzero vector $\boldsymbol{x}$ equals zero, therefore $x_v > 0$, $x_u < 0$ for some suitable $v \in M$ and $u \in L$, i.e. $p_{ui} = 0$ for each $i \in A$ and $p_{vi} = 0$ for each $i \in B$. Hence $\sum_{i=1}^{k} p_{ui} p_{vi} = 0$, which means that the product $\boldsymbol{PP'}$ is not positive.

Souhrn

O KONVERGENCI HOMOGENNÍCH MARKOVOVÝCH ŘETĚZCŮ

PETR KRATOCHVÍL

Nechť $\boldsymbol{p}_t$ značí vektor rozložení absolutních pravděpodobností v nerozložitelném aperiodickém homogenním Markovově řetězci s konečným prostorem stavů. Profesor Alladi Ramakrishnan navrhl následující ostrou nerovnost pro normy rozdílů

$$\|\boldsymbol{p}_{t+2} - \boldsymbol{p}_{t+1}\| < \|\boldsymbol{p}_{t+1} - \boldsymbol{p}_t\| \, .$$

V článku je dokázána nutná a postačující podmínka pro platnost této nerovnosti, což může být užitečné při zkoumání charakteru konvergence rozložení v markovových řetězcích.

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