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BOUNDS OF THE ROOTS OF THE REAL POLYNOMIAL

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Summary. An algorithm for the calculation of a lower bound of the absolute values of the roots of a real algebraic polynomial, of an arbitrary degree, is derived. An example is given to compare the bounds calculated by the method proposed and by other methods.

1. FORMULATION OF THE PROBLEM

Let us consider a real polynomial

$$P(x) = x^n - s_1 x^{n-1} + (-1)^2 s_2 x^{n-2} + \ldots + (-1)^{n-2} s_{n-2} x^2 + + (-1)^{n-1} s_{n-1} x + (-1)^n s_n \,, \tag{1.1}$$

where $s_1, s_2, \ldots, s_n$ are symmetric functions of roots and $s_n \neq 0$. In what follows, consider the equation $P(x) = 0$. We define a substitution

$$x = \frac{(-1)^{n-1} s_n}{x_0} \,; \tag{1.2}$$

then we have

$$\left[\frac{(-1)^{n-1} s_n}{x_0}\right]^n - s_1 \left[\frac{(-1)^{n-1} s_n}{x_0}\right]^{n-1} + (-1)^2 s_2 \left[\frac{(-1)^{n-1} s_n}{x_0}\right]^{n-2} + \ldots + + (-1)^{n-2} s_{n-2} \left[\frac{(-1)^{n-1} s_n}{x_0}\right]^2 + (-1)^{n-1} s_{n-1} \left[\frac{(-1)^{n-1} s_n}{x_0}\right] + (-1)^n s_n = 0 \,.$$

Let us divide this equation by the expression

$$\frac{(-1)^n s_n}{x_0^2}$$

and then extract before the brackets the factor $(-1)^{n-1} s_n$ from those members in which it is contained. So we have

(1.3)

$$(-1)^{n-1} s_n \left\{ \left[\frac{(-1)^{n-1} s_n}{x_0} \right]^{n-2} - s_1 \left[\frac{(-1)^{n-1} s_n}{x_0} \right]^{n-3} + \ldots + (-1)^{n-2} s_{n-2} \right\} +$$

$$+ (-1)^{n-1} s_{n-1} x_0 - x_0^2 = 0 .$$

If we set

$$(1.4) \qquad x_1 = \left[\frac{(-1)^{n-1} s_n}{x_0} \right]^{n-2} - s_1 \left[\frac{(-1)^{n-1} s_n}{x_0} \right]^{n-3} + \ldots + (-1)^{n-2} s_{n-2} ,$$

then we obtain, after a small arrangement, the equation (1.3) in the form

$$(1.5) \qquad x_0^2 + x_0 s_{n-1}(-1)^{n-2} - x_1 s_n(-1)^{n-1} = 0 .$$

Theorem 1.1. *Let* $s_n \neq 0$. *Then for every solution* (x, x_0, x_1) *of the system*

$$x_0^2 + x_0 s_{n-1}(-1)^{n-2} - x_1 s_n(-1)^{n-1} = 0 ,$$

$$(1.6) \qquad x_1 = \left[\frac{(-1)^{n-1} s_n}{x_0} \right]^{n-2} - s_1 \left[\frac{(-1)^{n-1} s_n}{x_0} \right]^{n-3} + \ldots + (-1)^{n-2} s_{n-2} ,$$

$$x x_0 = (-1)^{n-1} s_n ,$$

x is a root of the polynomial (1.1).

The proof is clear.

In the plane (x_0, x_1) the first equation of the system (1.6) represents a parabola, the second equation yields an algebraic curve with the asymptote

$$(1.7) \qquad x_1 = (-1)^{n-2} s_{n-2} .$$

We are going to do further considerations in the plane (x_0, x_1).

Lemma 1.1. *The function*

$$(1.8) \qquad f(x_0) = \frac{x_0(x_0 - |s_{n-1}|)}{|s_n|} , \quad s_n \neq 0 ,$$

has in the open interval $(0, \infty)$ *the following characteristics*:

a) f *is continuous*;

b) *for* $x_0 > |s_{n-1}|$, f *is positive*;

c) *for* $x_0 > |s_{n-1}|/2$, f *is monotonously ascending.*

These characteristics are obvious, so we will not prove them.

Lemma 1.2. *The function*

$$(1.9) \qquad g(x_0) = \left(\frac{|s_n|}{x_0} \right)^{n-2} + |s_1| \left(\frac{|s_n|}{x_0} \right)^{n-3} + \ldots + |s_{n-3}| \frac{|s_n|}{x_0} + |s_{n-2}|$$

has in the open interval $(0, \infty)$ *the following characteristics*:

a) g *is continuous and positive*;

b) g *is monotonously descending*;

c) g *has an asymptote* $x_1 = |s_{n-2}|$, *such that* $g(x_0) > |s_{n-2}|$;

d) *for the real coordinate* $\bar{x}_0$ *of the point of intersection with the curve* (1.8) *we have*: $\bar{x}_0 > |s_{n-1}|$.

The characteristics a), b), c) of Lemma 1.2 are obvious, d) follows from the characteristic b) of Lemma 1.1 and from the characteristic a) of Lemma 1.2.

Theorem 2.1. *Consider the polynomial* (1.1), *where at least one of the coefficients* s_{n-1}, s_{n-2} *is different from zero. Let the function* $g(x_0)$ *be given by the relation* (1.9). *Then for the absolute value* $|\lambda|$ *of the roots of the polynomial* (1.1) *the following inequality holds*:

$$|\lambda| \geqq \frac{2|s_n|}{|s_{n-1}| + \sqrt{(|s_{n-1}|^2 + g(x_0)\,|s_n|)}}, \quad x_0 \in (0, \infty)\,. \tag{1.10}$$

Proof. Solving the quadratic equation in the relation (1.6) for the unknown x_0, we obtain

$$(x_0)_{1,2} = \frac{-s_{n-1}(-1)^{n-2} \pm \sqrt{(s_{n-1}^2 + 4x_1 s_n(-1)^{n-1})}}{2}\,. \tag{1.11}$$

Hence for the absolute value $|x_0|$ we get

$$|x_0| \leqq \frac{|s_{n-1}| + \sqrt{(|s_{n-1}|^2 + 4|x_1|\,|s_n|)}}{2} \leqq \tag{1.12}$$

$$\leqq \frac{|s_{n-1}| + \sqrt{(|s_{n-1}^2|^2 + 4\,g(x_0)\,|s_n|)}}{2} = \bar{x}_0\,,$$

where $g(x_0)$ is given by the relation (1.9) and $\bar{x}_0$ is the real coordinate of the point of intersection of the curves given by the relations (1.8) and (1.9). But from the third equation of the system (1.6) we have

$$|x| = \frac{|s_n|}{|x_0|}, \tag{1.13}$$

and according to Theorem 1.1

$$|x| = |\lambda|\,, \tag{1.14}$$

which yields

$$|x_0| = \frac{|s_n|}{|\lambda|}\,.$$

Let us substitute the last relation into the inequality (1.12); then the statement (1.10) directly follows. This completes the proof.

2. THE ALGORITHM FOR THE CALCULATION OF A LOWER BOUND OF THE ROOTS OF THE POLYNOMIAL (1.1).

The value $\bar{x}_0$ in the relation (1.12), as has already been mentioned, is in fact the coordinate of the real intersection point of the curves $f(x_0)$ and $g(x_0)$ described in Lemma 1.1 and Lemma 1.2. Because we do not know how to calculate the coordinate precisely, we propose how to calculate an upper bound of the point $\bar{x}_0$. We proceed so that we find two points x_{01}, x_{02}, such that the inequalities

$$f(x_{01}) < g(x_{01}), \quad f(x_{02}) > g(x_{02}) \tag{2.1}$$

are satisfied. Then, taking into account the properties of the functions $f(x_0)$, $g(x_0)$, we conclude that the point x_{02} is an upper bound of the point $\bar{x}_0$. We proceed in three steps:

1. We find the coordinate of the intersection point of the asymptote c) in (1.9) with the parabola (1.8), obtaining the solution

$$x_{01} = \frac{|s_{n-1}| + \sqrt{(|s_{n-1}|^2 + 4|s_{n-2}|\,|s_n|)}}{2}, \tag{2.2}$$

where s_{n-1}, s_{n-2} are not both simultaneously equal to zero. It satisfies the relation

$$f(x_{01}) < g(x_{01}), \tag{2.3}$$

because $f(x_{01}) = |s_{n-2}|$ and, according to c) in Lemma 1.2, $|s_{n-2}| < g(x_{01})$.

2. We calculate the value of the function (1.9) at the point x_{01}:

$$g(x_{01}) = \left|\frac{s_n}{x_{01}}\right|^{n-2} + |s_1| \left|\frac{s_n}{x_{01}}\right|^{n-3} + \ldots + |s_{n-3}| \left|\frac{s_n}{x_{01}}\right| + |s_{n-2}|. \tag{2.4}$$

3. We calculate the coordinate x_{02} of the point of intersection of the parabola (1.8) with the line

$$x_1 = g(x_{01}) = f(x_{02}). \tag{2.5}$$

We obtain

$$x_{02} = \frac{|s_{n-1}| + \sqrt{(|s_{n-1}|^2 + 4\,g(x_{01})\,|s_n|)}}{2}. \tag{2.6}$$

We claim that the point x_{02} is already an upper bound of the point $\bar{x}_0$. The relations (2.2) and (2.6) and c) in (1.9) imply the inequality

$$x_{01} < x_{02}. \tag{2.7}$$

As a consequence of Lemma 1.2 we have $g(x_{01}) > g(x_{02})$ and the relation (2.5) yields

$$f(x_{02}) > g(x_{02}). \tag{2.8}$$

The points x_{01}, x_{02} given by the relations (2.2) and (2.6), respectively, satisfy the conditions (2.1) and so, because of the relations (1.13) and (1.14), we obtain

$$(2.9) \qquad |\lambda| > \left|\frac{s_n}{x_{02}}\right| = \frac{2|s_n|}{|s_{n-1}| + \sqrt{(|s_{n-1}|^2 + 4\,g(x_{01})\,|s_n|)}}.$$

The relation (2.9) gives a lower bound of the absolute values of the roots of the polynomial (1.1).

Remark 2.1. Let us substitute

$$(2.10) \qquad x = \frac{1}{y}$$

into the polynomial (1.1). It is known that after an arrangement we get a polynomial with the roots

$$\frac{1}{\lambda_i}.$$

If we apply the above algorithm (computation of a lower bound of the absolute values of the roots of the polynomial (1.1)) to this polynomial then the upper bound of the absolute values of the roots of the polynomial (1.1) is the reciprocal value to the value (2.9).

As an interesting illustration, the results of the method proposed are demonstrated on a simple example, and these are compared with the results of other known methods, in particular with Westerfield's method and the method using A, B as maximum values.

Example. Consider the equation

$$(2.11) \qquad x^3 - 2{\cdot}10^2x^2 - 5{\cdot}10^4x + 6{\cdot}10^6 = 0\,.$$

Exact roots: $x_1 = 100$, $x_2 = -200$, $x_3 = 300$. After the substitution

$$x = \frac{1}{y}$$

into (2.11) we get

$$(2.12) \qquad y^3 - \frac{5}{6\,.\,10^2}\,y^2 - \frac{2}{6\,.\,10^4}\,y + \frac{1}{6\,.\,10^6} = 0\,.$$

1. *Westerfield's method.*

We calculate

$$q_r = \sqrt[r]{|s_r|}\,, \quad r = 1, 2, \ldots, n\,,$$

and arrange these values in a non-increasing sequence:

$$q_{k1} \geqq q_{k2} \geqq \ldots \geqq q_{kn}\,. \quad \text{Then} \quad |\lambda| \leqq q_{k1} + q_{k2}\,.$$

Lower bound: using (2.12), $q_1 = 8{\cdot}3333 \,.\, 10^{-3}$, $q_2 = 5{\cdot}7735 \,.\, 10^{-3}$, $q_3 = 5{\cdot}5032 \,.\, 10^{-3}$; $|\lambda| \geqq 70{\cdot}887798$.

Upper bound: using (2.11), $q_1 = 200$, $q_2 = 223{\cdot}60679$, $q_3 = 181{\cdot}71205$; $|\lambda| \leqq$ $\leqq 423{\cdot}60679$.

2. *Method using A, B as maximum values.*

Lower bound:

$$|\lambda| > \frac{1}{1 + \dfrac{B}{|s_n|}}, \quad B = \max\left(1, |s_1|, \ldots, |s_{n-1}|\right).$$

Upper bound:

$$|\lambda| < 1 + A, \quad A = \max\left(|s_1|, |s_2|, \ldots, |s_n|\right). \; |\lambda| > 0{\cdot}9917355;$$

$$|\lambda| < 6 \,.\, 10^6 + 1 \,.$$

3. *Proposed method.*

Lower bound: using (2.11),

$$x_{01} = 67720{\cdot}018\,, \quad g(x_{01}) = 288{\cdot}6\,, \quad x_{02} = 73544{\cdot}83\,; \quad |\lambda| > 81{\cdot}58\,.$$

Upper bound: using (2.12),

$$y_{01} = \left(1 + \sqrt{(6)}\right) . \, 60000^{-1}\,, \quad g(y_{01}) = 0{\cdot}0112323\,, \quad y_{02} = 0{\cdot}0630329 \,.\, 10^{-3}\,;$$

$$|\lambda| < 378{\cdot}19\,.$$

Table 1

Method	Lower bound	Upper bound
Using A, B	0·9917355	$6 \,.\, 10^6 + 1$
Westerfield's	70·887798	423·60679
Proposed	81·58	378·19

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Súhrn

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ODHAD ABSOLÚTNYCH HODNÔT KOREŇOV REÁLNEHO POLYNÓMU

V práci je odvodený algoritmus na výpočet dolnej hranice absolútnych hodnôt koreňov reálneho algebraického polynómu ľubovoľného stupňa. Na príklade sa porovnávajú hodnoty hraníc vypočítané navrhovanou metódou s hodnotami vypočítanými podľa iných metód.

Резюме

ОЦЕНКА АБСОЛЮТНЫХ ЗНАЧЕНИЙ КОРНЕЙ РЕАЛЬНОГО АЛГЕБРАИЧЕСКОГО ПОЛИНОМА

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В этой работе выведенный алгоритм для вычисления нижней границы абсолютных значений корней реального алгебраического полинома любого порядка. На примере сравниваются результаты вычисленны указанным методом и дальнейшими методами.

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