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TWO-PARAMETRIC MOTIONS IN E_3

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Summary. The paper deals with the local differential geometry of two-parametric motions in the Euclidean space. The first part of the paper contains contemporary formulation of classical results in this area together with the connection to the elliptical differential geometry. The remaining part contains applications: Necessary and sufficient conditions for splitting of a two-parametric motion into a product of two one-parametric motions, characterization of motions with constant invariants and some others. The case of rolling of two isometric surfaces is treated in detail.

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A. PRELIMINARIES

Let G be a Lie group of dimension m with Lie algebra $\mathfrak{G}$. By a p-dimensional motion in G, $p < m$, we mean an immersion g of a p-dimensional manifold X into G, $g: X \to G$. For a moment let us suppose that G is realized as a group of matrices, acting in a vector space V in a natural manner. Let us choose two copies of V, the moving space $\bar{V}$ and the fixed space V. In each of them we select a base $\bar{\mathscr{R}}_0 = \{\bar{f}_1, \ldots, \bar{f}_n\}$ and $\mathscr{R}_0 = \{f_1, \ldots, f_n\}$, respectively. Then G acts as a group of linear maps from $\bar{V}$ into V by the rule $g(\bar{\mathscr{R}}_0) = \mathscr{R}_0 . g$, where the product means formal multiplication of the row $\mathscr{R}_0$ by the matrix g. By a frame in $\bar{V}(V)$ we mean any base $\bar{\mathscr{R}}(\mathscr{R})$ such that $\bar{\mathscr{R}} = \bar{\mathscr{R}}_0 . g$ ($\mathscr{R} = \mathscr{R}_0 . g$), respectively, for some $g \in G$. The group G is then identified with the set of all frames in $\bar{V}$ or in V. Any p-dimensional motion $g(X)$ in G then determines a p-parametric system of linear maps from $\bar{V}$ into V by the rule $g(X)(\bar{\mathscr{R}}_0) = \mathscr{R}_0 g$, where $g(X) = g$ is the corresponding matrix.

Let us determine what happens if we choose different frames $\bar{\mathscr{R}}_0 = \bar{\mathscr{R}}_1 . \bar{\gamma}$ and $\mathscr{R}_0 = \mathscr{R}_1 . \gamma$, $\bar{\gamma}, \gamma \in G$, in $\bar{V}$ and V, respectively, as the basic ones. Then $g(X)(\bar{\mathscr{R}}_0) = g(X)(\bar{\mathscr{R}}_1\bar{\gamma}) = g(X)\,\bar{\mathscr{R}}_1\bar{\gamma} = \mathscr{R}_1\gamma g$ and so $g(X)\,\bar{\mathscr{R}}_1 = \mathscr{R}_1\gamma g\bar{\gamma}^{-1}$. If we write $g(X) . \bar{\mathscr{R}}_1 = \mathscr{R}_1\tilde{g}$, we get $\tilde{g} = \gamma g\bar{\gamma}^{-1}$. Motions g and $\tilde{g}$ have to be considered as equivalent. This leads to the following definition:

Definition 1. *Let G be a Lie group. Let us consider the homogeneous space $G_0 = G \times G/\mathrm{Diag}(G \times G)$, where $G \times G$ acts on G by the rule $(g_1, g_2)\, g = g_1 g g_2^{-1}$, the natural projection π is $\pi(g_1, g_2) = g_1 g_2^{-1}$, the origin is the unit element e of G, the isotropy group of e is $G_i = \mathrm{Diag}(G \times G)$. G_i is isomorphic with G and G_0 is identified with G as a manifold. By a p-dimensional motion g in G we understand an immersion g of a p-dimensional manifold X into G_0.*

For each closed subgroup H of G we have the corresponding homogeneous space G/H, with the transformation group G acting from the left, $g(g_1H) = gg_1H$, $g, g_1 \in G$. If $g(X)$ is a p-dimensional motion in G, then for each point $x_0 \in G/H$ we have the set $g(X)\, x_0$, which is called the *trajectory* of x_0 under $g(X)$. The trajectory of x_0 is in general not an immersed submanifold of G/H as the dimension of the tangent space of the trajectory of x_0 at $g(t)\, x_0 = x$, $t \in X$, is $\mathrm{p} - \dim\, (\mathfrak{G}_x \cap \omega(g_* X_t))$, where $\mathfrak{G}_x$ is the isotropy algebra of x and ω is the left invariant form on G. The kinematic geometry of the motion g then studies the relations between the properties of $g(X)$ and $g(X)\, x_0$ for various x_0 and H.

The group G has a natural representation in $\mathfrak{G}$ by the adjoint action. This means that for any p-dim. motion $g(X)$ (if Ad is an isomorphism of Lie algebras) we get a motion in $\mathfrak{G}$, ad $\mathrm{g}(X)$. By the natural projection $\tilde{\pi}$ from $\mathfrak{G}$ to the projective space P_{m-1} modelled on $\mathfrak{G}$ we get a motion in P_{m-1}. (Here, as well as in the sequel, all considerations are local.)

Further, let H be a closed subgroup of G with the Lie algebra $\mathfrak{H}$. Let $N(\mathfrak{H}) = \mathfrak{H}$, where $N(\mathfrak{H})$ denotes the normalizer of $\mathfrak{H}$ in the group ad$G$. Then the set of all subspaces of $\mathfrak{G}$ of the form ad$g(\mathfrak{H})$ is locally equimorphic with G/H and it is a submanifold of $\mathrm{Gr}\,(\mathfrak{G}, \dim H)$ – the Grassmanian manifold of all subspaces of dimension $\dim H$ in $\mathfrak{G}$, on which G acts by the induced action. This submanifold may serve as a model for the study of the properties of trajectories of points of G/H directly in $\mathfrak{G}$ or in P_{m-1}.

Example. Let $\mathscr{E}_3$ be the 6-dimensional Lie group of congruences of the 3-dim. Euclidean space E_3, let $\mathfrak{E}_3$ be its Lie algebra with the invariant Killing and Klein quadratic forms. We can construct the corresponding projective space P_5 together with the induced action. The manifold of all straight lines of E_3 is then naturally immersed in P_5 as the Klein quadric, and the trajectories of lines from E_3 may be regarded as subsets of the Klein quadric.

The Lie algebra of $G \times G$ is $\mathfrak{G} \times \mathfrak{G}$, the isotropy algebra $\mathfrak{G}_i$ is $\mathfrak{G}_i = \{(X, X) \mid X \in \mathfrak{G}\}$ and may be identified with $\mathfrak{G}$. The adjoint representation of the isotropy group G_i is ad $g(X, Y) = (\mathrm{ad}\, gX, \mathrm{ad}\, gY)$. Let us denote $\mathbf{m} = \{(X, -X) \mid X \in \mathfrak{G}\}$. Then we have an ad G invariant splitting $\mathfrak{G} \times \mathfrak{G} = \mathfrak{G}_i + \mathbf{m}$. $\mathfrak{G}_i$ and $\mathbf{m}$ are orthogonal with respect to the Killing form and $\mathbf{m}$ determines a connection on G, which is both left and right invariant.

Let us denote by φ_0 the Maurer-Cartan left invariant form on G, $\mathrm{d}\varphi_0 + \frac{1}{2}[\varphi_0, \varphi_0] = 0$. Then the Maurer-Cartan form on $G \times G$ can be written as (φ_0, ψ_0), where ψ_0 is another copy of φ_0 (with $\mathrm{d}\psi_0 + \frac{1}{2}[\psi_0, \psi_0] = 0$ again). Let us consider a p-dimensional motion $g(X)$ in G. Then by a lift of $g(X)$ we mean a (differentiable) mapping $\xi\colon G \to G \times G$ such that $\pi \circ \xi = \mathrm{id}$ on $g(X)$. By any lift ξ of $g(X)$ we get the induced forms φ, ψ on X, given by $(\varphi, \psi) = g_* \circ \xi_*(\varphi_0, \psi_0)$, which satisfy the same integrability conditions as φ_0 and ψ_0.

If the lift of the motion $g(X)$ is changed by $h(X) \in G$, we get new forms φ_1, ψ_1, where

$$(1) \qquad \varphi_1 = \operatorname{ad} h^{-1}\varphi + h^{-1}\,\mathrm{d}h\,, \quad \psi_1 = \operatorname{ad} h^{-1}\psi + h^{-1}\,\mathrm{d}h\,.$$

Let us denote

$$(2) \qquad \omega = \tfrac{1}{2}(\varphi - \psi)\,, \quad \eta = \tfrac{1}{2}(\varphi + \psi)\,, \quad \text{so} \quad \varphi = \omega + \eta\,, \quad \psi = \eta - \omega\,.$$

Then

$$(3) \qquad \omega_1 = \operatorname{ad} h^{-1}\omega\,, \quad \eta_1 = \operatorname{ad} h^{-1}\eta + h^{-1}\,\mathrm{d}h\,.$$

For any $v \in X_p$, $p \in X$ we have

$$(\varphi(v), \psi(v)) = \tfrac{1}{2}(\varphi(v) + \psi(v)\,, \varphi(v) + \psi(v)) + \\ + \tfrac{1}{2}(\varphi(v) - \psi(v), \psi(v) - \varphi(v))$$

and so $\omega(v) \in \mathbf{m}$, $\eta(v) \in \mathfrak{G}_i$. This means that $\omega(X_p)$ is a p-dimensional linear subspace of $\mathbf{m} \equiv \mathfrak{G}$ and by a change of the lift it changes by the group ad G.

The integrability conditions now are

$$(4) \qquad \mathrm{d}\omega = -\tfrac{1}{2}([\omega, \eta] + [\eta, \omega]), \quad \mathrm{d}\eta = -\tfrac{1}{2}([\omega, \omega] + [\eta, \eta])\,.$$

This is an immediate consequence of the definition.

We also see that we have an invariant differential quadratic form on any motion, which is induced by the invariant Killing form on $\mathfrak{G}$: $K(X, Y) = \operatorname{Tr} \operatorname{Ad} X \operatorname{Ad} Y$ for $X, Y \in \mathfrak{G}$. This form is given by the formula $(u, v) = K(\omega(u), \omega(v))$, where $u, v \in X_p$, $p \in X$. This form can be taken as the first fundamental form of the immersion.

Integrability conditions (4) can also be obtained by using the following formalism:

Let us denote by D the ring of the so called "double numbers", which is a two-dim. associative and commutative algebra over $\mathbf{R}$ with the unit element 1 and with the base 1, δ, where $\delta^2 = 1$. Let $\mathfrak{G}$ be a Lie algebra and let us consider the Lie algebra $D \otimes \mathfrak{G} = \mathfrak{G}^d$. For the corresponding M.C. form ω^d on $\mathfrak{G}^d$, $\omega^d = \omega + \delta\eta$, we must have $\mathrm{d}\omega^d + \frac{1}{2}[\omega^d, \omega^d] = 0$, as the integrability conditions are determined by the Lie structure of G only. In components we have

$$\mathrm{d}\omega + \delta\,\mathrm{d}\eta + \tfrac{1}{2}[\omega + \delta\eta, \omega + \delta\eta] = 0$$

and so

$$\mathrm{d}\omega + \tfrac{1}{2}([\omega, \omega] + [\eta, \eta]) = 0 \quad \text{and} \quad \mathrm{d}\eta + \tfrac{1}{2}([\omega, \eta] + [\eta, \omega]) = 0\,.$$

Further, as $D = \mathbf{R} \times \mathbf{R}$ (for $\varepsilon_1 = \frac{1}{2}(1 + \delta)$, $\varepsilon_2 = \frac{1}{2}(1 - \delta)$ we have $\varepsilon_1\varepsilon_2 = 0$, $\varepsilon_1^2 = \varepsilon_1$, $\varepsilon_2^2 = \varepsilon_2$), we get $D \otimes \mathfrak{G} = \mathfrak{G} \times \mathfrak{G}$. As $\varepsilon_1 + \varepsilon_2 = 1$, $\varepsilon_1 - \varepsilon_2 = \delta$, we have

$\omega^d = \omega + \delta\eta = (\varepsilon_1 + \varepsilon_2)\,\omega + (\varepsilon_1 - \varepsilon_2)\,\eta = \varepsilon_1(\omega + \eta) + \varepsilon_2(\omega - \eta) = \varepsilon_1\varphi + \varepsilon_2\psi$ and so (φ, ψ) is the Maurer-Cartan form on $G \times G$.

The formalism mentioned above was used for the first time by W. Blaschke in [1] for the group $O(3)$.

Remark. We may have more ad G invariant quadratic forms on $\mathfrak{G}$. For instance on the group $\mathscr{E}_3$ we have the Killing invariant form and the Klein invariant form. Also on compact nonsemisimple groups we have more invariant forms.

Remark. The classification of motions in the 1st order (and the specialization of the frame in the 1st order) means to find the fundamental domains for the action of ad G on p-dimensional subspaces of $\mathfrak{G}$, and to describe the orbits of this action (see [3] and [4]).

Remark. In the case of the spherical motion (immersions in $O(3)$) we have $G_0 = O(3) \times O(3)/O(3)$. Because $O(3) \times O(3) \cong O(4)$ (we mean local isomorphism), we get (locally) $G_0 = O(4)/O(3)$, which is (locally) the elliptic space. This correspondence may be realized by using the group of unit quaternions, as was done in [1] and [5]. In the case of Euclidean motions in E_3 the group $\mathscr{E}_3$ can be realized as the group of unit dual quaternions. The group $\mathscr{E}_3 \times \mathscr{E}_3$ acts then by the left and right multiplication of quaternions (see [1]).

B. REMARKS ON MOTIONS IN E_n

Let $g(X)$ be a Euclidean p-parametric motion in E_n. Let us choose fixed orthonormal frames $\bar{\mathscr{R}}_0$ in $\bar{E}_n$ and $\mathscr{R}_0$ in E_n. Then we may identify the elements of $\mathscr{E}_n \times \mathscr{E}_n$ with pairs of orthonormal frames by $\mathscr{R} = \mathscr{R}_0 g_1$, $\bar{\mathscr{R}} = \bar{\mathscr{R}}_0 g_2$, so that the pair (g_1, g_2) is identified with the pair $(\mathscr{R}, \bar{\mathscr{R}})$. The pair $(\mathscr{R}(X), \bar{\mathscr{R}}(X))$ is a lift of $g(X)$ iff $g(X) \,.\, (\bar{\mathscr{R}}(X)) = \mathscr{R}(X)$. This is easy to see, as $g(X)\,\bar{\mathscr{R}}_0 = \mathscr{R}_0 g$ and $g(X)\,\bar{\mathscr{R}}(X) = g(X) \,.\, \bar{\mathscr{R}}_0 g_2 = (g(X)\,\bar{\mathscr{R}}_0)\,g_2 = \mathscr{R}_0 g g_2 = \mathscr{R}_0 g_1$ and so $g = g_1 g_2^{-1} = \pi(g_1, g_2)$.

So let $(\mathscr{R}, \bar{\mathscr{R}})$ be a lift of $g(X)$. Then

$$\text{(5)} \qquad \mathrm{d}\mathscr{R} = \mathscr{R}\varphi\,, \quad \mathrm{d}\bar{\mathscr{R}} = \bar{\mathscr{R}}\psi \quad \text{and} \quad \mathrm{d}\varphi + \varphi \wedge \varphi = 0\,, \quad \mathrm{d}\psi + \psi \wedge \psi = 0$$

or, by using ω and η:

$$\text{(6)} \qquad \mathrm{d}\omega + \eta \wedge \omega + \omega \wedge \eta = 0\,, \quad \mathrm{d}\eta + \omega \wedge \omega + \eta \wedge \eta = 0\,.$$

In the group $\mathscr{E}_n$ we have the natural homomorphism $\chi\colon \mathscr{E}_n \to O(n)$, which we get by the restriction of the action of $\mathscr{E}_n$ on the vectors of E_n. This gives the mapping $\chi \circ g\colon X \to O(n)$, which is equivariant with respect to the homogeneous space structure $O(n) \times O(n)/\mathrm{Diag}\,(O(n) \times O(n) \cong O(n)$. We have to suppose that the rank of the mapping $\chi \circ g$ is constant on X. Then we get a spherical motion on some factor manifold of X(locally), which is associated with the space motion $g(X)$ and the rank of $\chi \circ g$ is the basic invariant of the motion (it determines the number of independent translations of the motion.). Let us call it the rank of the motion $g(X)$.

Further let $\bar{A}$ be a fixed point of $\bar{E}_n$. Then $\mathrm{d}\bar{A} = 0$. Let us write $\bar{A} = \bar{\mathscr{R}}X_{\bar{A}}$, where $X_{\bar{A}}$ is the column of coordinates of $\bar{A}$ in $\bar{\mathscr{R}}$. Then $0 = \mathrm{d}\bar{A} = \mathrm{d}\bar{\mathscr{R}}X_{\bar{A}} + \bar{\mathscr{R}}\,\mathrm{d}X_{\bar{A}} = \bar{\mathscr{R}}\,.$ $.\,(\psi X_{\bar{A}} + \mathrm{d}X_{\bar{A}})$ and so

$$\mathrm{d}X_{\bar{A}} = -\psi X_{\bar{A}}\,. \tag{7}$$

This means that the point $\bar{A}$ with the coordinates $X_{\bar{A}}$ belongs to the moving system iff (7) holds. (7) is a completely integrable system of differential equations, as $0 = \mathrm{d}^2X_{\bar{A}} = -\mathrm{d}\psi X_{\bar{A}} + \psi \wedge \mathrm{d}X_{\bar{A}} = -(\mathrm{d}\psi + \psi \wedge \psi)\,X_{\bar{A}}$.

Similarly, for a point A with coordinates X_A we have

$$\mathrm{d}X_A = -\varphi X_A\,. \tag{8}$$

The trajectory $A(X)$ of the point $\bar{A}$ is $A(X) = \mathscr{R}X_A$, where $X_A = X_{\bar{A}}$, because $g(\bar{\mathscr{R}}) = \mathscr{R}$ and $g(\bar{A}) = g(\bar{\mathscr{R}})\,X_{\bar{A}} = \mathscr{R}X_{\bar{A}}$.

For the tangent space of the trajectory $A(X)$ of $\bar{A}$ we get

$$\mathrm{d}A = \mathrm{d}\mathscr{R}X_A + \mathscr{R}\,\mathrm{d}X_A = \mathscr{R}\varphi X_A - \mathscr{R}\psi X_A = 2\mathscr{R}\omega X_A\,. \tag{9}$$

If we denote by Δ the ordinary differential, we get $\Delta^2 A = 2(\mathrm{d}\mathscr{R}\omega X_A + \mathscr{R}\,\Delta\omega X_A + \mathscr{R}\omega\,\Delta X_A) = 2\mathscr{R}(\varphi\omega - \omega\psi + \Delta\omega)\,X_A$, as $\Delta X_A = \mathrm{d}X_A$, and similarly for higher orders.

Let us write $\omega = \begin{pmatrix} 0, & 0 \\ \omega_0, & \omega_1 \end{pmatrix}$, where ω_0 is the column ω^i, ω_1 is the matrix ω^i_j, $i, j = 1, \ldots, n$. Then $X_A = (1, x^1, \ldots, x^n)^{\mathrm{T}}$ and

$$\omega X_A = \begin{pmatrix} 0, & 0 \\ \omega_0, & \omega_1 \end{pmatrix}\begin{pmatrix} 1 \\ x \end{pmatrix} = \omega_0 + \omega_1 x \quad \text{with} \quad x = (x^1, \ldots, x^n)^{\mathrm{T}}\,.$$

This gives

$$(\omega X_A)^i = \omega^i + \sum_{j=1}^{n} \omega^i_j x^j\,.$$

Let η^α, $\alpha = 1, \ldots, p$ be a base for 1-forms (locally). Then $\omega^i_j = a^i_{j\alpha}\eta^\alpha$, $\omega^i = b^i_\alpha \eta^\alpha$. The base of the tangent space of the trajectory of $\bar{A}$ at A is determined by the vectors v_α, $\alpha = 1, \ldots, p$ with the coordinates

$$v^i_\alpha = b^i_\alpha + \sum_{j=1}^{n} a^i_j x^j\,, \quad i = 1, \ldots, n\,.$$

The set of the singular points of trajectories of points is therefore given by the condition $\operatorname{rank}\,(b^i_\alpha + \sum_{j=1}^{n} a^i_{j\alpha} x^j) < p$, which is an intersection of algebraic surfaces of degree $\leqq p$.

C. CLASSIFICATION OF 2-PARAMETRIC MOTIONS IN E_3 OF RANK 2

Let $g(X)$ be a 2-parametric motion in E_3 with rank $\chi \circ g = 2$, let $(\mathscr{R}, \bar{\mathscr{R}})$ be its lift. Then we get forms ω and η on X as in the previous section. If we choose another lift of $g(X)$, say $(\mathscr{R}_1, \bar{\mathscr{R}}_1)$, where $\mathscr{R}_1 = \mathscr{R}h$, $\bar{\mathscr{R}}_1 = \bar{\mathscr{R}}h$, then for the new form $\tilde{\omega}$

we have $\tilde{\omega} = \mathrm{ad}\, h^{-1}\omega$ as in (3). Let

$$\omega = \begin{pmatrix} 0, & 0 \\ \omega_0, & \omega_1 \end{pmatrix}, \quad h = \begin{pmatrix} 1, & 0 \\ t, & \gamma \end{pmatrix} \quad \text{with} \quad \gamma \in O(3)\,. \tag{10}$$

Then

$$\tilde{\omega}_0 = \gamma^{\mathrm{T}}\omega_0 + \gamma^{\mathrm{T}}\omega_1 t\,, \quad \tilde{\omega}_1 = \gamma^{\mathrm{T}}\omega_1\gamma\,. \tag{11}$$

We denote

$$\omega_0 = \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix}, \quad \omega_1 = \begin{pmatrix} 0, & -\omega_1^2, & \omega_3^1 \\ \omega_1^2, & 0, & -\omega_2^3 \\ -\omega_3^1, & \omega_2^3, & 0 \end{pmatrix}. \tag{12}$$

If we use the isomorphism i: $\mathfrak{O}(3) \to V_3$ given by

$$\mathrm{i}(\omega_1) = (\omega_2^3, \omega_3^1, \omega_1^2)^{\mathrm{T}}\,, \quad \text{we get} \quad \mathrm{i}(\mathrm{ad}\,\gamma\omega_1) = \gamma\,\mathrm{i}(\omega_1)$$

and so $\mathrm{i}(\tilde{\omega}_1) = \gamma^{\mathrm{T}}\,\mathrm{i}(\omega_1)$.

The condition rank $\chi \circ g = 2$ means that $\dim \omega_1(X_p) = 2$, $p \in X$. As the group $\mathrm{O}(3)$ is transitive on 2-dim. subspaces of V_3, we may always find a lift of $g(X)$ such that $\omega_1^2 = 0$. Then $\omega_2^3 \wedge \omega_3^1 \neq 0$ and the remaining isotropy group is

$$\{\gamma\} = \left\{\begin{pmatrix} \cos\varphi, & \sin\varphi, & 0 \\ -\sin\varphi, & \cos\varphi, & 0 \\ 0, & 0, & 1 \end{pmatrix}\right\}, \tag{13}$$

and t is arbitrary. For the new form $\tilde{\omega}$ we get

$$\begin{aligned}
\tilde{\omega}_2^3 &= \omega_2^3 \cos\varphi + \omega_3^1 \sin\varphi\,, \\
\tilde{\omega}_3^1 &= -\omega_2^3 \sin\varphi + \omega_3^1 \cos\varphi\,, \\
\tilde{\omega}^1 &= (\omega^1 + \omega_3^1 t_3)\cos\varphi + (\omega^2 - \omega_2^3 t_2)\sin\varphi\,, \\
\tilde{\omega}^2 &= -(\omega^1 + \omega_3^1 t_3)\sin\varphi + (\omega^2 - \omega_2^3 t_2)\cos\varphi\,, \\
\tilde{\omega}^3 &= \omega^3 - \omega_3^1 t_1 + \omega_2^3 t_2\,.
\end{aligned}$$

Let us denote

$$\omega^1 = a\omega_3^1 + b\omega_2^3, \omega^2 = c\omega_3^1 + e\omega_2^3, \omega^3 = f\omega_3^1 + g\omega_2^3\,,$$

and similarly for $\tilde{\omega}$.

Computation gives

$$\begin{aligned}
\tilde{a} &= a\cos^2\varphi + (c - b)\sin\varphi\cos\varphi - e\sin^2\varphi + t_3\,, \\
\tilde{b} &= (a + e)\sin\varphi\cos\varphi + c\sin^2\varphi + b\cos^2\varphi\,, \\
\tilde{c} &= -(a + e)\sin\varphi\cos\varphi + c\cos^2\varphi + b\sin^2\varphi\,, \\
\tilde{e} &= -a\sin^2\varphi + e\cos^2\varphi + (c - b)\sin\varphi\cos\varphi - t_3\,, \\
\tilde{f} &= (f - t_1)\cos\varphi - (g + t_2)\sin\varphi\,, \\
\tilde{g} &= (f - t_1)\sin\varphi + (g + t_2)\cos\varphi.
\end{aligned} \tag{14}$$

From (14) we immediately see that $b + c$ is an invariant and we get

$$\tilde{a} - \tilde{e} = a - e - 2t_3,\ \tilde{a} + \tilde{e} = (a + e)\cos 2\varphi + (c - b)\sin 2\varphi\,,$$
$$\tilde{c} - \tilde{b} = -(a + e)\sin 2\varphi + (c - b)\cos 2\varphi\,.$$

This shows that we may always choose a lift of $g(X)$ such that $a = e = f = g = 0$, $b \geqq c$. If $b > c$, the lift is fixed up to a finite group, if $b = c$, the group (13) remains. Let us denote $b = \mathbf{v}$ and $c = \mathbf{w}$ in the sequel.

On X we have two invariant quadratic differential forms Φ and Ψ induced by the two invariant quadratic forms on $\mathfrak{E}_3$, the Killing form $(\omega_2^3)^2 + (\omega_3^1)^2 + (\omega_1^2)^2$ and the Klein form $\omega_2^3\omega^1 + \omega_3^1\omega^3 + \omega_2^3\omega^2$. They are

$$\Phi = (\omega_2^3)^2 + (\omega_3^1)^2\,, \quad \Psi = \mathbf{v}(\omega_2^3)^2 + \mathbf{w}(\omega_3^1)^2\,.$$

Remark. The Klein form determines the Klein quadric, which is the image of the space of the straight lines of E_3 in $\mathfrak{E}_3$.

With respect to the form Ψ we distinguish elliptic, hyperbolic, parabolic and flat motions. The first three constitute the general case of a motion of rank 2, where we have already defined the canonical frame of the immersion, the flat case must be treated separately.

1. *The general case.* We have

$$\omega_1^2 = \omega^3 = 0\,, \quad \omega^1 = \mathbf{v}\omega_2^3\,, \quad \omega^2 = \mathbf{w}\omega_3^1\,, \quad \mathbf{v} > \mathbf{w}.$$

The integrability conditions are

$$(15)\qquad \begin{aligned} d\varphi^1 &= -\varphi^2 \wedge \varphi_1^2 + \varphi^3 \wedge \varphi_3^1\,, & d\varphi_1^2 &= \varphi_3^1 \wedge \varphi_2^3\,,\\ d\varphi^2 &= \varphi^1 \wedge \varphi_1^2 - \varphi^3 \wedge \varphi_2^3\,, & d\varphi_2^3 &= \varphi_1^2 \wedge \varphi_3^1\,,\\ d\varphi^3 &= -\varphi^1 \wedge \varphi_3^1 + \varphi^2 \wedge \varphi_2^3\,, & d\varphi_3^1 &= \varphi_2^3 \wedge \varphi_1^2 \end{aligned}$$

and similarly for ψ. In terms of ω and η we get

$$(16)\qquad \begin{aligned} d\omega^1 &= -\eta^2 \wedge \omega_1^2 + \eta^3 \wedge \omega_3^1 - \omega^2 \wedge \eta_1^2 + \omega^3 \wedge \eta_3^1\,,\\ d\omega^2 &= \eta^1 \wedge \omega_1^2 - \eta^3 \wedge \omega_2^3 + \omega^1 \wedge \eta_1^2 - \omega^3 \wedge \eta_2^3\,,\\ d\omega^3 &= -\eta^1 \wedge \omega_3^1 + \eta^2 \wedge \omega_2^3 - \omega^1 \wedge \eta_3^1 + \omega^2 \wedge \eta_2^3\,,\\ d\omega_1^2 &= \eta_3^1 \wedge \omega_2^3 + \omega_3^1 \wedge \eta_2^3\,,\\ d\omega_2^3 &= \eta_1^2 \wedge \omega_3^1 + \omega_1^2 \wedge \eta_3^1\,,\\ d\omega_3^1 &= \eta_2^3 \wedge \omega_1^2 + \omega_2^3 \wedge \eta_1^2\,,\\ d\eta^1 &= -\eta^2 \wedge \eta_1^2 + \eta^3 \wedge \eta_3^1 - \omega^2 \wedge \omega_1^2 + \omega^3 \wedge \omega_3^1\,,\\ d\eta^2 &= \eta^1 \wedge \eta_1^2 - \eta^3 \wedge \eta_2^3 + \omega^1 \wedge \omega_1^2 - \omega^3 \wedge \omega_2^3\,,\\ d\eta^3 &= -\eta^1 \wedge \eta_3^1 + \eta^2 \wedge \eta_2^3 - \omega^1 \wedge \omega_3^1 + \omega^2 \wedge \omega_2^3\,,\\ d\eta_1^2 &= \eta_3^1 \wedge \eta_2^3 + \omega_3^1 \wedge \omega_2^3\,,\\ d\eta_2^3 &= \eta_1^2 \wedge \eta_3^1 + \omega_1^2 \wedge \omega_3^1\,,\\ d\eta_3^1 &= \eta_2^3 \wedge \eta_1^2 + \omega_2^3 \wedge \omega_1^2\,. \end{aligned}$$

Using Cartan's lemma we get

$$
\begin{aligned}
(17)\qquad & \eta_3^1 = \alpha\omega_2^3 + \beta\omega_3^1\,, \quad \eta_2^3 = -\beta\omega_2^3 - \gamma\omega_3^1\,, \quad \eta_1^2 = a_1\omega_2^3 + a_2\omega_3^1\,,\\
& \eta^1 = (\mathbf{w}\beta - n)\,\omega_2^3 - p\omega_3^1\,, \quad \eta^2 = m\omega_2^3 + (n - \mathbf{v}\beta)\,\omega_3^1\,,\\
& \eta^3 = b_1\omega_2^3 + b_2\omega_3^1\,,\\
& \mathrm{d}\omega_2^3 = a_1\omega_2^3 \wedge \omega_3^1, \quad \mathrm{d}\omega_3^1 = a_2\omega_2^3 \wedge \omega_3^1\,,\\
& b_1 = -(\mathbf{v})_2 + a_1(\mathbf{v} - \mathbf{w}), \quad b_2 = (\mathbf{w})_1 - a_2(\mathbf{v} - \mathbf{w})\,,\\
& -(a_1)_2 + (a_2)_1 + a_1^2 + a_2^2 + 1 = \beta^2 - \alpha\gamma\,,\\
& (\beta)_2 - (\gamma)_1 - 2a_1\beta + a_2(\alpha - \gamma) = 0\,,\\
& -(\alpha)_2 + (\beta)_1 + 2a_2\beta + a_1(\alpha - \gamma) = 0\,,\\
& -(b_1)_2 + (b_2)_1 + b_1a_1 + b_2a_2 + \alpha p + \gamma m - 2\beta n + (\mathbf{v} + \mathbf{w})(1 + \beta^2) = 0\,,\\
& (\mathbf{w}\beta - n)_2 + (p)_1 - a_1[\beta(\mathbf{v} + \mathbf{w}) - 2n] + a_2(p - m) + b_1\beta - b_2\alpha = 0\,,\\
& -(m)_2 + (n - \mathbf{v}\beta)_1 + a_1(m - p) + a_2[2n - \beta(\mathbf{v} + \mathbf{w})] - b_1\gamma + b_2\beta = 0\,.
\end{aligned}
$$

2. *The flat case* ($\mathbf{v} = \mathbf{w}$). We have

$$\omega_1^2 = \omega^3 = 0\,, \quad \omega^1 = \mathbf{v}\omega_2^3, \quad \omega^2 = \mathbf{v}\omega_1^3.$$

Further,

$$
\begin{aligned}
\eta_3^1 &= \alpha\omega_2^3 + \beta\omega_3^1\,, & \tilde{\eta}_2^3 &= \eta_2^3\cos\varphi + \eta_3^1\sin\varphi\,,\\
\eta_2^3 &= -\beta\omega_2^3 - \gamma\omega_3^1\,, & \tilde{\eta}_3^1 &= -\eta_2^3\sin\varphi + \eta_3^1\cos\varphi\,.
\end{aligned}
$$

Computation yields

$$
\begin{aligned}
\tilde{\alpha} &= 2\beta\sin\varphi\cos\varphi + \alpha\cos^2\varphi + \gamma\sin^2\varphi\,,\\
\tilde{\beta} &= (\gamma - \alpha)\sin\varphi\cos\varphi - \beta\sin^2\varphi + \beta\cos^2\varphi\,,\\
\tilde{\gamma} &= -2\beta\sin\varphi\cos\varphi + \alpha\sin^2\varphi + \gamma\cos^2\varphi\,.
\end{aligned}
$$

So $\alpha + \gamma = \tilde{\alpha} + \tilde{\gamma}$ is an invariant and

$$
\begin{aligned}
(18)\qquad & \tilde{\beta} = \beta\cos 2\varphi - \tfrac{1}{2}(\alpha - \gamma)\sin 2\varphi\,,\\
& \tfrac{1}{2}(\tilde{\alpha} - \tilde{\gamma}) = \beta\sin 2\varphi + \tfrac{1}{2}(\alpha - \gamma)\cos 2\varphi.
\end{aligned}
$$

This means that we may always choose such a lift that $\beta = 0$ and $\alpha - \gamma \geqq 0$. If $\alpha - \gamma > 0$, we are finished, if $\beta = \alpha - \gamma = 0$, (13) remains.

The integrability conditions change to

$$
\begin{aligned}
(19)\qquad & \eta_1^2 = a_1\omega_2^3 + a_2\omega_3^1\,, \quad \mathrm{d}\omega_2^3 = a_1\omega_2^3 \wedge \omega_3^1\,,\\
& \eta^1 = -n\omega_2^3 - p\omega_3^1\,, \quad \mathrm{d}\omega_3^1 = a_2\omega_2^3 \wedge \omega_3^1\,,\\
& \eta^2 = m\omega_2^3 + n\omega_3^1\,,\\
& \eta^3 = -(\mathbf{v})_2\omega_2^3 + (\mathbf{v})_1\omega_3^1\,,\\
& (a_2)_1 - (a_1)_2 + a_1^2 + a_2^2 + 1 = -\alpha\gamma\,, \quad (\gamma)_1 = a_2(\alpha - \gamma)\,,\\
& (\alpha)_2 = a_1(\alpha - \gamma),
\end{aligned}
$$

$$(\mathbf{v})_{11} + (\mathbf{v})_{22} - (\mathbf{v})_2\, a_1 + (\mathbf{v})_1\, a_2 + \alpha p + \gamma m + 2\mathbf{v} = 0,$$
$$-(n)_2 + (p)_1 + 2a_1 n + a_2(p - m) - (\mathbf{v})_1 = 0\,,$$
$$-(m)_2 + (n)_1 + (m - p)\, a_1 + 2a_2 n + (\mathbf{v})_2 = 0;$$

3. *The case* $\mathbf{v} = \mathbf{w}$, $\beta = \alpha - \gamma = 0$.

This case is treated in a similar way as the previous one. The result is: $n = 0$, $m - p \geqq 0$, so

$$\omega_1^2 = \omega^3 = 0, \quad \omega^1 = \mathbf{v}\omega_2^3, \quad \omega^2 = \mathbf{v}\omega_3^1, \quad \eta_3^1 = \alpha\omega_2^3\,,$$
$$\eta_2^3 = -\alpha\omega_3^1, \quad \eta^1 = -p\omega_3^1, \quad \eta^2 = m\omega_2^3.$$

The integrability conditions are obtained by a mere specialization of (19). If $n = m - p = 0$, (13) remains.

4. *The case* $\mathbf{v} = \mathbf{w}$, $\beta = \alpha - \gamma = 0$, $n = m - p = 0$. We have

$$\omega_1^2 = \omega^3 = 0\,, \quad \omega^1 = \mathbf{v}\omega_2^3\,, \quad \omega^2 = \mathbf{v}\omega_3^1\,, \quad \eta_3^1 = \alpha\omega_2^3\,, \quad \eta_2^3 = -\alpha\omega_3^1\,,$$
$$\eta^1 = -p\omega_3^1\,, \quad \eta^2 = p\omega_2^3\,.$$

Computation yields $\tilde{\eta}^3 = \eta^3$ and

$$(\tilde{\mathbf{v}})_2 = (\mathbf{v})_2 \cos\varphi - (\mathbf{v})_1 \sin\varphi\,, \quad (\tilde{\mathbf{v}})_1 = (\mathbf{v})_2 \sin\varphi + (\mathbf{v})_1 \cos\varphi\,.$$

The lift can be changed to $(\mathbf{v})_1 = 0$, $(\mathbf{v})_2 \geqq 0$; if $(\mathbf{v})_2 = 0$, (13) remains. The integrability conditions are obtained again from (19).

5. *The singular case*, for which the specialization cannot be completed. We have

$$\omega_1^2 = \omega^3 = \eta^3 = 0\,, \quad \omega^1 = \mathbf{v}\omega_2^3\,, \quad \omega^2 = \mathbf{v}\omega_3^1\,, \quad \eta_3^1 = \alpha\omega_2^3\,, \tag{20}$$
$$\eta_2^3 = -\alpha\omega_3^1\,, \quad \eta^1 = -p\omega_3^1, \quad \eta^2 = p\omega_2^3\,, \quad \mathbf{v} = \text{const.}$$

The integrability conditions are

(21)

$$\alpha = \text{const.}, \quad p = \text{const.} \quad \mathbf{v} = -\alpha p\,, \quad d\omega_2^3 = a_1\omega_2^3 \wedge \omega_3^1\,, \quad d\omega_3^1 = a_2\omega_2^3 \wedge \omega_3^1\,,$$
$$(a_2)_1 - (a_1)_2 + a_1^2 + a_2^2 + 1 = -\alpha^2\,. \quad \text{(13) remains.}$$

D. DISCUSSION OF THE CASE 5

Let us find out what motions we get in the case when the specialization of the frame cannot be completed. In order to simplify notation in this section we shall write for ω: $\omega = \begin{pmatrix} \omega^1, & \omega^2, & \omega^3 \\ \omega_2^3, & \omega_3^1, & \omega_1^2 \end{pmatrix}$. The other matrices from $\mathfrak{E}_3$ will be denoted in a similar way.

Let us consider a motion which satisfies conditions (20) from the case 5. Then the set of all tangent frames over our motion is given by the following system of Pfaffian equations:

$$\omega_1^2 = \omega^3 = \eta^3 = 0\,, \quad \omega^1 - \mathrm{v}\omega_2^3 = 0\,, \quad \omega^2 - \mathrm{v}\omega_3^1 = 0\,, \quad \eta_3^1 - \alpha\omega_2^3 = 0\,,$$
$$\eta_2^3 + \alpha\omega_3^1 = 0\,, \quad \eta^1 + p\omega_3^1 = 0\,, \quad \eta^2 - p\omega_2^3 = 0 \quad \text{with} \quad \mathrm{v} = -\alpha p,\, \alpha, p = \text{const.}$$

It is easy to see that this system is completely integrable. So it has 3 − dim. integral manifolds. The distribution D_3 for these integral manifolds has left translates to $\mathfrak{E}_3 \times \mathfrak{E}_3$ given by

$$\omega = \begin{pmatrix} \mathrm{v}x_2^3, & \mathrm{v}x_3^1, & 0 \\ x_2^3, & x_3^1, & 0 \end{pmatrix} \quad \eta = \begin{pmatrix} -px_3^1, & px_2^3, & 0 \\ -\alpha x_3^1, & \alpha x_2^3, & y_1^2 \end{pmatrix}.$$

Then

$$\varphi = \begin{pmatrix} \mathrm{v}x_2^3 - px_3^1, & \mathrm{v}x_3^1 + px_2^3, & 0 \\ x_2^3 - \alpha x_3^1, & x_3^1 + \alpha x_2^3, & y_1^2 \end{pmatrix}, \quad \psi = \begin{pmatrix} -px_3^1 - \mathrm{v}x_2^3, & px_2^3 - \mathrm{v}x_3^1, & 0 \\ -\alpha x_3^1 - x_2^3, & \alpha x_2^3 - x_3^1, & y_1^2 \end{pmatrix},$$

where x_2^3, x_3^1, y_1^2 are arbitrary functions on $\mathscr{E}_3 \times \mathscr{E}_3$.

Let us write $\varphi \equiv \varphi(x_2^3, x_3^1, y_1^2)$, $\psi \equiv \psi(x_2^3, x_3^1, y_1^2)$ and let us denote

$$\begin{aligned} \varphi((1 + \alpha^2)^{-1/2}, 0, 0) &= X_1\,, & \psi((1 + \alpha^2)^{-1/2}, 0, 0) &= Y_1\,, \\ \varphi(0, (1 + \alpha^2)^{-1/2}, 0) &= X_2\,, & \psi(0, (1 + \alpha^2)^{-1/2}, 0) &= Y_2\,, \\ \varphi(0, 0, 1) &= X_3\,, & \psi(0, 0, 1) &= Y_3\,. \end{aligned}$$

Then $Z_i = (X_i, Y_i)$, $i = 1, 2, 3$ is from $\mathfrak{E}_3 \times \mathfrak{E}_3$ and Z_1, Z_2, Z_3 is a base for D_3. It is easy to compute that $[Z_i, Z_j] = \varepsilon_{ijk}Z_k$. This shows that the integral manifolds of D_3 are the translates of a group G_3 isomorphis with $O(3)$ and the investigated motion is the sphere $O(3)/O(2)$.

Let us further denote by π_i: $\mathscr{E}_3 \times \mathscr{E}_3 \to \mathscr{E}_3$ the projection on the i-th factor, $i = 1, 2$. Then $\pi_1 G_3$ and $\pi_2 G_3$ are isomorphic groups, isomorphic to $O(3)$. The isomorphism between them is given by α: $\pi_1 G_3 \to \pi_2 G_3$, $\alpha = \pi_2 \circ \pi_1^{-1}$. Easy computation shows that $\pi_1 G_3$ is the group preserving the point $p\boldsymbol{e}_3$, $\pi_2 G_3$ is the group preserving the point $-p\boldsymbol{e}_3$, the groups $\pi_1 O(2)$ and $\pi_2 O(2)$ are the groups of rotations round the $\boldsymbol{e}_3$ axis.

Further on, if the investigated motion is M, we may locally write $G_3 = M \,.\, O(2)$, $\pi_1(G_3) = \pi_1(M) \,.\, O(2)$, $\pi_2(G_3) = \pi_2(M) \,.\, O(2)$, with $\pi_1(M) \,.\, \pi(M)^{-1} = M$.

$$\pi_1(M) = \exp t_1 X_1 \,.\, \exp t_2 X_2\,, \quad \pi_2(M) = \exp t_1 Y_1 \,.\, \exp t_2 Y_2$$

and so M can be locally expressed as the manifold

$$M = \exp t_1 X_1 \,.\, \exp t_2 X_2 \,.\, \exp(-t_2 Y_2) \,.\, \exp(-t_1 Y_1)\,,$$

where $\exp t_1 X_1$ is the rotation round $A_1 = p\boldsymbol{e}_3 + \lambda(\boldsymbol{e}_1 + \alpha\boldsymbol{e}_2)$,
$\exp t_2 X_2$ is the rotation round $A_2 = p\boldsymbol{e}_3 + \lambda(-\alpha\boldsymbol{e}_1 + \boldsymbol{e}_2)$,
$\exp t_1 Y_1$ is the rotation round $A_3 = -p\boldsymbol{e}_3 + \lambda(-\boldsymbol{e}_1 + \alpha\boldsymbol{e}_2)$,
$\exp t_2 Y_2$ is the rotation round $A_4 = -p\boldsymbol{e}_3 + \lambda(-\alpha\boldsymbol{e}_1 - \boldsymbol{e}_2)$.

The natural homomorphism χ associates with every space motion of rank 2 a 2-parametrical spherical motion. The space $O(3) \times O(3)/\text{Diag. } O(3)$ is locally isomorphic with the space $S^3 = O(4)/O(3)$ which is (locally) the 3-dim. elliptic space, and so each property of surfaces in elliptical geometry is at the same time a property of space motions. For this reason we present here a short review of the local surface theory in S^3.

The material of this section is basically a transcription of results on the elliptic surface theory from [1].

The isomorphism between $\mathfrak{O}(3) \times \mathfrak{O}(3)$ and $\mathfrak{O}(4)$ is explicitly described in the following way:

Let $X = (x^i_j)$, $Y = (y^i_j)$ be any two vectors from $\mathfrak{O}(3)$, $i, j = 1, 2, 3$. Denote $x_k = \frac{1}{2}\varepsilon_{ijk}x^j_i$, $y_k = \frac{1}{2}\varepsilon_{ijk}y^j_i$. Let us define the matrix $Z \in \mathfrak{O}(4)$, $Z = (z^\alpha_\beta)$, $\alpha, \beta = 0, 1, 2, 3$, by the formulas $z^0_i = \frac{1}{2}(x_i - y_i) = -z^i_0$, $z^i_j = \frac{1}{2}(x^i_j + y^i_j)$, $z^0_0 = 0$. Direct computation shows that the described mapping is an isomorphism of Lie algebras.

Let us now have the space S^3, realized as the unit sphere of a vector space V_4 with the Euclidean scalar product, and let $g: X \to S^3$ be a two-dimensional immersed submanifold. By an adapted frame $\mathscr{R} = \{\boldsymbol{e}_0, \ldots, \boldsymbol{e}_3\}$ of the first order of the submanifold $g(X)$ we understand any orthonormal frame such that $\boldsymbol{e}_0 = g(X)$ and $\boldsymbol{e}_1, \boldsymbol{e}_2$ span $g_*(X_x)$ for $x \in X$. Then we have

$$\mathrm{d}\mathscr{R} = \mathscr{R} \begin{pmatrix} 0, & -\omega^3_2, & -\omega^1_3, & 0 \\ \omega^3_2, & 0, & -\eta^2_1, & \eta^1_3 \\ \omega^1_3, & \eta^2_1, & 0, & -\eta^3_2 \\ 0, & -\eta^1_3, & \eta^3_2, & 0 \end{pmatrix}$$

for the adapted frames of the first order; the notation is justified by the described isomorphism between $\mathfrak{O}(3) \times \mathfrak{O}(3)$ and $\mathfrak{O}(4)$.

The integrability conditions are given in (17). The remaining isotropy group is

$$O(2) \cong \begin{pmatrix} 1, & 0, & 0 \\ 0, & O(2), & 0 \\ 0, & 0, & 1 \end{pmatrix},$$

its action is described by (18). From (18) we immediately see that

$$\Phi = (\omega^3_2)^2 + (\omega^1_3)^2\,,$$

$$\Phi_1 = \omega^3_2\eta^3_2 + \omega^1_3\eta^1_3 = -\beta(\omega^3_2)^2 + (\alpha - \gamma)\,\omega^1_3\omega^3_2 + \beta(\omega^1_3)^2$$

and

$$\Phi_2 = \omega^3_2\eta^1_3 - \omega^1_3\eta^3_2 = \alpha(\omega^3_2)^2 + 2\beta\omega^1_3\omega^3_2 + \gamma(\omega^1_3)^2$$

are invariant quadratic forms. Any direction in the tangent plane of $g(X)$ is given by the equation $\omega^3_2 \cos\varphi + \omega^1_3 \sin\varphi = 0$.

The directions given by $\Phi_1(\varphi) = 0$ are called the main curvature directions, the directions $\Phi_2(\varphi) = 0$ are called the asymptotic directions. Further, $\alpha + \gamma = 2H_0$, $1 + \alpha\gamma - \beta^2 = K_0$ are invariants of the submanifold, the mean and Gauss curvatures. The integrability conditions (17) also show that K_0 is an invariant of the first form Φ only. The main curvature directions are always real and orthogonal, they are eigendirections of Φ_2. The asymptotic directions are real and distint for $K_0 < 1$, they coincide for $K_0 = 1$ and are imaginary for $K_0 > 1$. The surfaces with $K_0 = 1$ are developable surfaces, as we shall show later on. The form $\Phi_3 = (\eta_2^3)^2 + (\eta_3^1)^2$ is invariant as well and it is the length element of the surface described by the normal e_3.

The frame can be specialized to $\beta = 0$ provided $\frac{1}{2}(\alpha - \gamma)^2 + \beta^2 \neq 0$, as we know from Section B, case 2. If $\beta = 0$, $\alpha = \gamma$, we have a spherical point, if this condition is satisfied on a neighbourhood, we get a part of a sphere, as then $\mathrm{d}(e_1 - (1/\alpha)\, e_3) = 0$ with $\alpha =$ const. by virtue of the integrability conditions.

If we denote $\tan\varphi = \lambda$, we get $\lambda = (-1/2\beta)\{\alpha - \gamma \pm [(\alpha - \gamma)^2 + 4\beta^2]^{1/2}\}$ for the main directions and $\lambda = (1/\alpha)\{\beta \pm [\beta^2 - \alpha\gamma]^{1/2}\}$ for the asymptotic directions.

For the eigendirections of Φ_1 we get $\tan 2\varphi = 2\beta/(\alpha - \gamma)$ and so $\lambda = (-1/2\beta)\,.\,\{\alpha - \gamma \pm [(\alpha - \gamma)^2 + 4\beta^2]^{1/2}\}$. For the eigendirections of Φ_2 we get $\tan 2\varphi = (\gamma - \alpha)/2\beta$, so $\lambda = 1/(\alpha - \gamma)\,.\,\{-2\beta \pm [(\alpha - \gamma)^2 + 4\beta^2]^{1/2}\}$.

a) Let us determine all *the surfaces with a transitive group of isometries*. The possibilities are:

1. $\beta = 0, \alpha = \gamma$. This is a sphere of radius $\pi/2 - \mu$, where $\alpha = \tan\mu$.

2. Let $\alpha - \gamma > 0$, $\alpha > 0$, $\alpha =$ const., $\gamma =$ const. The integrability conditions give $(\gamma)_1 = 0 = a_2(\alpha - \gamma)$, $(\alpha)_2 = 0 = a_1(\alpha - \gamma)$, so $a_1 = a_2 = 0$, $\alpha\gamma = -1$. Then $\mathrm{d}\omega_2^3 = \mathrm{d}\omega_3^1 = 0$. Denote $\omega_2^3 = \mathrm{d}u$, $\omega_3^1 = \mathrm{d}v$. The Frenet formulas for the surface are

$$\begin{aligned} \mathrm{d}e_0 &= \mathrm{d}u\,e_1 + \mathrm{d}v\,e_2\,, & \mathrm{d}e &= -\mathrm{d}v\,e_0 + 1/\alpha\,\mathrm{d}v\,e_3\,,\\ \mathrm{d}e_1 &= -\mathrm{d}u\,e_0 - \alpha\,\mathrm{d}u\,e_3\,, & \mathrm{d}e_3 &= \mathrm{d}u\,e - 1/\alpha\,\mathrm{d}v\,e_2\,. \end{aligned}$$

Integration yields

$$e_0 + \alpha e_3 = \frac{1}{\cos\mu}(f_1 \cos u + f_2 \sin u)\,,$$

$$-\,e_0 + \frac{1}{\alpha}\,e_3 = \frac{1}{\sin\mu}(f_3 \cos v + f_3 \sin v)\,,$$

where $\alpha = \tan\mu$ and $\{f_1, \ldots, f_4\}$ is a fixed orthonormal base. So

$$e_0 = \cos\mu(f_0 \cos u + f_1 \sin u) + \sin\mu(f_2 \cos v + f_3 \sin v)$$

and the equation of the surface is

$$\frac{x_0^2 + x_1^2}{\cos^2\mu} = \frac{x_2^2 + x_3^2}{\sin^2\mu}\,.$$

Such a surface is called Clifford's quadric.

b) *Developable surfaces.* A surface is called developable, if $K_0 = 1$. Let us investigate these surfaces in more detail. $K_0 = 1$ implies $\alpha\gamma = 0$. Let $\gamma = 0$, $\alpha \neq 0$. The integrability conditions give $a_2 = 0$, $(\alpha)_2 = a_1\alpha$, $(a_1)_2 = a_1^2 + 1$. $\mathrm{d}\omega_3^1 = 0$, so denote $\omega_3^1 = \mathrm{d}v$. Further, $\mathrm{d}\eta_3^1 = \mathrm{d}(\alpha\omega_2^3) = \mathrm{d}\alpha \wedge \omega_2^3 + \alpha a_1\omega_2^3 \wedge \omega_3^1 = (\alpha)_2\omega_3^1 \wedge$ $\wedge\, \omega_2^3 + \alpha a_1\omega_2^3 \wedge \omega_3^1 = 0$. So let $\eta_3^1 = \mathrm{d}u$. Then $a_1 = \tan(v + g(u))$, $\alpha = \{h(u)\,.$ $.\cos[v + g(u)]\}^{-1}$, where $h(u)$, $g(u)$ are arbitrary functions, $\eta_2^3 = 0$, $\omega_2^3 = \mathrm{d}u/\alpha$.

Frenet formulas for the frame $\mathscr{R} = \{e_i\}$, $i = 0, \ldots, 3$, are

$$\mathrm{d}e_0 = e_1 \frac{\mathrm{d}u}{\alpha} + e_2\,\mathrm{d}v\,, \qquad \mathrm{d}e_2 = -e_0\,\mathrm{d}v - e_1 \frac{a_1}{\alpha}\,\mathrm{d}u\,,$$

$$\mathrm{d}e_1 = -e_0 \frac{\mathrm{d}u}{\alpha} + e_2 \frac{a_1}{\alpha}\,\mathrm{d}u - e_3\,\mathrm{d}u,\ \mathrm{d}e_3 = e_1\,\mathrm{d}u\,.$$

Integration with respect to v gives $e_0 = f_0(u)\cos v + f_1(u)\sin v$, where f_0, f_1 are fixed orthonormal vectors, and so the curves $u = \text{const.}$ are great circles (lines of the elliptic geometry), $e_3 = e_3(u)$ is a function of u only, the edge of the regression is given by $\cos(v + g(u)) = 0$, its tangent vector is e_2, which coincides with the tangent vector of the circle. This means that developable surfaces in elliptic geometry have similar properties as those in the Euclidean geometry.

c) *Bianchi's developable surfaces.* Bianchi's developable surfaces in elliptical geometry are surfaces which can be developed into the Euclidean plane. Such surfaces are characterized by the condition $K_0 = 0$. Let us have such a surface. Then $\alpha\gamma =$ $= -1$, let $\alpha > 0$, $\gamma = 1/\alpha$. Then $(\alpha)_2 = a_1(\alpha + 1/\alpha)$, $(\alpha)_1 = \alpha^2 a_2(\alpha + 1/\alpha)$, $(a_2)_1 -$ $- (a_1)_2 + a_1^2 + a_2^2 = 0$. Further, $\mathrm{d}\eta_1^2 = \mathrm{d}(a_1\omega_2^3 + a_2\omega_3^1) = 0$.

Let us now suppose that our surface is determined by a two-parametric spherical motion. Then

$$\varphi = \begin{pmatrix} 0 & , -a_1\omega_2^3 - a_2\omega_3^1\,, & \alpha\omega_2^3 + \omega_3^1 \\ a_1\omega_2^3 + a_2\omega_3^1\,, & 0 & , -\omega_2^3 - \dfrac{1}{\alpha}\omega_3^1 \\ -\alpha\omega_2^3 - \omega_3^1 & , \omega_2^3 + \dfrac{1}{\alpha}\omega_3^1 & , \quad 0 \end{pmatrix} = (\varphi_j^i)\,.$$

Let us write $\eta_1^2 = \mathrm{d}\vartheta$ and let us rotate the frame by the angle ϑ. Then

$$\tilde{\varphi} = \begin{pmatrix} 0 & , -\varphi_1^2 - \mathrm{d}\vartheta & , \quad \varphi_3^1\cos\vartheta - \varphi_2^3\sin\vartheta \\ \varphi_1^2 - \mathrm{d}\vartheta & , \quad 0 & , -\varphi_3^1\sin\vartheta - \varphi_2^3\cos\vartheta \\ -\varphi_3^1\cos\vartheta + \varphi_2^3\sin\vartheta, & \varphi_3^1\sin\vartheta + \varphi_2^3\cos\vartheta, & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0, & 0, & \tilde{\varphi}_3^1 \\ 0, & 0, & -\tilde{\varphi}_2^3 \\ -\tilde{\varphi}_3^1, & \tilde{\varphi}_2^3, & 0 \end{pmatrix}.$$

The integrability conditions give

$$\tilde{\varphi}_3^1 \wedge \tilde{\varphi}_2^3 = 0\,, \quad \mathrm{d}\tilde{\varphi}_3^1 = 0\,, \quad \mathrm{d}\tilde{\varphi}_2^3 = 0$$

and so

$$\tilde{\varphi}_3^1 = \mathrm{d}u\,, \quad \tilde{\varphi}_2^3 = \lambda\,\mathrm{d}u\,, \quad \mathrm{d}\tilde{\varphi}_2^3 = \mathrm{d}\lambda \wedge \mathrm{d}u = 0\,, \quad \lambda = \lambda(u)\,.$$

This means that $\tilde{\varphi}$ is a function of u only. Similarly, $\tilde{\psi}$ is also a function of one variable v and so $g_1 = g_1(u)$, $g_2 = g_2(v)$ and $g(u, v) = g_1(u) \,.\, g_2^{-1}(v)$. This shows than Bianchi's developable surface is a product of two one-parametric motions. The converse of this statement will be discussed later.

A special case of a Bianchi's developable surface is Clifford's quadric. For it we have $\alpha\omega_2^3 + \omega_3^1 = \mathrm{d}u$, $\alpha\omega_2^3 - \omega_3^1 = \mathrm{d}v$. This yields $\mathrm{d}(\boldsymbol{e}_1 + \alpha\boldsymbol{e}_2) = 0$, $\mathrm{d}(-{}^{-}\boldsymbol{e}_1 + \alpha\,{}^{-}\boldsymbol{e}_2) = 0$ and so Clifford's quadric is a product of rotations round the axes determined by the vectors $\boldsymbol{e}_1 + \alpha\boldsymbol{e}_2$ and $-{}^{-}\boldsymbol{e}_1 + \alpha\,{}^{-}\boldsymbol{e}_2$. Easy computation shows that Clifford's quadric $(x_0^2 + x_1^2)/\cos^2\mu = (x_2^2 + x_3^2)/\sin^2\mu$ is the product of rotations round (intersecting) axes with the angle 2μ.

F. GENERAL PROPERTIES OF 2-PAR. MOTIONS OF RANK 2

The origins of the kinematic geometry of two-parametric motions in E_3 go back to the second half of the 19th century. According to [6] the first paper published on this subject was by Th. Schönemann in 1855, later contributions were from A. Manheim, A. Ribeaucour, A. Cayley, G. Darboux and others. A short review of their results is in [6], pp. 151–154, 239–243. As we shall need some of those "classical" results in our further considerations, we shall present what is necessary in an abbreviated form. Whenever convenient, we shall also present proofs. The reader may also consult [2], which deals with similar problems.

To simplify notations, we shall write $\omega_2^3 = \omega_1$, $\omega_3^1 = \omega_2$. We shall also use the notation from C) for matrices from $\mathfrak{E}_3$. Then

$$\text{(22)} \quad \omega = \begin{pmatrix} \mathbf{v}\omega_1, & \mathbf{w}\omega_2, & 0 \\ \omega_1, & \omega_2, & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} r\omega_1 - p\omega_2, & m\omega_1 + s\omega_2, & b_1\omega_1 + b_2\omega_2 \\ -\beta\omega_1 - \gamma\omega_2, & \alpha\omega_1 + \beta\omega_2, & a_1\omega_1 + a_2\omega_2 \end{pmatrix}$$

and

$$\text{(23)} \qquad \varphi = \begin{pmatrix} (r + \mathbf{v})\,\omega_1 - p\omega_2, & m\omega_1 + (s + \mathbf{w})\,\omega_2, & b_1\omega_1 + b_2\omega_2 \\ (1 - \beta)\,\omega_1 - \gamma\omega_2, & \alpha\omega_1 + (\beta + 1)\,\omega_2, & a_1\omega_1 + a_2\omega_2 \end{pmatrix},$$

$$\psi = \begin{pmatrix} (r - \mathbf{v})\,\omega_1 - p\omega_2, & m\omega_1 + (s - \mathbf{w})\,\omega_2, & b_1\omega_1 + b_2\omega_2 \\ -(\beta + 1)\,\omega_1 - \gamma\omega_2, & \alpha\omega_1 + (\beta - 1)\,\omega_2, & a_1\omega_1 + a_2\omega_2 \end{pmatrix},$$

where $r = \beta\mathbf{w} - n$, $s = n - \beta\mathbf{v}$.

The two-dimensional subspace $\omega(T_x(X)) \subset \mathfrak{E}_3$ characterizes the instantaneous one-parametric motions of the given two-parametric motion at a given instant x. Vectors

$$X = \begin{pmatrix} \mathbf{v}, & 0, & 0 \\ 1, & 0, & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0, & \mathbf{w}, & 0 \\ 0, & 1, & 0 \end{pmatrix}$$

form an orthonormal base of $\omega(T_x(X))$ with respect to the Killing form. Any one-parametric subgroup of the instantaneous motion is determined by the vector $Z = X \,.$ $. \cos \varphi + Y \sin \varphi = \begin{pmatrix} \mathbf{v} \cos \varphi, & \mathbf{w} \sin \varphi, & 0 \\ \cos \varphi, & \sin \varphi, & 0 \end{pmatrix}$. The axis $\mathbf{z}$ of such a motion is $\mathbf{z} =$ $= \lambda(e_1 \cos \varphi + e_2 \sin \varphi) + e_3(\mathbf{v} - \mathbf{w}) \cos \varphi \sin \varphi, \lambda \in \mathbf{R}$, the parameter $\mathbf{v}_0$ is $\mathbf{v}_0 =$ $= \mathbf{v} \cos^2 \varphi + \mathbf{w} \sin^2 \varphi = \Psi(\varphi)$, so $\mathbf{v}$ and $\mathbf{w}$ are extremal values of the parameter of the instantaneous one-parameter motions. All axes $\mathbf{z}$ generate a ruled surface

$$z(x^2 + y^2) = (\mathbf{v} - \mathbf{w})\, xy \,,$$

known as Cayley's cylindroid of Plücker's conoid. Its properties can be derived from its equation.

The canonical frames $\bar{\mathscr{R}} = \{\bar{A}, \bar{e}_1, \bar{e}_2, \bar{e}_3\}$ and $\mathscr{R} = \{A, e_1, e_2, e_3\}$ gradually coincide during the motion as $g(X)\,\bar{\mathscr{R}}(X) = \mathscr{R}(X)$. This means that we have two surfaces $\bar{A}(X)$ and $A(X)$ in E_3, which are in a 1-1 correspondence and such that $g(\bar{A}) = A$. Similarly, we have three line congruences $\bar{L}_i = \bar{A} + \mu\, \bar{e}_i$, $L_i = A + \mu e_i$, $\mu \in \mathbf{R}$, such that $g(X)\bar{L}_i(X) = L_i(X)$, $i = 1, 2, 3$. Let us call the surface $\bar{A}(A)$ the moving (fixed) polar surface; the endpoint of $\bar{e}_3(e_3)$ is called the moving (fixed, respectively) spherical pole.

It is known ([5]) that the correspondence $\bar{e}_3 \to e_3$ is volume preserving:

$$\mathrm{d}\omega_1^2 = \tfrac{1}{2}\, \mathrm{d}(\varphi_1^2 - \psi_1^2) = \tfrac{1}{2}(\varphi_3^1 \wedge \varphi_2^3 - \psi_3^1 \wedge \psi_2^3) = 0 \,,$$

$$\mathrm{d}\, \bar{e}_3 = \psi_3^1\, \bar{e}_1 - \psi_2^3\, \bar{e}_2 \,, \quad \mathrm{d}e_3 = \varphi_3^1 e_1 - \varphi_2^3 e_2$$

and the corresponding volume elements are $\psi_3^1 \wedge \psi_2^3$ and $\varphi_3^1 \wedge \varphi_2^3$.

Straightforward computation also shows that the congruences $\bar{L}_3$ and L_3 have the same mean curvature of the second fundamental form.

The correspondence between $\bar{A}$ and A has no special properties; the tangent spaces of the polar surfaces at $g(\bar{A})$ and A are in general different. The difference between the first fundamental forms $\bar{F}_1$ of $\bar{A}$ and F_1 of A is

$$\bar{F}_1 - F_1 = 4[\mathbf{v} r \omega_1^2 + (m\mathbf{w} - p\mathbf{v})\, \omega_1 \omega_2 + \mathbf{w} s \omega_2^2] \,,$$

and so the correspondence is not an isometry.

On Cayley's cylindroid we have two exceptional straight lines u and v, for which the instantaneous motion is a rotation. They correspond to the asymptotic directions of Ψ (in the elliptic case they are not real) and are given by the equations

(24) $$z^2 + \mathbf{v}\mathbf{w} = 0 \,, \quad xz - \mathbf{w}y = 0 \,, \quad \mathbf{v}x + yz = 0 \,.$$

In the case $\mathbf{v} = \mathbf{w} = 0$ we get the whole plane $z = 0$, such a motion is a rolling of two isometric surfaces (Ribaucour, Thévenet, see [6]).

The tangent plane of the trajectory of a point $\bar{P}$ at P is given in the frame $\mathcal{R}$ by the form

$$\text{(25)} \qquad \mathrm{d}P = \omega P = (\mathbf{v}\omega_1 + z\omega_2)\, e_1 + (-z\omega_1 + \mathbf{w}\omega_2)\, e_2 + (y\omega_1 - x\omega_2)\, e_3\,,$$

where x, y, z are the coordinates of P in $\mathcal{R}$. The normal vector of the trajectory of $\bar{P}$ at P is determined by the vector

$$\text{(26)} \qquad \boldsymbol{n} = (xz - y\mathbf{w})\, e_1 + (yz + x\mathbf{v})\, e_2 + (z^2 + \mathbf{vw})\, e_3\,,$$

as is easily computed from (25). This means that the points of the lines u and v are singular points of their trajectories. All normals of trajectories of all points at a given instant form a congruence of lines, it is the congruence of all lines which intersect u and v.

The second order properties of the trajectory of a point are derived from the formulas for the differentials ΔP and $\Delta^2 P$ of the trajectory of $\bar{P}$ at P. In $\mathcal{R}$ we have (25) and according to (9) we obtain $\Delta^2 P = 2(\varphi\omega - \omega\psi + \Delta\omega)\, P$.

The first or second fundamental form of the trajectory of P is $\Phi_P = (\Delta P, \Delta P)$ or $\Psi_P = (\Delta^2 P, N)$, respectively, where N is the unit vector of the normal of the trajectory, determined by (26). This gives the possibility to express the mean and Gauss curvatures of the trajectory. The set of all points which are parabolic points of their trajectory is a surface of the 6th degree (see [2], the fact was already known to Manheim), each normal of the trajectory intersects this surface in 6 points, four of which are on u and v. Further details will be discussed later on.

G. SPECIAL MOTIONS

a) *Splitting*

We say that a space motion $g(u, v)$ splits (into two one-parametric motions) if there exist two one-parametric motions $k_1(u)$ and $k_2(v)$ such that $g(u, v) = k_1(u) \,.\, k_2(v)$.

Lemma 1. *A motion $g(u, v)$ splits iff there is a lift (g_1, g_2) of g such that $\varphi = a(u)\, \mathrm{d}u$, $\psi = b(v)\, \mathrm{d}v$.*

Proof. Let $g(u, v) = k_1(u) \,.\, k_2(v)$. Then $(k_1(u), k_2^{-1}(v))$ is a lift of g and we have $\varphi = k_1^{-1}\, \mathrm{d}k_1(u)$, $\psi = k_2(v)\, \mathrm{d}k_2^{-1}(v) = -\mathrm{d}k_2 \,.\, k_2^{-1}$, as $\mathrm{d}k^{-1} = -k^{-1}\, \mathrm{d}k \,.\, k^{-1}$ for any k. Conversely, let $\varphi = a(u)\, \mathrm{d}u$, $\psi = b(v)\, \mathrm{d}v$. Then we have $g_1 = g_1(u)$, $g_2 = g_2(v)$ for the corresponding lift (g_1, g_2) and so $g(u, v) = g_1(u)\, g_2^{-1}(v)$.

Lemma 2. *A motion $g(u, v)$ splits iff there is a form ϑ on $g(u, v)$ with values in $\mathfrak{E}_3$ such that* $\mathrm{d}\vartheta = \vartheta \wedge \vartheta$ *and*

$$(\varphi_j^i + \vartheta_j^i) \wedge (\varphi_l^k + \vartheta_l^k) = 0\,, \quad (\psi_j^i + \vartheta_j^i) \wedge (\psi_l^k + \vartheta_l^k) = 0\,,$$
$$(\varphi_j^i + \vartheta_j^i) \wedge (\varphi^k + \vartheta^k) = 0\,, \quad (\psi_j^i + \vartheta_j^i) \wedge (\psi^k + \vartheta^k) = 0\,,$$

where i, j, k, l = 1, 2, 3 *and* φ, ψ *correspond to any lift of* g.

Proof. Let φ, ψ correspond to any lift of a motion $g(u, v)$. Then for the change of the lift we have $\tilde{\varphi} = h^{-1}\varphi h + h^{-1}\,\mathrm{d}h, \tilde{\psi} = h^{-1}\psi h + h^{-1}\,\mathrm{d}h$, where $h(u, v)$ is the matrix of the change. The motion splits according to Lemma 1 iff there is a matrix $h(u, v) \in \mathscr{E}_3$ such that $\tilde{\varphi} = a(w_1)\,\mathrm{d}w_1$, $\tilde{\psi} = b(w_2)\,\mathrm{d}w_2$, where w_1 and w_2 are functions of u and v. This happens iff $\tilde{\varphi}_j^i \wedge \tilde{\varphi}_l^k = 0$, $\tilde{\varphi}_j^i \wedge \tilde{\varphi}^k = 0$, $\tilde{\varphi}^i \wedge \tilde{\varphi}^j = 0$ and similarly for $\tilde{\psi}$. As one of the forms $\tilde{\varphi}_j^i$ is different from zero, the condition $\tilde{\varphi}^i \wedge \tilde{\varphi}^j = 0$ is superfluous and similarly for $\tilde{\psi}$. Further, let us denote $\vartheta = \mathrm{d}h\,.\,h^{-1}$. Then $\mathrm{d}\vartheta = -\mathrm{d}h \wedge \mathrm{d}(h^{-1}) = \mathrm{d}h \wedge h^{-1}\,\mathrm{d}h\,.\,h^{-1} = \mathrm{d}h\,.\,h^{-1} \wedge \mathrm{d}h\,.\,h^{-1} = \vartheta \wedge \vartheta$, where we have used that $\sigma_1 \wedge h\sigma_2 = \sigma_1 h \wedge \sigma_2$ for any matrix forms σ_1 and σ_2 and any matrix h. We have also $h\tilde{\varphi}h^{-1} = \varphi + \vartheta$ and $(\varphi_j^i + \vartheta_j^i) \wedge (\varphi_l^k + \vartheta_l^k) = \sum_{\alpha,\beta,\gamma,\delta} h_\alpha^i \tilde{\varphi}_\beta^\alpha (h^{-1})_j^\beta \wedge h_\gamma^k \tilde{\varphi}_\delta^\gamma (h^{-1})_l^\delta = 0$ and similarly for the others. As $\tilde{\varphi} = h^{-1}(\varphi + \vartheta)\,h$ and the equation $\mathrm{d}h = \vartheta h$ is completely integrable, the converse is also true. This completes the proof.

The forms $\tilde{\varphi}$ and $\tilde{\psi}$ are defined uniquely up to the initial conditions by the form ϑ. Indeed, if h_1 and h are two solutions of $\mathrm{d}h = \vartheta h$, we get $h_1 = h\gamma$, where $\gamma = \text{const.}$ For the corresponding $\tilde{\varphi}_1$, $\tilde{\varphi}$ we get $\tilde{\varphi}_1 = h_1^{-1}(\varphi + \vartheta)\,h_1 = \gamma^{-1}\tilde{\varphi}\gamma$ and also $\tilde{\psi}_1 = \gamma^{-1}\tilde{\psi}\gamma$. This means, that $\tilde{\varphi}$ and $\tilde{\psi}$ are determined up to the adjoint action, but their mutual position is invariant. If we write $k_1^{-1}\,\mathrm{d}k_1 = \tilde{\varphi}$, $\tilde{k}_1^{-1}\,\mathrm{d}\tilde{k}_1 = \gamma^{-1}\tilde{\varphi}\gamma$ and similarly for k_2, we obtain by analogous considerations that $\tilde{k}_1 = \alpha k_1 \gamma$ and $\tilde{k}_2 = \gamma^{-1}k_2\beta$, where α, β, γ are constant matrices. This show that ϑ determines the splitting of g uniquely up to the position of the factors k_1 and k_2.

Theorem 1. *The motion $g(X)$ of rank 2 splits iff there exist functions A, B, C on X such that*

$$AC - B^2 = K_0\,, \quad A\gamma - 2B\beta + C\alpha = -2K_0\,,$$
$$-A(p + \gamma\mathbf{w}) - C(m + \alpha\mathbf{v}) + 2Bn = L_0 + K_0(\mathbf{v} + \mathbf{w})\,,$$
$$(B)_2 - (C)_1 = a_2(C - A) + 2Ba_1\,, \quad -(A)_2 + (B)_1 = a_1(C - A) - 2Ba_2,$$
$$-2(\mathbf{v})_2\,(\beta + B) + (\mathbf{v})_1\,(\gamma + C) + [-(\mathbf{w})_1 + 2a_2(\mathbf{v} - \mathbf{w})]\,(\alpha + A) = 0\,,$$
$$-2(\mathbf{w})_1\,(\beta + B) + [-(\mathbf{v})_2 + 2a_1(\mathbf{v} - \mathbf{w})]\,(\gamma + C) + (\mathbf{w})_2\,(\alpha + A) = 0\,,$$

where $K_0 = 1 + \alpha\gamma - \beta^2$, $L_0 = \alpha p + \gamma m - 2\beta n + (\mathbf{v} + \mathbf{w})\,(1 + \beta^2)$.

Proof. We use Lemma 2. The proof is straightforward computation.

Theorem 2. *A two-par. spherical motion $g(X)$ splits iff there exists a surface $h(X)$ in the Euclidean space E_3 such that it is isometric with $g(X)$ and for its second* *fundamental form* $F = A(\omega_1)^2 + 2B\omega_1\omega_2 + C(\omega_2)^2$ *we have* $A\gamma - 2B\beta + C\alpha = -2K_0$.

Proof. Let us define forms α^i, α^i_j, $i, j = 1, 2, 3$, $\alpha^i_j + \alpha^j_i = 0$, in the following way: $\alpha^1 = \omega_1$, $\alpha^2 = \omega_2$, $\alpha^3 = 0$, $\alpha^2_1 = -\vartheta^2_1$, $\alpha^1_3 = \vartheta^1_3$, $\alpha^3_2 = \vartheta^3_2$. Then it is easy to check that the forms α^i, α^i_j define a frame of an isometric surface in E_3 iff the equations for splitting of a spherical motion are satisfied with the exception of the second one: $AC - B^2 = K_0$ is the isometry condition, the other two equations are the integrability conditions. The second fundamental form of h is $F = \alpha^1\alpha^1_3 - \alpha^2\alpha^3_2 = \omega_1(A\omega_1 + B\omega_2) + \omega_2(B\omega_1 + C\omega_2) = A(\omega_1)^2 + 2B\omega_1\omega_2 + C(\omega_2)^2$.

Surfaces satisfying the conditions from Theorem 2 will be simply called associated.

Remark. The notion of associated surfaces is independent of the choice of the tangent frame, as the equation $A\gamma - 2B\beta + C\alpha = -2K_0$ in invariant with respect to rotations round the third axis of the frame.

Remark. Elliptical surfaces with $K_0 = 0$ (Bianchi's developable surfaces) are associated with the Euclidean plane, so spherical motions with $K_0 = 0$ always split, which was proved in [6] (see D)c)).

Let us now look for cases with $K_0 = 0$, for which the splitting is not unique. In such a case we must have an associated surface with $F \neq 0$. We may suppose that we have chosen such a frame that $B = 0$. It is the canonical frame of the associated surface. This frame is unique, if $A \neq C$. We have $AC = 0$. Let $A = 0$. Then $C\alpha = 0$. $\alpha \neq 0$ gives the already known trivial solution, so we must have $\alpha = 0$. Then $K_0 = 1 - \beta^2 = 0$. Using the integrability conditions, we finally obtain

$$(C)_1 = (\gamma_1) = a_1 = a_2 = \alpha = A = B = 0\,, \quad \beta = \pm 1\,, \quad C \neq 0\,.$$

Integration shows that the associated surface is a cylinder; the splitting will be described later on.

Now we shall show that solutions for the splitting of a spherical motion, other than with $K_0 = 0$, exist. So let $g(u, v) = g_1(u) \,.\, g_2^{\mathrm{T}}(v)$ be a product of two one-par. motions, let u and v be canonical parameters. Then $(g_1(u), g_2(v))$ is a lift of this motion and $\varphi(u) = g_1^{\mathrm{T}}(u)\,\mathrm{d}g_1(u) = a(u)\,\mathrm{d}u$, $\psi(v) = g_2^{\mathrm{T}}(v)\,\mathrm{d}g_2(v) = b(v)\,\mathrm{d}v$ with $a^2 = b^2 = 1$. Let us denote $(a, b) = \cos\boldsymbol{\alpha}$, $\boldsymbol{\alpha} = \boldsymbol{\alpha}(u, v)$, $\sin\boldsymbol{\alpha} \neq 0$.

To find invariants K_0, H_0 of the motion $g(u, v)$, we have to change the lift (g_1, g_2) to a tangent one. Let $h = (h_1, h_2, h_3) \in O(3)$ be the matrix of the change with columns h_1, h_2, h_3. Then $\tilde{\varphi} = h^{\mathrm{T}}\varphi h + h^{\mathrm{T}}\,\mathrm{d}h$ and similarly for ψ. We need such a change that $\tilde{\omega}^2_1 = \tilde{\varphi}^2_1 - \tilde{\psi}^2_1 = [h^{\mathrm{T}}(\varphi - \psi)\,h]^2_1 = 0$, so $(h_3, a) = (h_3, b) = 0$, where a and b are written in the vector form, see D). This yields $h_3 = (a \times b)/\sin\boldsymbol{\alpha}$. Let us choose (for symmetry reasons) $h_1 = (a + b)/2\cos(\alpha/2)$, , $h_2 = (a - b)/2\sin(\alpha/2)$.

After some computation we arrive at $K_0 = -4\sin^{-4}\boldsymbol{\alpha}|a, b, a'| \,.\, |a, b, b'|$, $H_0 = \frac{1}{2}(\alpha + \gamma) = \cos\boldsymbol{\alpha}\sin^{-3}\boldsymbol{\alpha}(\sin^2\boldsymbol{\alpha} + |a, b, b'| - |a, b, a'|)$, where the prime denotes the derivative.

Let now $K_0 = 0$. Then $|a, b, a'| = 0$ or $|a, b, b'| = 0$. If $b =$ const. or $a =$ const., we have $K_0 = 0$. So let $b' \neq 0$, $a' \neq 0$, $|a, b, a'| = 0$. Then $(b(v), a \times a'(u_0)) = 0$ for fixed u_0 and all v. This means that $b(v)$ lies in a plane and $a \times a'$ is constant, and so a and b lie in the same plane.

Theorem 3. *Let $K_0 = 0$. Then for $g(u, v) = g_1(u) \,.\, g_2(v)$ one of the following situations occurs*: a) *g_1 is a rotation with a fixed axis, g_2 is arbitrary*; b) *g_1 is arbitrary, g_2 is a rotation with a fixed axis*; c) *g_1 is a rolling of a curve on a great circle and g_2 is a rolling of the same great circle on a curve.*

Proof. Let $K_0 = 0$, $a(u)$, $b(v)$ be not constant. Then a and b lie in the same plane for all u and v, and $a(u)$ describes the fixed centrode of $g_1(u)$, $b(v)$ describes the fixed centrode of the motion inverse to $g_2(v)$ and $a(u)$, $b(v)$ lie on the same circle. The converse is obvious.

Remark. Theorem 3 shows that the product of any two spherical motions different from those in Theorem 3 gives a solution of the equations for the functions A, B. C in Theorem 2, for which $K_0 \neq 0$.

Remark. We shall show that Bianchi's developable surfaces are characterized as splitting spherical motions such that the curves $u =$ const. and $v =$ const. are asymptotic curves (see [1]). Indeed, let $g(u, v) = g_1(u)\, g_2(v)$ and let $dv = 0$. Then $\Phi(dv = 0) = (-|a, b, a'|\,(du)^2)/2 \sin \alpha$ and so $\Phi(dv = 0) = 0$ iff $K_0 = 0$ and similarly for $u =$ const. As we see from the formula, one of the curves $u =$ const., $v =$ const. is enough.

It remains to describe the cases with $K_0 = 0$ for which the splitting is not unique. We have $\beta = \pm 1$, $\alpha = (\gamma)_1$, $\alpha = (\gamma)_1 = a_1 = a_2 = 0$. Let $\beta = 1$. Then

$$\psi = \begin{pmatrix} 0, & 0, & 0 \\ 0, & 0, & 2\, du + \gamma\, dv \\ 0, & -2\, du - \gamma\, dv, & 0 \end{pmatrix}$$

and g_2 is a rotation round the first axis. The change of the frame $h(v)$ which gives the different splitting is again a rotation round the first axis and we have

$$g(u, v) = g_1(v)\, g_2^{-1}(u) = g_1(v)\, h(v) \,.\, [h^{-1}(v)\, g_2^{-1}(u)] = g_3(v)\, g_2^{-1}(u + v)\,,$$

where $g_3(v) = g_1(v)\, h(v)$ is a one-parametric motion and $g_2(u)\, h(v) = g_2(u + v)$. The case $\beta = -1$ is similar. This shows that the splitting in the case $K_0 = 0$ is not unique in the cases a) and b) of Theorem 3.

b) *Motions with a two-dimensional group of automorphisms*

Let us find all 2-par. motions which have a 2-dim. transitive group of automorphisms. This excludes the already discussed case 5, where the group of automorphisms has dimension 3. If a motion has a 2-dim. transitive group of automorphisms and if it has a canonical frame, then its invariants must be constant. Let us consider the general case first. Then we have

$$2a_1\beta + a_2(\alpha - \gamma) = 0\,, \quad a_1(\alpha - \gamma) + 2a_2\beta = 0\,.$$

Let $\beta = 0$, $\alpha = \gamma$. Then $a_1^2 + a_2^2 + 1 = -\alpha^2$, which is impossible. So $4\beta^2 + (\alpha - \gamma)^2 \neq 0$ and $a_1 = a_2 = 0$. Then $b_1 = b_2 = K_0 = L_0 = 0$ and the conditions for splitting are satisfied, so the motion is a product of two one-parametric motions. In the case 2 we get similar result, cases 3 and 4 are impossible.

Theorem 4. *A two-parametric motion of rank* 2 *with a two-dimensional transitive group of automorphisms exists only if* $4\beta^2 + (\alpha - \gamma)^2 \neq 0$. *Such a motion is determined by constants* $\mathbf{v}$, $\mathbf{w}$, α, β, γ, m, n, p, *where* $K_0 = L_0 = 0$. *It splits into a product of two one-parametric motions. The spherical image of such a motion is the Clifford's quadric.*

c) *Rolling of two isometric surfaces*

Theorem 5. *A two-parametric motion is a rolling of two isometric surfaces iff* $\mathbf{v} = \mathbf{w} = 0$, $n^2 - pm \neq 0$. *The two surfaces are then the polar surfaces of the motion.*

Proof. Let $A(u, v)$ and $\bar{A}(u, v)$ be two isometric surfaces, $B(s)$ and $\bar{B}(s)$ two corresponding curves on A and $\bar{A}$, respectively, such that $B(0) = A(0, 0)$ and $\bar{B}(0) = A(0, 0)$, let s be the arc. Then for the rolling of $\bar{A}$ on A which carries $\bar{B}$ into B we must have $g(s)\,\bar{B}(s) = B(s)$ and $g(s)\,\bar{B}'(s) = B'(s)$. The derivative of the first equation gives $g'(s)\,\bar{B}(s) + g(s)\,\bar{B}'(s) = B'(s)$. This yields $0 = g'(s)\,\bar{B}(s) = g'(s) \,.\, g^{-1}(s)\,B(s)$. Hence we see that the instantaneous motion is a rotation (there exists a point with velocity zero). As this is true for any curve on $A(u, v)$, all instantaneous motions are rotations and so $\mathbf{v} = \mathbf{w} = 0$.

Conversely, let $\mathbf{v} = \mathbf{w} = 0$. Then we have the case 2 and we may suppose that $\beta = 0$. Then $\eta^3 = 0$, $b_1 = b_2 = \alpha p + \gamma m = 0$. The polar surfaces contact, as $\omega^3 = \eta^3 = 0$, and their correspondence is an isometry since the difference of the first fundamental forms of $\bar{A}$ and A is zero, as we see from F). From (23) we see that the tangent vectors of the corresponding curves also correspond if $n^2 = pm$; the polar surfaces have a singular point.

Remark. The rolling of two isometric surfaces is in general determined if an isometric correspondence between these two surfaces is given. The problem of unicity of such a correspondence was solved by E. Cartan. We shall simply suppose that such a correspondence is already given.

Now we shall discuss the connection between the invariants of the motion and the invariants of the polar surfaces.

Let $\bar{A}(u, v)$ and $A(u, v)$ be two surfaces in an isometric correspondence. Let $\bar{\mathcal{R}}$ and $\mathcal{R}$ be tangent frames of $\bar{A}$ and A, respectively, which correspond to each other in the isometry. Let us consider a motion $g(u, v)$ such that $g(\bar{\mathcal{R}}) = \mathcal{R}$. ($g$ is not necessarily a two-par. motion.) $(\mathcal{R}, \bar{\mathcal{R}})$ is a lift of $g(u, v)$ and for this lift we have $\mathrm{d}\mathcal{R} = \mathcal{R}\varphi$, $\mathrm{d}\bar{\mathcal{R}} = \bar{\mathcal{R}}\psi$.

As $\varphi^1 = \psi^1$, $\varphi^2 = \psi^2$, we have
$\varphi_3^1 = a\varphi^1 + b\varphi^2$, $\varphi_2^3 = -b\varphi^1 - c\varphi^2$, $\psi_3^1 = \bar{a}\varphi^1 + \bar{b}\varphi^2$, $\psi_2^3 = -\bar{b}\varphi^1 - \bar{c}\varphi^2$, where $K = ac - b^2 = \bar{a}\bar{c} - \bar{b}^2$. Let us denote $b - \bar{b} = b_1$, $a - \bar{a} = a_1$, $c - \bar{c} = c_1$, $a + \bar{a} = a_2$, $c + \bar{c} = c_2$. For the change of the tangent frame we get $\tilde{b}_1 = \frac{1}{2}(c_1 - a_1)$. $\sin 2\alpha + b_1 \cos 2\alpha$, where α is the angle of rotation of the frame. This shows that we may always change the frame to $b = \bar{b}$. (If $c_1 = a_1$ the frame is not unique.) The resulting motion has rank 2 if ω_3^1 and ω_2^3 are linearly independent. This yields $\omega_3^1 \wedge \omega_2^3 = (\varphi_3^1 - \psi_3^1) \wedge (\varphi_2^3 - \psi_2^3) = -a_1c_1\varphi^1 \wedge \varphi^2 \neq 0$ and $a_1c_1 \neq 0$.

1. Let us look for what pairs of surfaces we have $a_1c_1 = 0$. Let $a_1 = 0$. Then also $ac = \bar{a}\bar{c}$. If $a \neq 0$, then the surfaces are identical, $g(u, v) = e$. We may therefore suppose that $a = \bar{a} = 0$, $c \neq \bar{c}$, $b = \bar{b}$. The integrability conditions give $\mathrm{d}\varphi^1 = r\varphi^1 \wedge \varphi^2$, $\mathrm{d}\varphi^2 = s\varphi^1 \wedge \varphi^2$ for some functions r and s; further, $(b)_1 = -2bs + cr = -2bs + \bar{c}r$ and $r = 0$. Let us write $\varphi^1 = \mathrm{d}u$. Then for the curves $\varphi^2 = 0$ we get $\mathrm{d}A = \mathrm{d}u e_1$, $\mathrm{d}e_1 = \varphi_1^2 e_2 - \varphi_3^1 e_3 = 0$ and so they are straight lines. Along those curves we also have $\mathrm{d}(\bar{e}_3 - e_3) = 0$ and so $\bar{e}_3(u) = e_3(u)$. This means that the surfaces are ruled surfaces, which contact along a generating line. Such surfaces are axoids of a one-parametric motion. (If two isometric ruled surfaces have another position, they still may determine a two-parametric motion.)

Theorem 6. *Two surfaces in an isometric correspondence define a 2-parametric motion of rank 2 iff they are not identical or if they are not two ruled surfaces which contact along a generating line.*

2. Let $a_1c_1 \neq 0$, $a_1 - c_1 \neq 0$. To find invariants of the motion we have to rotate the tangent frame again. Let us denote the angle of rotation by ϑ, let $M^2 = b^2(a_1 + c_1)^2 + a_2^2c_1^2$. Computation yields

$$\alpha = (a_1c_1)^{-1}\left[b(c_1 - a_1) + M\right], \quad \gamma = (a_1c_1)^{-1}\left[b(c_1 - a_1) - M\right],$$

$$m = -2(a_1c_1)^{-1}(c_1 \sin^2\vartheta + a_1\cos^2\vartheta), \quad p = -2(a_1c_1)^{-1}(c_1\cos^2\vartheta + a_1\sin^2\vartheta),$$

$$n = (a_1c_1)^{-1}(a_1 - c_1)\sin 2\vartheta,$$

where

$$\operatorname{cotan}\vartheta = (a_2c_1)^{-1}\left[-b(a_1 + c_1) + M\right].$$

In particular,

$$K_0 = \alpha\gamma + 1 = -4K(a_1c_1)^{-1}, \quad n^2 - pm = -4(a_1c_1)^{-1}.$$

3. Let $a_1c_1 \neq 0$, $a_1 - c_1 = 0$. Discussion of this case shows that for $a = c$ we get two opposite spheres with the same radius; it is a special case of the case 5. For $a \neq c$ we get two surfaces which are symmetric with respect to the common tangent plane; the main directions correspond to each other. The associated spherical motion is a minimal surface, $\alpha + \gamma = 0$.

d) *Motions with two curves as trajectories*

In this section we shall describe all 2-parametric motions of rank 2 which have at least two curves as trajectories of points. The tangent plane of the trajectory of a point is given by (25), the tangent space has dimension less then 2 if (24) is satisfied. If $\mathbf{v} = \mathbf{w} = x = y = z = 0$, the dimension is zero, it is the origin, which is fixed by a spherical motion. We leave this case out.

If a space motion has two curves as trajectories, then the corresponding spherical motion must have one curve as a trajectory of a point. Let us investigate this case first. So let $g(u, v)$ be a spherical motion such that the point $P = (x, y, z)$ has a curve as its trajectory. From (24) we get $z = 0$ and (7) yields

$$(27) \qquad \mathrm{d}x = \psi_1^2 y\,, \quad \mathrm{d}y = -\psi_1^2 x\,, \quad -\psi_3^1 x + \psi_2^3 y = 0\,.$$

(27) may have a nontrivial solution only if $K_0 = 0$ and the motion splits into a product of two one-parametric motions; let us find them. $\mathrm{d}\psi_1^2 = K_0 \omega_2^3 \wedge \omega_3^1$ gives $\mathrm{d}\psi_1^2 = 0$ and we may choose such a frame that $a_1 = a_2 = 0$. Then $\psi_1^2 = 0$ and $x = \text{const}$, $y = \text{const}$. Now it is easy to see that we get the case b) of Theorem 3.

Let us return to the original problem.

a) Let $\mathbf{v} = \mathbf{w} = 0$ and let $g(u, v)$ be a space motion with two curves as trajectories of points. As the corresponding spherical motion must have one curve as a trajectory, we shall use the preceding results. Let us also keep the same frame for the spherical motions. Points $P = (X, Y, Z)$ with curves as trajectories must satisfy the equations

$$Z = 0\,, \quad \mathrm{d}X = -\psi^1\,, \quad \mathrm{d}Y = -\psi^2\,, \quad X = -\mu Y\,, \quad \mu = \alpha^{-1}(\beta + 1) = \text{const.}$$

If $(X, Y, 0)$ is a solution of these equations, then $(X + C, Y + C, 0)$, $C = \text{const.}$, is also a solution. The integrability conditions show that the motion splits. Frenet formulas show that the moving polar surface is a straight line. The fixed polar surface is a curve. The points of the moving polar surface have curves as trajectories since the second factor of the motion is a rotation about the moving polar surface.

b) Let $\mathbf{v} \neq 0$, $-\mathbf{v}\mathbf{w} \geqq 0$. Denote $\lambda = -\mathbf{v}^{-1/2}\,\mathbf{w}^{1/2}$. Points which have curves as trajectories must lie of the line $Z = \lambda \mathbf{v}$, $X = -\lambda Y$ (we change the orientation of $\boldsymbol{e}_3$ if necessary). The functions (X, Y, Z) must solve (7) and we know, that $(X + C, Y + C, Z)$, $C = \text{const.}$, must be also a solution. We obtain

$$(28) \qquad \psi_2^3 = -\lambda\psi_3^1\,, \quad \lambda\,\mathrm{d}b = -\psi^3\,, \quad \mathrm{d}Y = -\psi^2 - \lambda^2 b\psi_3^1\,, \quad \mathrm{d}\lambda = 0\,,$$
$$\psi^1 + \lambda\psi^2 + \lambda b\psi_3^1(\lambda^2 + 1) = 0\,.$$

The solution of (28) together with the integrability conditions yields that $K_0 = L_0 = 0$, $\alpha, \beta, \gamma, m, n, p\, \mathbf{v}, \mathbf{w}$ are functions of one variable only, where $\alpha, \mathbf{v}, \mathbf{n}$, can be chosen arbitrarily and the others can be expressed from them.

Theorem 7. *A space motion whose two trajectories are curves is given by three arbitrary functions of one variable and one nonnegative constant* $(\alpha, \mathbf{v}, n, \lambda)$.

Theorem 8. *Any 2-parametric space motion whose two trajectories are curves splits into a product $g(u, v) = g_1(u)\, g_2(v)$, where $g_2(v)$ is a rotation and the points of the axis of rotation have curves as trajectories.*

Proof. The proof is computational and we leave it out.

As an application we shall describe the two-parametric motion of rank 2 which has two skew straight lines as trajectories. Such a motion is a motion from Theorem 7, such that there are two solutions for Y which describe straight lines. The tangent vector of the trajectory at the point $(-\lambda Y, Y, \lambda \mathbf{v})$ is the vector $(\mathbf{v}, -\lambda\mathbf{v}, Y)$. This means that the unit vector $\mathbf{a}$ of the straight trajectory is $\mathbf{a} = \mu^{-1}(\mathbf{v}, -\lambda\mathbf{v}, Y)$, where $\mu = (\mathbf{v}^2\omega\varkappa + Y^2)^{1/2}$, $\omega = (1 + \lambda^2)^{1/2}$. The vector $\mathbf{a}$ belongs to the fixed system and therefore it satisfies (8).

As a result we must have two vectors $\mathbf{a} = (x, y, z)$, $y = -\lambda x$, such that

$$\mathrm{d}x = -z\,\mathrm{d}u\,, \quad \mathrm{d}y = -\alpha^{-1}(\beta - 1)\, z\,\mathrm{d}u\,, \quad \mathrm{d}z = x\,\mathrm{d}u + \alpha^{-1}(\beta - 1)\, y\,\mathrm{d}u\,.$$

Substitution for y yields $\beta = 0$, $\alpha = \lambda^{-1}$, $\alpha = \text{const}$. Further, we have $\mathrm{d}Y = -n\,\mathrm{d}u$. Derivatives with respect to u will be denoted by a prime. We have

$$x = \mathbf{v}\mu^{-1}\,, \quad y = -\lambda\mathbf{v}\mu^{-1}\,, \quad z = Y\mu^{-1}\,, \quad \mu\mu' = \mathbf{v}\mathbf{v}'\,\omega^2 - Yn\,.$$

The derivative of $\mathbf{v} = x\mu$ yields $Y^2 + Y\mathbf{v}' + \mathbf{v}^2\omega^2 + \mathbf{v}n = 0$. We have $Y_{1,2} = \frac{1}{2}(-\mathbf{v}' \pm D^{1/2})$, where $Y_1 - Y_2 = D^{1/2} = 2\varkappa = \text{const.}$, $Y_{1,2} = \pm\varkappa - \mathbf{v}'/2$, $D = (\mathbf{v}')^2 - 4(\mathbf{v}^2\omega^2 + \mathbf{v}n) = 4\varkappa^2$. As $Y' = -n = -\mathbf{v}''/2$, we have $(\mathbf{v}')^2 - 2\mathbf{v}\mathbf{v}'' = 4\varkappa^2 + 4\mathbf{v}^2\omega^2$. The general solutions is $\mathbf{v} = C + B\omega^{-1}\sin 2\omega u$, where C is an integration constant (the other one was absorbed in u), $B = (\varkappa^2 + C^2\omega^2)^{1/2}$. $Y = \pm\varkappa - B\cos 2\omega u$, $n = -2\omega B \sin 2\omega u$ and the invariants of the motion are determined. To complete the computation we have to show that $\mathbf{a}$ satisfies (8), and this is easy.

For the angle φ of straight trajectories we have $\cos\varphi = C\,\omega B^{-1}$, their shortest distance d satisfies $d = 2\varkappa BC^{-1}\,\omega^{-2}|\lambda^2 - 1|$.

e) *Rolling of two curves*

Let $\bar{c}(u)$ and $c(u)$ be two curves in the spaces $\bar{E}_3$ and E_3, respectively, let u be the arc, $\bar{\mathcal{R}}$, $\mathcal{R}$ their Frenet frames, $\bar{k}$, k the curvatures, $\bar{\tau}$, τ the torsions, respectively. Let $h(v)$ be the rotation round the first axis. Then the motion which realizes the rolling of $\bar{c}(u)$ on $c(u)$ is the 2-par. motion $g(u, v)$ such that $g(\bar{\mathcal{R}}(u)\, h^{-1}(v)) = \mathcal{R}(u)\, h(v)$. For the lift $(\mathcal{R}\,.\,h, \bar{\mathcal{R}}\,.\,h^{-1})$ of $g(u, v)$ we have

$$(29)\qquad \varphi = \begin{pmatrix} \mathrm{d}u\ , & 0\ , & 0 \\ \tau - \mathrm{d}v, & -k\sin v\,.\,\mathrm{d}u, & k\cos v\,.\,\mathrm{d}u \end{pmatrix},$$

$$\psi = \begin{pmatrix} \mathrm{d}u\ , & \ , & 0 \\ \bar{\tau} + \mathrm{d}v, & \bar{k}\sin v\,.\,\mathrm{d}u, & \bar{k}\cos v\,.\,\mathrm{d}u \end{pmatrix},$$

$$\omega = \tfrac{1}{2}\begin{pmatrix} 0\ , & 0\ , & 0 \\ \tau - \bar{\tau} - 2\,\mathrm{d}v, & -(k + \bar{k})\sin v\,.\,\mathrm{d}u, & (k - \bar{k})\cos v\,.\,\mathrm{d}u \end{pmatrix}.$$

Theorem 9. *The two-parametric motion of rank two, given as a rolling of two curves with* $k^2 + \bar{k}^2 \neq 0$, *is characterized by the conditions* $\mathbf{v} = \mathbf{w} = 0$, $n^2 = mp$. *The polar surfaces of such a motion form the rolling pair of curves and we have* $\alpha p + \gamma n = 0$ *and* $(K_0)_2 \,.\, \sqrt{(\alpha)} = (K_0)_1 \sqrt{(\alpha - \gamma)}$.

Proof. Let $g(u, v)$ be given as above. It has rank two iff one of the products $\omega_1^2 \wedge \omega_2^3 = -\frac{1}{2}(k - \bar{k}) \cos v \, du \wedge dv$ and $\omega_3^1 \wedge \omega_2^3 = \frac{1}{2}(k + \bar{k}) \sin v \, du \wedge dv$ is different from zero. This yields $k^2 + \bar{k}^2 \neq 0$. The rest follows from (29), (23) and (19). (To be continued.)

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Souhrn

DVOJPARAMETRICKÉ POHYBY V E_3

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Článek se zabývá lokální diferenciální geometrií dvojparametrických pohybů euklidovského prostoru. Prvá část se zabývá přeformulováním klasických výsledků této disciplíny do současného geometrického jazyka spolu s uvedením souvislosti s eliptickou diferenciální geometrií.

Dále jsou uvedeny některé aplikace: Nutné a postačující pomínky pro rozklad dvojparametrického pohybu na součin dvou jednoparametrických pohybů, jsou charakterizovány pohyby s konstantními invarianty, pohyby mající dvě křivkové trajektorie a pohyby určené jako odvalování dvou křivek. Podrobně je rozebrán případ odvalování dvou isometrických ploch.

Резюме

ДВУХПАРАМЕТРИЧЕСКИЕ ДВИЖЕНИЯ В E_3

ADOLF KARGER

Статья занимается локальной дифференциальной геометрией двухпараметрических движений в пространстве E_3. В первой части изложены классические результаты этой области из точки зрения современной геометрии и их связи с эллиптической геометрией. Остальная часть занимается приложениями. Она содержит необходимые и достаточные условия для разложения движения в произведение двух однопараметрических движений, описание характеристических свойств движений с постоянными инвариантами некоторые другие результаты.

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