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ON GENERALIZED DIFFERENCE EQUATIONS

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Summary. In this paper linear difference equations with several independent variables are considered, whose solutions are functions defined on sets of n-dimensional vectors with integer coordinates. These equations could be called partial difference equations. Existence and uniqueness theorems for these equations are formulated and proved, and interconnections of such results with the theory of linear multidimensional digital systems are investigated.

Numerous examples show essential differences of the results from those of the theory of (one-dimensional) difference equations. The significance and the correct formulation of initial conditions for the solution of partial difference equations is established and methods are described, which make it possible to construct the solution algorithmically. Extensions of the theory to some special nonlinear partial difference equations are also considered.

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1. INTRODUCTION

Linear difference equations form a classical topic of mathematical analysis. These equations are usually written in the form

$$\sum_{i=0}^{n} a_i(k) f(n - i + k) = g(k) . \tag{1.1}$$

The solution of such a difference equation is a sequence f satisfying equation (1.1) with given coefficients $a_i(k)$, given input $g(k)$ for $k \in N$, and satisfying some initial (or boundary) conditions.

Here, N is the set of positive integers, a, f, g are sequences, i.e., mappings of the type $f: N \to \boldsymbol{R}$ or $f: N \to \boldsymbol{C}$. If the values of f on a certain set (e.g. for $k = 0, 1, \ldots$ $\ldots, n - 1$) are given, the difference equation (1.1) has exactly one solution, etc.

In the last decades these difference equations have found important applications closely connected to numerical analysis and, in general, to the so-called digital systems. Finite difference methods in solving partial difference equations led to investigations of simple difference equations in several variables. During the last

$10-20$ years digital processing of "two dimensional signals" – functions defined on different types of grids, such as planar images of various origin – have become very useful and important. So the classical problem of difference equations, now in a more general setting, has been brought forward again. The analysis and applications of these more general difference equations revealed that well-known techniques and theorems can be used here only with caution, and moreover, that two and more dimensional problems may have some unexpected properties. Therefore it seems reasonable to reformulate the basic existence and uniqueness theorems for difference equations to make the necessary generalizations possible and, furthermore, to formulate basic theorems on difference equations, occurring in the so-called $n - D$ digital systems theory. Such investigations seem to be even more important in connection with the widening field of applications on $n - D$ digital filtering (recursive filters on the so-called nonsymmetric halfplanes [5], difference equations on non-rectangular grids [3], nonlinear filtering techniques [5], [2], and many related problems). The use of the n-dimensional Z-transform for the description of $n - D$ systems should also be based on a consistent theory of difference equations and their systems, since such basis could give more reliable results in the so-called frequency domain description of $n - D$ digital filters. We could also mention qualitative investigations, namely the most important problem of BIBO stability (bounded input – bounded output) of linear systems [5].

Our approach starts with the single equation

$$\sum_{\beta \in B} a_\beta(\alpha) f(\alpha + \beta) = x(\alpha)\,, \tag{1.2}$$

where $\alpha \in A \subset Z^n$, $B \subset Z^n$, $x\colon A \to \boldsymbol{C}$, $a\colon A \times B \to \boldsymbol{C}$, $a_\beta(\alpha) \neq 0$ for all $\alpha \in A$, $\beta \in B$, $f\colon A + B \to \boldsymbol{C}$.

In this formulation the following notation has been used: Z is the set of integers, Z^n the set of their n-tuples, i.e. $\alpha \in Z^n$ means $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i \in Z$. $\boldsymbol{C}$ is the set of complex numbers, $A + B = \{\gamma \in Z^n\colon \gamma = \alpha + \beta,\ \alpha \in A,\ \beta \in B\}$. In equation (1.2) we shall always suppose n fixed and B to be a finite set, $2 \leqq \operatorname{card} B = |B| < \infty$.

For $x(\alpha) \equiv 0$ the equation is called homogeneous, for $a_\beta(\alpha)$ independent of α for all $\beta \in B$ it is with constant coefficients. The mappings f, x will also be called sequences. A sequence $f\colon A + B \to \boldsymbol{C}$ is called a solution of (1.2) if for given a and x the sequence f satisfies equation (1.2).

In equation (1.2) it is supposed that the Z^n has a "natural" algebraic structure, namely Z^n is a (torsion-free Abelian) group with component-wise addition as its group operation. Such group can be endowed with an order relation $\leqq$ so that for any $\alpha, \beta \in Z^n$ either $\alpha \leqq \beta$ or $\beta \leqq \alpha$ holds true. Moreover, we shall suppose that $\alpha \leqq \beta$ implies $\alpha + \gamma \leqq \beta + \gamma$ for any $\gamma \in Z^n$. Such orders do exist: one such example is the lexicographical order. The group Z^n endowed with a fixed order $\leqq$ will be denoted by $(Z^n, \leqq)$. For subsets of ordered groups we shall write $A \subset (Z^n, \leqq)$, for $\alpha, \beta \in (Z^n, \leqq)$, $\alpha < \beta$ will stand for $\alpha \leqq \beta$ and $\alpha \neq \beta$.

Any sequence f satisfying equation (1.2) for all $\alpha \in A$ is called its solution. These solutions may perhaps be subject to some additional requirements. Accordingly we shall distinguish here two types of solutions; their definitions are derived from conditions of computability of the value $f(\alpha)$ for arbitrary $\alpha \in A + B$.

Definition 1.1. *A solution f is called a C-solution of (1.2) with given sets A, B and with a given mapping $a: A \times B \to \mathbf{C}$, $a_\beta(\alpha) \neq 0$ for all $\beta \in B$, if its value $f(\alpha)$ for any $\alpha + A + B$ can be obtained by a finite number of arithmetic operations from (1.2), provided the values $f(\alpha)$ are given on a fixed subset $C \subset A + B$.*

We aim at proving a uniqueness theorem for the solutions of at least some classes of equations (1.2). In earlier investigations the so-called recursively computable or RC solution has been considered, which is inherently unique. To formalize the definition of the corresponding class of equations we formulate the following definition (see also [2], [3]).

Definition 1.2. *The equation (1.2) with given sets $A, B \subset Z^n$ is said to have an* RC-*solution if there exists an order $\leqq$ in the set $A + B$ so that the value $f(\alpha)$, $\alpha \in A + B$, of this solution can be obtained by a finite number of arithmetic operations from equation (1.2) provided the values $f(\alpha')$ for $\alpha' < \alpha$ are known.*

Remark 1.3. In Definition 1.2 the value $f(\alpha)$ evidently depends on the coefficients a. The main distinction between Definitions 1.1 and 1.2 can be characterized as follows: To have a C-solution is a property of one single equation with fixed coefficients a, while to posses an RC-solution is a property of a class of equations defined by the sets A, B only (independently of the coefficients a). Although it is not quite consequent, we shall speak of an RC-solution for a single equation, if this equation belongs to this class. In this sense each RC-solution is a C-solution, but not conversely. This is easily revealed by comparing Theorem 2.1 and Example 3.6. Moreover, any RC-solution is unique, while a certain fixed equation (1.2) may have nonunique C-solutions or no C-solution at all. In what follows we are mainly interested in RC-solutions of equation (1.2). We shall see, that the existence of RC-solutions corresponds to an initial value problem for equation (1.2).

2. BASIC THEOREMS

Theorem 2.1. *In equation (1.2) let A, B be nonempty sets, $A, B \subset Z^n$, B finite with at least two elements. Then there exists a set $C \subset A + B$ such that for any given function $g: C \to \mathbf{C}$ and for any coefficients $a_\beta(\alpha)$ equation (1.2) has exactly one* RC-*solution $f: A + B \to \mathbf{C}$ satisfying the (initial) conditions $f(\alpha) = g(\alpha)$ for all $\alpha \in C$.*

The proof of this theorem is based on two lemmas.

Lemma 2.2. *For any nonempty finite set $B \subset Z^n$ there exists an n-tuple of integers $(N_1, N_2, \ldots, N_n)$ so that the mapping $h: h(z) = N_1 z_1 + N_2 z_2 + \ldots + N_n z_n$ is one-to-one on the set B.*

We can prove this assertion by induction:

a) Let $n = 1$. Then for any $N_1 \neq 0$, $h(z) = N_1 z_1$ is one-to-one on B.

b) Fix $n > 1$ and let us assume the assertion to be true for $n - 1$, $(B \subset Z^{n-1})$.

Denote $B' = \{(\beta_1, \ldots, \beta_{n-1}), \exists \beta_n \in Z, (\beta_1, \ldots, \beta_n) \in B\}$. As $B \neq 0$, $|B| < \infty$ we have $B' \neq 0$, $|B'| < \infty$. That is why there exist integers $N_1, \ldots, N_{n-1}$ so that the form $\sum_{i=1}^{n-1} N_i z_i$ is one-to-one on B'.

Now it is easy to see that for any two elements $\beta, \beta' \in B$ with $\beta_n \neq \beta'_n$ there exists exactly one number $N(\beta, \beta')$ such that

$$\sum_{i=1}^{n-1} N_i \beta_i + N(\beta, \beta')\, \beta_n = \sum_{i=1}^{n-1} N_i \beta'_i + N(\beta, \beta')\, \beta'_n \,.$$

Since B is finite we can choose an $N_n \in Z$ so that $N_n \neq N(\beta, \beta')$ for all $\beta, \beta' \in B$. It is now trivial to show that $\beta = \beta'$ whenever

$$\sum_{i=1}^{n} N_i \beta_i = \sum_{i=1}^{n} N_i \beta'_i \,, \quad \text{and} \quad \beta, \beta' \in B \,.$$

Indeed, $\beta_n \neq \beta'_n$ would imply $N_n = N(\beta, \beta')$ and therefore $\beta_n = \beta'_n$. But this means

$$\sum_{i=1}^{n-1} N_i \beta_i = \sum_{i=1}^{n-1} N_i \beta'_i$$

whence $\beta_i = \beta'_i$ also for all $1 \leqq i < n$ and the statement is proved.

Lemma 2.3. *Any set $A \subset Z^n$ can be endowed with an order $\prec$ so that A becomes a well-ordered set and, moreover, to any finite set $B \subset Z^n$, $2 \leqq$ card B there exists a mapping $\beta: A \to B$ such that*

$$\alpha' + \beta(\alpha') \in \alpha + B \quad \textit{implies} \quad \alpha' \prec \alpha \quad \textit{for all} \quad \alpha, \alpha' \in A \,. \tag{2.1}$$

Proof. From Lemma 2.2 it follows, that there exists a function $h(z) = N_1 z_1 + N_2 z_2 + \ldots + N_n z_n$, $N_i \in Z$, $z = (z_1, z_2, \ldots, z_n) \in Z^n$, which attains its strict minimum and strict maximum on the set B, i.e., there exist $\beta^0, \beta^1 \in B$ so that

$$h(\beta^1) < h(\beta)\,, \quad h(\beta^0) > h(\beta) \quad \text{for all} \quad \beta \in B \,. \tag{2.2}$$

Now we can define

$$A_i = \{\alpha \in A: |2h(\alpha) + 1| = i,\ i = 1, 3, 5, \ldots\} \,.$$

Certainly $\bigcup_{i=0}^{\infty} A_i = A$. Now let us take an arbitrary but fixed set of orders $\leqq_i$ so that each $(A_i, \leqq_i)$ becomes a well-ordered set. (Such orders certainly exist [1].) Define the following order $\prec$:

$$\text{(2.3)} \qquad \text{for} \quad \alpha' \in A_j\,, \quad \alpha \in A_i \quad \text{there is} \quad \alpha' \prec \alpha$$
$$\text{iff either} \quad j < i \quad \text{or} \quad \alpha' \leqq_i \alpha\,.$$

Obviously, A is well ordered by $\prec$. Furthermore, we put

$$\text{(2.4)} \qquad \begin{aligned} A_0 &= \{\alpha \in A\colon h(\alpha) \geqq 0\} \\ A_1 &= \{\alpha \in A\colon h(\alpha) < 0\} \end{aligned}$$

and define the mapping $\beta\colon A \to B$ by

$$\beta(\alpha) = \begin{cases} \beta^0 & \text{for} \quad \alpha \in A_0 \\ \beta^1 & \text{for} \quad \alpha \in A_1\,. \end{cases}$$

It remains to be verified that the order $\prec$ and the mapping β satisfy condition (2.1). Assuming $\alpha' + \beta(\alpha') = \alpha + \beta$ for some $\beta \in B$ we have to discuss four cases:

$$\begin{aligned} &\alpha', \alpha \in A_0 \\ &\alpha', \alpha \in A_1 \\ &\alpha' \in A_0\,, \quad \alpha \in A_1 \\ &\alpha' \in A_1\,, \quad \alpha \in A_0\,. \end{aligned}$$

The linearity of the function h implies

$$h(\alpha') + h(\beta(\alpha')) = h(\alpha) + h(\beta)\,.$$

Now $\alpha' \in A_0$, $\alpha \in A_1$ yields $\beta(\alpha') = \beta^0$,

$$h(\alpha) < 0 \leqq h(\alpha') \quad \text{and} \quad h(\beta) \leqq h(\beta^0)\,;$$

therefore $h(\alpha) + h(\beta) < h(\alpha') + h(\beta(\alpha'))$, which contradicts the assumption. Similarly, the fourth case cannot occur, either. Let the first case be analyzed. For $\alpha', \alpha \in A_0$ we have

$$h(\alpha') = h(\alpha) + h(\beta) - h(\beta^0)\,, \quad \text{whence} \quad h(\alpha') \leqq h(\alpha)\,.$$

Since $0 \leqq h(\alpha')$, we obtain

$$|2\,h(\alpha') + 1| \leqq |2\,h(\alpha) + 1|\,.$$

If the equality sign holds true, we have $h(\alpha) = h(\alpha')$ and, consequently $h(\beta) = h(\beta^0)$, whence $\beta = \beta^0$ nad $\alpha = \alpha'$. The strict inequality immediately implies $\alpha' \prec \alpha$.

In a similar way the case $\alpha', \alpha \in A_1$ can be handled; the mapping β satisfies condition (2.1) and the lemma is proven.

Proof of Theorem 2.1. Recalling Lemma 2.3, it is sufficient to prove the theorem assuming that A is a well-ordered set and that there exists a mapping $\beta\colon A \to B$, satisfying condition (2.1).

In the sequel let the following set C be considered

(2.5) $$C = (A + B) \setminus \bigcup_{\alpha \in A} \{\alpha + \beta(\alpha)\} .$$

Define $\Gamma_\alpha = \bigcup_{\alpha' < \alpha} \{\alpha' + \beta(\alpha')\}$ for all $\alpha \in A$ and denote $B^0(\alpha) = B \setminus \beta(\alpha)$. From (2.1) it easily follows that for all $\alpha, \alpha' \in A$

(2.6) $$\alpha' + \beta(\alpha') \in \alpha + B^0(\alpha) \quad \text{implies} \quad \alpha' < \alpha .$$

Furthermore, for all $\alpha \in A$ we have

(2.7) $$\alpha + B^0(\alpha) \subset \Gamma_\alpha \cup C ,$$

or equivalently

$$(\alpha + B^0(\alpha)) \cap \bigcup_{\alpha \leqq \alpha'} \{\alpha' + \beta(\alpha')\} = \emptyset ;$$

this is immediately evident from (2.6). Moreover, for $\alpha' < \alpha$ we have $\Gamma_{\alpha'} \subset \Gamma_\alpha$. The sum over B in equation (1.2) can now be decomposed and we obtain

(2.8) $$f(\alpha + \beta(\alpha)) = [a_{\beta(\alpha)}(\alpha)]^{-1} (x(\alpha) - \sum_{\beta \in B^0(\alpha)} a_\beta(\alpha) f(\alpha + \beta)) .$$

Since A is well-ordered and the mapping $\alpha \to \alpha + \beta(\alpha)$ is one-to-one[1]), the sets Γ_α are also well-ordered. Suppose that the values of f have been computed on the set Γ_α (and given on the set C), i.e. $f(\alpha' + \beta(\alpha'))$ are known for all $\alpha' < \alpha$. Then (2.8) yields the still unknown value $f(\alpha + \beta(\alpha))$, since on the right-hand side only the values of f on either the set Γ_α or the set C occur and $\alpha + \beta(\alpha) \notin \Gamma_\alpha$.

By the induction principle for well-ordered sets [1] we may conclude, that all the values of f on the set $A + B$ are recursively computable. Since every point $\xi \in (A + B) \setminus C$ can be uniquely represented as $\xi = \alpha + \beta(\alpha)$ with the mapping β fixed, the recursively constructed function f is indeed a unique solution of (1.2) and our proof is completed.

Remark 2.4. From the proof it follows that the assumption $a_\beta(\alpha) \neq 0$ for all $\alpha \in A$, $\beta \in B$ can be weakened to $a_{\beta(\alpha)}(\alpha) \neq 0$ for all $\alpha \in A$.

Remark 2.5. The proof remains unchanged, if in (2.4) the sets A_0, A_1 are defined by

$$A_0 = \{\alpha \in A,\ h(z) \geqq m\} ,$$
$$A_1 = \{\alpha \in A,\ h(z) < m\}$$

for an arbitrary fixed integer m. As a result we obtain a "shifted" set C.

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[1]) Indeed, from $\alpha \neq \alpha'$ and $\alpha + \beta(\alpha) = \alpha' + \beta(\alpha')$ a contradiction can easily be derived. From (2.1) and from $\alpha + \beta(\alpha) \in \alpha' + B$ follows $\alpha \leqq \alpha'$ and from $\alpha' + \beta(\alpha') \in \alpha + B$ follows $\alpha' \leqq \alpha$. Hence $\alpha = \alpha'$.

Remark 2.6. Although the idea of the RC-solution is based on a step-by-step construction of the solution f, the proof has some "nonconstructive" parts. Therefore the algorithm of determining the value $f(\xi)$ for an arbitrary $\xi \in A + B$ is not self-evident. Bellow it will be shown, how such an algorithm can be derived from Theorem 2.1.

The system-theoretical point of view considers equation (1.2) with $g(\alpha) = 0$ for all $\alpha \in C$ as a linear system. For linear systems with one variable the notion of causality is of basic importance. For multidimensional systems such as those described by equation (1.2) the notion of causality seemed to be less important [5]. Our next theorem, which in a certain sense is the converse of Theorem 2.1, shows that a correctly described linear system given by a difference equation of the type (1.2) implies causality as a certain ordering of the set A. More precisely:

Theorem 2.7. *Let $A, B, C \subset Z^n$ be nonempty sets with $2 \leqq \operatorname{card} B < \infty$ and $C \subset A + B$ such that the following condition is satisfied: for each mapping $a: A \times B \to C$ with $a_\beta(\alpha) \neq 0$ for all $\alpha \in A$, $\beta \in B$, and for each sequence $x: A \to C$ there exists one and only one sequence $f: A + B \to C$ such that*

$$\sum_{\beta \in B} a_\beta(\alpha) f(\alpha + \beta) = x(\alpha)$$

for all $\alpha \in A$ and with $f(\gamma) = 0$ for all $\gamma \in C$.

Then there exists: an order $\leqq$ such that A is well-ordered with respect to $\leqq$, and a mapping $\beta: A \to B$ satisfying the implication (as in (2.1))

$$\alpha' + \beta(\alpha') \in \alpha + B \Rightarrow \alpha' \leqq \alpha$$

so that (as in (2.5))

$$C = (A + B) \setminus \bigcup_{\alpha \in A} \{\alpha + \beta(\alpha)\} .$$

Proof. From given A, B, C let the following sets be defined

$\Gamma_0 = C$,
$\Lambda_0 = \{\alpha \in A$ such that there exists exactly one $\beta \in B$ with $\alpha + \beta \notin \Gamma_0\}$,
$\Gamma_1 = \Gamma_0 \cup (\Lambda_0 + B)$,
$\Lambda_k = \{\alpha \in A$ such that there exists exactly one $\beta \in B$ with $\alpha + \beta \notin \Gamma_k\}$,
$\Gamma_{k+1} = \Gamma_k \cup (\Lambda_k + B)$.

Evidently

$$\Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \ldots \subset \Gamma_k \subset \ldots \tag{2.9}$$

$$\Gamma_{k+1} = C \cup (\Lambda_0 + B) \cup \ldots \cup (\Lambda_k + B)$$

and

$$\Lambda_i \cap \Lambda_j = \emptyset \quad \text{for} \quad i \neq j . \tag{2.9a}$$

We shall now prove by contradiction that

$$A = \bigcup_{k=0}^{\infty} A_k . \tag{2.10}$$

Suppose that

$$A \setminus \bigcup_{k=0}^{\infty} A_k \neq \emptyset$$

and, moreover, suppose that for all $\alpha \in A$ belonging to this nonempty set there exists two distinct values $\beta_1, \beta_2 \in B$ such that

$$\alpha + \beta_i \notin \bigcup_{k=0}^{\infty} \Gamma_k , \quad i = 1, 2 .$$

Then there do exist nonzero coefficients $a_\beta(\alpha)$ such that

$$\sum_{\substack{\beta \in B \\ \alpha+\beta \notin \cup \Gamma_k}} a_\beta(\alpha) = 0 \quad \text{for all} \quad \alpha \in A \setminus \bigcup_{k=0}^{\infty} A_k .$$

Put $f_1(\alpha + \beta) = \begin{cases} 0 \text{ for } \alpha + \beta \in \bigcup \Gamma_k \\ 1 \text{ for the other elements of } A + B \end{cases}$

and $f_2(\alpha + \beta) = 0$ for all $\alpha + \beta \in A + B$. For $x = 0$ both these functions satisfy the chosen equation (1.2) with the corresponding initial conditions and from the assumptions of the theorem we obtain $f_1 = f_2$. Therefore $A + B = \bigcup \Gamma_k$, which contradicts our assumption that $\alpha + \beta_i \notin \bigcup \Gamma_k$.

On the other hand for $\alpha \in A \setminus \bigcup A_k$ the condition $\alpha + B \subset \bigcup \Gamma_k$ cannot be satisfied, since if f and f' are solutions of (1.2) for inputs x and x' respectively, with $x = x'$ on the set $\bigcup A_k$, then evidently $f \equiv f'$ on the set $\bigcup \Gamma_k$.

Summarizing, we obtain that there exists $\bar{\alpha} \in A \setminus \bigcup A_k$ such that a single $\bar{\beta} \in B$ can be found satisfying

$$\bar{\alpha} + \bar{\beta} \notin \bigcup_{k=0}^{\infty} \Gamma_k .$$

In this case (see (2.1)) there must exist an index j for which this conclusion holds true with $\bigcup \Gamma_k$ replaced by Γ_j. This would mean $\bar{\alpha} \in A_j$, which contradicts the assumption

$$\bar{\alpha} \in A \setminus \bigcup_{k=0}^{\infty} A_k .$$

Therefore (2.10) is true.

To continue the proof of Theorem 2.7 we define a mapping $\beta: A \to B$ by

$$\alpha + \beta(\alpha) \notin \Gamma_k , \quad \alpha \in A_k . \tag{2.11}$$

From the proof of formula (2.10) and from (2.9a) it follows, that (2.11) indeed defines a mapping.

Now we define an order relation $\leqq$ on the set A: for $\alpha' \in A_i$, $\alpha \in A_j$ we shall say that

$$\alpha' \leqq \alpha \text{ iff either } i < j \text{ or } (i = j \text{ and } \alpha' \leqq_i \alpha), \tag{2.12}$$

where $\leqq_i$ are arbitrary but fixed orders on the corresponding (mutually disjoint) sets A_i.

We have to prove that the mapping (2.11) and the order given by (2.12) satisfy the implication (2.1). To this end let $\alpha' + \beta(\alpha') = \alpha + \beta$ with $\alpha' \in A_i$, $\alpha \in A_j$. We immediately have $\alpha' + \beta(\alpha') \notin \Gamma_i$ and $\alpha + \beta \in \Gamma_{j+1}$ and therefore $i \leqq j$. For $i < j$ there remains nothing to be proven. For $\alpha, \alpha' \in A_i$ and $\alpha \neq \alpha'$ we have $\alpha + \beta \notin \Gamma_i$ and therefore $\beta = \beta(\alpha)$. This gives $\alpha' + \beta(\alpha') = \alpha + \beta(\alpha)$ and for the equation(1.2) this means that the value $x(\alpha)$ depends on the values of $x(\bar{\alpha})$ with

$$\bar{\alpha} \in \bigcup_{j=0}^{i-1} A_j \cup \{\alpha'\}.$$

Since this would be a contradiction, we can conclude that either $i < j$ or $\alpha = \alpha'$, hence $\alpha' \leqq \alpha$.

The last part of the proof consists in proving (2.5). First let us prove that

$$\bigcup_{\alpha \in A} \{\alpha + \beta(\alpha)\} \supset (A + B) \setminus C.$$

Choose $\gamma \in (A + B) \setminus C$ and let $i = \min\{k: \gamma \in A_k + B\}$. For $\alpha \in A_i$ and $\alpha + \beta = \gamma$ we obtain $\alpha + \beta \notin \Gamma_i$ and therefore $\beta = \beta(\alpha)$. Indeed, if

$$\gamma \in \Gamma_i = C \cup (A_0 + B) \cup (A_1 + B) \cup \ldots \cup (A_{i-1} + B),$$

it would be $\gamma \in A_m + B$ for some index m with $0 \leqq m < i$, which contradicts the definition of the index i. Therefore $\gamma = \alpha + \beta(\alpha)$, $\alpha \in A$ and the inclusion is proved. The converse inclusion follows from the fact that $\alpha + \beta(\alpha) \notin \Gamma_k$ for $\alpha \in A_k$ and $C \subset \Gamma_k$.

Formula (2.5) is proved. Hence, the proof of Theorem 2.7 is complete.

Corollary 2.8. *Under the assumptions of Theorem* 2.7 *we have*

$$|(A + B) \setminus C| = |A| \quad \textit{and therefore if} \quad |A| < \infty \quad \textit{then} \tag{2.13}$$

$$|C| = |A + B| - |A|,$$

where $|A| = \operatorname{card} A$.

Proof. The mapping $E: A \to (A + B) \setminus C$ defined by

$$E(\alpha) = \alpha + \beta(\alpha)$$

is evidently one-to-one and surjective. Formula (2.13) follows from the assumption $C \subset A + B$.

In Theorem 2.7 the proof of formula (2.10) depends on the fact that a is a mapping from $A \times B$, i.e. the equations considered are with variable coefficients. Supposing $a: B \to \boldsymbol{C}$ in fact weakens the assumption of this theorem. Although a statement similar to that of Theorem 2.7 for equations with constant coefficients would be important for multivariable linear systems theory, its proof would probably require some other type of reasoning.

This theorem also emphasizes the distinction between C- and RC-solutions of difference equations: the existence of a C-solution, being the property of a single equation, cannot impose any order relation on the set A. On the other hand, RC-solutions as a property of a class of equations may determine an order in the calculations of values of the solution or, furthermore, an algorithm.

It becomes also evident, that algorithms with various extent of paralellisms for the evaluation of $f(\alpha)$ can be constructed, although the initial set C is, in general, not finite. In fact, for the evaluation of $f(\alpha)$ at a fixed point $\alpha \in A + B$ only a finite number of initial values $g(\alpha)$, $\alpha \in C$, has to be used. This is immediately clear from the proof of Theorem 2.7: any $\alpha \in (A + B) \setminus C$ can be uniquely written as $\alpha = \alpha' + \beta(\alpha')$ with $\alpha' \in A_j$. Since B is a finite set, to calculate $f(\alpha)$ from (1.2) we need to know only $|B| - 1$ values of f. Therefore, if $f(\alpha)$ for $\alpha \in \Gamma_k$ has to be calculated,the values of $g(\alpha)$, $\alpha \in C$ on not more than $(|B| - 1)^k$ points must be used. In fact, this estimate is rather pessimistic. We do not want to go into details of construction of an effective algorithm of evaluation, although we consider this problem to be very important for practical applications of multidimensional digital systems.

3. SOME EXAMPLES AND FURTHER RESULTS

Equation (1.2) is a generalization of (1.1) even in the one-dimensional case; unexpected results stem from comparatively very simple problems.

Example 3.1. Let equation

$$2f_{n+1} - f_n = 0\,, \quad n \in A\,, \tag{3.1}$$

be considered for $A = \{n \in Z\colon 4 < n \neq 2^k\}$. Here, B has two elements, $B = \{0, 1\}$, and two different sets C can be considered: $C_1 = \{n \in Z\colon n = 2^k + 1, k = 2, 3, 4, \ldots\}$, $C_2 = \{n \in Z\colon n = 2^{k+1} - 1,\ k = 2, 3, 4, \ldots\}$. For a given function $g\colon C_i \to \boldsymbol{C}$ let the RC-solution of equation (3.1) be denoted $f^{(1)}$ and $f^{(2)}$, respectively.

Since $|f_{i+1}| = |f_i|/2$ for all $i \in A$, we obtain

$$\sup_{\gamma \in A+B} |f^{(1)}(\gamma)| \leqq \sup_{\gamma \in C_1} |g(\gamma)|\,,$$

i.e. bounded initial values imply bounded solutions.

For the function $f^{(2)}$ put $g(\gamma) = 1$ for all $\gamma \in C_2$. We successively obtain

$$f^{(2)}(2^{k+1} - 1) = g(2^{k+1} - 1) = 1\,, \quad f^{(2)}(2^{k+1} - 2) = 2f^{(2)}(2^{k+1} - 1) = 2$$

and finally

$$f^{(2)}(2^n + 1) = 2^{2^n - 2} \quad \text{for} \quad n \geqq 2\,,$$

whence for n sufficiently large we have

$$f^{(2)}(2^n + 1) \geqq 2^n\,.$$

The solution $f^{(2)}$ is unbounded.

This example shows that boundedness of the solution on a set $A + B$ may substantially depend on the initial set C; this conclusion remains true a fortiori in multidimensional cases.

Example 3.2. In equation (1.2) let $x = 0$, $\sup\limits_{\gamma \in C} |g(\gamma)| < \infty$ and let the coefficients $a_\beta(\alpha)$ satisfy for all $\alpha \in A$ the condition

$$|a_{\beta(\alpha)}(\alpha)| > \sum_{\beta \in B^0} |a_\beta(\alpha)|\,,$$

where $\beta(\alpha)$ is the mapping defined by (2.1) and $B^0 = B \setminus \{\beta(\alpha)\}$. Then the solution f is bounded, i.e. $\sup\limits_{\gamma \in A+B} |f(\gamma)| < \infty$.

The proof follows from (2.8).

In view of Theorems 2.1, 2.7 we may state that the correct choice of the "initial" set C for a given partial difference equation (1.2) is essential for the existence and uniqueness of its RC-solution irrespectively of the coefficients a_β, input x and initial values g. We shall say that he triple (A, B, C) with such a choice of the set C defines a class of well-posed initial value problems for the partial difference equation (1.2). Conclusions concerning the solutions of partial difference equations have to be formulated mostly within this class and therefore the construction of the initial set C from given sets A, B becomes important. This construction was hitherto mostly indirect. For practical purposes a direct construction will be described.

Construction 3.3. Let sets $A, B \subset Z^n$ be given and let $\leqq$ denote the lexicographic order. For all integers $m = 0, 1, 2, \ldots, 2^n - 1$ let their binary representation

$$m = \sum_{i=0}^{n-1} m_i 2^i\,, \quad m_i = 0, 1\,,$$

be considered. Let the given set A be decomposed into 2^n subsets A_m, defined by

$$A_m = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \in A : (\alpha_i \geqq 0 \text{ iff } m_i = 0) \text{ for all } i = 0, 1, 2, \ldots, n - 1\}\,.$$

Now the set Z^n will be endowed with 2^n different orders $\leqq_m$ as follows: for any $\alpha, \alpha' \in Z^n$ we have $\alpha \leqq_m \alpha'$ if there exists an index $k \in \{1, 2, \ldots, n\}$ such that $\alpha_i = \alpha'_i$ for all $i < k$ and $(1 - 2m_k)\,\alpha_k < (1 - 2m_k)\,\alpha'_k$. All these orders are invariant under addition and therefore $(Z^n, \leqq_m)$ is an ordered group for all $m = 0, 1, \ldots, 2^n - 1$. Since B is a finite set, it contains its maximal element with respect to the order $\leqq_m$. Let this element be denoted by β^m.

It can be proved by an argument resembling that of the proof of Theorem 2.1, that the set

$$C = (A + B) \setminus \bigcup_{m=0}^{2^n - 1} (A_m + \beta^m) \tag{3.2}$$

in (A, B, C) forms a class of well-posed initial value problems, i.e. that C is a "correct" initial set. Even more, the sets A_m are well ordered and the value $f(\alpha)$ for any $\alpha \in A + B$ can be computed by an algorithm, which is based on the orders $\leqq_m$ defined above.

For the sake of brevity we do not give the details of the proof here; instead, in the sequel, some examples will be shown.

Example 3.4. Let H be an upper triangular $(n \times n)$ nonsingular matrix of integers and let P be an $(n \times n)$ permutation matrix. Denoting by $\leqq$ the lexicographic order as in the previous construction, we may introduce the order $\prec$ in Z^n as follows: for $\alpha, \alpha' \in Z^n$ we have

$$\alpha \prec \alpha' \quad \text{iff} \quad PH\alpha \leqq PH\alpha' \, .$$

$(Z^n, \prec)$ is an ordered group. For $\alpha \in A$, $\beta \in B$ we may consider $\alpha^* = PH\alpha$, $\beta^* = PH\beta$ and use Construction 3.3. We obtain such a set C, that (A, B, C) forms a class of well-posed initial value problems.

Construction 3.3 and Example 3.4 impose a certain ordering on the computation of the values $f(\alpha)$. These orderings may yield different ways of parallel computation and may imply different algorithms. For equation (1.2) this procedure can be interpreted as a change of the independent variables.

Example 3.5. Consider a well-posed class (A, B, C) of initial value problems for partial difference equations (1.2) with $g(\alpha) = 0$ for all $\alpha \in C$. Then (1.2) defines a linear system

$$f = Tx \, .$$

If equation (1.2) has constant coefficients, the system is shift-invariant [2], [6] in the following sense: The equations belonging to the class $(A + \xi, B, C + \xi)$ for any $\xi \in Z^n$ are well -posed and

$$Tx(\alpha + \xi) = f(\alpha + \xi) \, .$$

For these linear, shift-invariant systems we may assume without loss of generality, that $0 \in A$. If a sequence δ is defined by

$$\delta(\alpha) = \begin{cases} 1 & \text{for} \quad \alpha = 0 \\ & \text{for} \quad 0 \neq \alpha \in A \end{cases}$$

then the solution h of the equation

$$\sum_{\beta \in B} a_\beta h(\alpha + \beta) = \delta(\alpha)$$

with $h(\alpha) = 0$ for all $\alpha \in C$ is usually called the impulse response of the corresponding LSI system.

For the sequence x^*

$$x^*(\alpha) = \begin{cases} x(\alpha) & \text{for} \quad \alpha \in A \\ 0 & \text{for} \quad \alpha \notin A \end{cases}$$

define now the function f^* by

$$f^*(\alpha + \beta) = \sum_{\alpha' \in A} h(\alpha' + \beta)\, x^*(\alpha' - \alpha) \tag{3.3}$$

for all $\beta \in B$, $\alpha, \alpha' \in A$. Multiplying this equation by a_β and summing these equations for all $\beta \in B$, we obtain

$$\sum_{\beta \in B} a_\beta f^*(\alpha + \beta) = \sum_{\alpha' \in A} \delta(\alpha')\, x^*(\alpha' - \alpha) = x^*(\alpha)$$

for all $\alpha \in A$. We may conclude from the uniqueness of the solution of (1.2) that $f = f^*$ and therefore

$$f = h * x\,, \tag{3.4}$$

where $*$ denotes the operation of convolution, defined by (3.3).

In Section 1. C- and RC-solutions of (1.2) have been defined. In the following example we present an idea, how we can construct equations as in (1.2), which have unique or nonunique C-solutions, but have no RC-solution.

Example 3.6. Choose $n = 2$ and let

$A = \{(\alpha_1, \alpha_2) \colon \alpha_1 \geqq 0,\ \alpha_2 \geqq 0,\ \alpha_i \in Z\}$,

$B = \{(0, 0),\ (1, 0),\ (0, 1),\ (1, 1)\}$,

$C = \{(0, q) \colon q = 0, 1, 2, \ldots\} \cup \{(k, 0) \colon k = 2, 3, 4, \ldots\} \cup \{(2, 1)\}$.

The corresponding difference equation with constant coefficients a_{kq} reads

$$a_{00} f(\alpha) + a_{10} f(\alpha + e_1) + a_{01} f(\alpha + e_2) + a_{11} f(\alpha + e_1 + e_2) = 0\,, \tag{3.5}$$

where

$$e_1 = (1, 0)\,, \quad e_2 = (0, 1)\,.$$

Let

$$\Delta = \begin{vmatrix} a_{10} & a_{11} \\ a_{00} & a_{01} \end{vmatrix}\,.$$

For $\Delta \neq 0$ we may conclude: given any function $g \colon C \to \boldsymbol{C}$, there exists one and only one $f: A + B \to \boldsymbol{C}$ satisfying the given equation (3.5) and the condition $f(\alpha) = g(\alpha)$ for all $\alpha \in C$. Indeed, $f(1, 0)$ and $f(1, 1)$ can be calculated from a system of linear equation with a (nonzero) determinant Δ. The remaining values $f(\alpha)$, $\alpha \in A$ can be calculated recursively.

For $\Delta = 0$ either $a_{00}^2 g(0, 0) = a_{11} a_{01} g(2, 1)$ and e.g. the value $f(1, 0)$ can be chosen arbitrarily (an infinite number of solutions) or the given function g does not satisfy this condition and no values of $f(1, 0)$, $f(1, 1)$ satisfy equation (3.5) (no solution exists).

Similar, more involved "boundary value" problems as in Example 3.6 might be constructed and investigated; these investigations are beyond the scope of this paper.

Summarizing the theorems and examples discussed so far we can conclude: Each element of a well-posed class (A, B, C) of initial value problems for partial difference equations defines a linear and causal system. If its coefficients a_β are constants, these systems are shift invariant and convolutional (see [6]).

4. NONLINEAR EQUATIONS

The method of proof of Theorems 2.1 and 2.7 offers the following generalization. Let us suppose we are given a map of B onto the set $\{1, 2, \ldots, M\}$, $M = \operatorname{card} B < \infty$, and a set of functions $F: \boldsymbol{C}^M \to \boldsymbol{C}$, $\alpha \in A$. Then we can consider the equation

$$F_\alpha(f(\alpha + \beta^1), \ldots, f(\alpha + \beta^M)) = x(\alpha)\,, \quad \alpha \in A\,.$$

We can extend our Theorem 2.1 also to this type of equations, provided the functions F are in some sense well-behaved. Indeed, it is not difficult to see, that the following theorem can be proved in exactly the same way as Theorem 2.1.

Theorem 4.1. *Let sets A, B satisfy conditions as in (2.1) (see the proof of Theorem 2.1) and let the function $F_\alpha: \boldsymbol{C}^M \to \boldsymbol{C}$ for any $\alpha \in A$ have the following property: for any vector $w \in \boldsymbol{C}^M$ there exists exactly one value y such that*

$$F_\alpha(w_1, w_2, \ldots, w_{i-1}, y, w_{i+1}, \ldots, w_M) = w_i\,,$$

where $\beta^i = \beta(\alpha)$. Then for any $x: A \to \boldsymbol{C}$ and $g: C \to \boldsymbol{C}$ there exists exactly one function $f: A + B \to \boldsymbol{C}$ such that

$$F_\alpha(f(\alpha + \beta^1), f(\alpha + \beta^2), \ldots, f(\alpha + \beta^M)) = x(\alpha)\,, \quad \alpha \in A$$

and $f(\gamma) = g(\gamma)$ for all $\gamma \in C$. Here, similarly as above,

$$C = (A + B) \setminus \bigcup_{\alpha \in A} \{\alpha + \beta(\alpha)\}\,.$$

A number of examples, where F satisfies the assumptions of the above theorem can be given.

Example 4.2. i) For

$$F_\alpha(w_1, w_2, \ldots, w_M) = \sum_{i=1}^{M} A_{\alpha,i} w_i^{p(\alpha,i)},$$

with $A_{\alpha,i} \in C$, $p(\alpha, i) \in N$, it is sufficient to assume $(A_{\alpha,i} \neq 0$, $p(\alpha, i)$ odd whenever $\beta(\alpha) = \beta^i)$.

ii) Formal discretization of a single nonlinear hyperbolic partial differential equation yields the following equation

$$\frac{1}{h}(f(\alpha + e_{n+1}) - f(\alpha)) + G(f(\alpha)) \sum_{i=1}^{n} \frac{1}{k_i}(f(\alpha + e_k) - f(\alpha)) = x(\alpha).$$

Here, G is a nonzero one-variable function, the set B consists of the point 0 and of the points e_k, $k = 1, 2, \ldots, n + 1$, which have all the coordinates except the k-th one equal to zero, the k-th coordinate being equal to one. A choice of the set A leads to the construction of the set C as in the previous cases, and subsequently, to the construction of a unique solution, provided $\beta(\alpha) \neq 0$.

Since only existence problems for nonlinear initial value problems are dealt with here, these examples are rather formal and we will not pursue this theme any further.

5. CONCLUSIONS

In this paper generalizations of difference equations have been dealt with, such that existence and uniqueness theorems could be formulated and proved. These theorems seem to be important in a rapidly growing area of multidimensional digital signal processing and, more generally, in the theory of multidimensional discrete systems. They may find applications also in numerical treatment of some partial differential equations by finite difference methods.

References

[1] *P. S. Alexandrov:* Introduction to general theory of sets and functions (Russian) Czech translation, Academia Praha, 1954.

[2] *N. K. Bose:* Applied Multidimensional Systems Theory. Van Nostrand, N.Y. 1982.

[3] *D. E. Dudgeon, R. M. Mercereau:* Multidimensional Digital Signal Processing, Prentice-Hall, Engelwood Cliffs, 1984.

[4] *A. O. Gelfond:* Finite difference calculus (in Russian). Nauka, Moscow, 1967.

[5] *T. S. Huang* (Editor): Two Dimensional Digital Signal Processing. Vol. I., Springer, Berlin, 1981.

[6] *B. Pondělíček:* On compositional and convolutional systems. Kybernetika (Prague), 17 (1981), No 4, pp. 277—286.

[7] *S. L. Sobolev:* Introduction to theory of cubature formulae (in Russian). Nauka, Moscow, 1974.

Souhrn

O ZOBECNĚNÝCH DIFERENČNÍCH ROVNICÍCH

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Článek formuluje věty o existenci a unicitě řešení lineárních diferenčních rovnic o n nezávislých proměnných. Kromě významných odlišností od případu jedné proměnné sleduje souvislosti zkoumaného předmětu s teorií n-dimensionálních digitálních systémů. Výsledků je také použito na některé nelineární diferenční rovnice.

Резюме

ОБ ОБОБЩЕННЫХ УРАВНЕНИЯХ В КОНЕЧНЫХ РАЗНОСТЯХ

MIROSLAV BOSÁK, JIŘÍ GREGOR

Формулируются теоремы существования и единственности решения для линейных уравнений в конечных разностях функций многих переменных. Рассматриваются особенности, которыми этот случай отличается от обыкновенных уравнений в конечных разностях, но внимание уделяется также вопросам теории n-мерных систем цифровой обработки сигналов. Результаты применяются также к некоторым нелинейным уравнениям в конечных разностях упомянутого типа.

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