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ON JOINT DISTRIBUTION IN QUANTUM LOGICS I. COMPATIBLE OBSERVABLES

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Summary. The notion of a joint distribution in σ-finite measures of observables of a quantum logic defined on some system of σ-independent Boolean sub-σ-algebras of a Boolean σ-algebra is studied. In the present first part of the paper the author studies a joint distribution of compatible observables. It is shown that it may exists, although a joint observable of compatible observables need not exist.

Keywords: Quantum logic, observable, compatibility, measure.

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This and the subsequent papers are devoted to the notion of a joint distribution of observables in a σ-finite measure on a quantum logic for a given system of observables defined on some collection of σ-independent Boolean sub-σ-algebras.

In this paper we study the problem of existence of a joint distribution for mutually compatible observables in a measure. It is shown that in this case the joint distribution in a measure may exists; however, a joint observable need not exist.

We postpone a detailed study of the existence of a joint distribution in a measure for noncompatible observables to a subsequent paper.

1. PRELIMINARIES

Assume that the set, $\mathscr{L}$, of all experimentally verifiable propositions of a physical system forms a *quantum logic*. So, according to [1], we suppose that $\mathscr{L}$ is a σ-lattice with the first and the last elements 0 and 1, respectively, with an orthocomplementation $\perp: a \to a^\perp$, $a, a^\perp \in \mathscr{L}$, which satisfies: (i) $(a^\perp)^\perp = a$ for any $a \in \mathscr{L}$; (ii) if $a < b$, then $b^\perp < a^\perp$; (iii) $a \vee a^\perp = 1$ for any $a \in \mathscr{L}$; (iv) if $a < b$, then $b = a \vee (b \wedge a^\perp)$ (the orthomodular law).

In particular, the notion of an orthomodular lattice (abbreviation OML), that is, a lattice $\mathscr{L}$ with (i)–(iv) above, is also of interest.

Two elements $a, b \in \mathscr{L}$ are (i) *orthogonal*, and we write $a \perp b$, if $a < b^{\perp}$; (ii) *compatible*, and we write $a \leftrightarrow b$, if there are three mutually orthogonal elements $a_1, b_1, c \in \mathscr{L}$ such that $a = a_1 \vee c$, $b = b_1 \vee c$. It is known that $a \leftrightarrow b$ iff $a = (a \wedge b) \vee (a \wedge b^{\perp})$.

Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be logics. A map $h: \mathscr{L}_1 \to \mathscr{L}_2$ is called a *σ-homomorphism* of $\mathscr{L}_1$ into $\mathscr{L}_2$ if (i) $h(1) = 1$; (ii) $h(a) \perp h(b)$ whenever $a \perp b$, $a, b \in \mathscr{L}_1$; (iii) $h(\bigvee_{i=1}^{\infty} a_i) = \bigvee_{i=1}^{\infty} h(a_i)$ for any $\{a_i\}_{i=1}^{\infty} \subset \mathscr{L}_1$, $a_i \perp a_j$, $i \neq j$. The kernel of a σ-homomorphism h is the set $Ker\ h := \{a \in \mathscr{L}_1 : h(a) = 0\}$.

An OML $\mathscr{L}$ (logic $\mathscr{L}$) is called a *Boolean algebra* (*Boolean σ-algebra*) if the distributive law holds on $\mathscr{L}$, that is, for all $a, b, c \in \mathscr{L}$, $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$. Due to [1, Corollary 6.15], the notion of a Boolean algebra(σ-algebra) coincides with that in [2]. The notions of sub OML, sublogic subalgebra and sub-σ-algebra of $\mathscr{L}$ are defined in a straightforward way, see [1, 2. 3], for instance.

Physical quantities of physical systems are identified with the observables of a quantum logic. Let $\mathscr{A}$ be a Boolean algebra and $\mathscr{L}$ an OML. We say that a map $x: \mathscr{A} \to \mathscr{L}$ is an *$\mathscr{A}$-observable* of $\mathscr{L}$ if (i) $x(1) = 1$; (ii) $x(E) \perp x(F)$ whenever $E \wedge F = 0$, $E, F \in \mathscr{A}$; (iii) $x(F \vee F) = x(E) \vee x(F)$ if $E \wedge F = 0$, $E, F \in \mathscr{A}$. If $\mathscr{A}$ is a Boolean σ-algebra and $\mathscr{L}$ is a quantum logic, then an $\mathscr{A}$-observable x of $\mathscr{L}$ is called an *$\mathscr{A}$-σ-observable* of $\mathscr{L}$ if $x(\bigvee_{i=1}^{\infty} E_i) = \bigvee_{i=1}^{\infty} x(E_i)$ for any $\{E_i\}_{i=1}^{\infty} \subset \mathscr{A}$, $E_i \wedge E_j = 0$, $i \neq j$. (Shortly observable, σ-observable, respectively, if $\mathscr{A}$ is clear from the context.)

The case which is of great importance for the quantum mechanics occurs when $\mathscr{A}$ is a Boolean (σ-)algebra of subsets of a set X, in particular, when $X = R^1$ and $\mathscr{A} = \mathscr{B}(R^1)$ is the Borel σ-algebra of subsets of the real line R^1.

The range of an $\mathscr{A}$-(σ)-observable x, $\mathscr{R}(x) := \{x(E): E \in \mathscr{A}\}$, is a Boolean sub-(σ-)algebra of $\mathscr{L}$. A Boolean σ-algebra $\mathscr{B}$ is *separable* if it is generated by countably many elements. $\mathscr{B}$ is a separable sub-σ-agebra of $\mathscr{L}$ iff there is a $\mathscr{B}(R^1)$-σ-observable x such that $\mathscr{B} = \mathscr{R}(x)$ [1, Lemma 6.16].

An $\mathscr{A}$-observable x and a $\mathscr{B}$-observable y are *compatible* if $x(E) \leftrightarrow y(F)$ for any $E \in \mathscr{A}$, $F \in \mathscr{B}$. It is known [1, Lemma 6.14, Corollary 6.15] that if x_t is an $\mathscr{A}_t$-(σ-)observable of $\mathscr{L}$ and $\{x_t : t \in T\}$ are mutually compatible observables, then there is a Boolean sub-(σ-)algebra of $\mathscr{L}$ containing all ranges $\mathscr{R}(x_t)$, $t \in T$.

We shall identify physical states with measures. A map $m: \mathscr{L} \to [0, \infty]$ is a *measure* if (i) $m(0) = 0$; (ii) $m(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} m(a_i)$ whenever $a_i \perp a_j$, $i \neq j$. A measure m is (i) finite if $m(1) < \infty$; (ii) a *state* if $m(1) = 1$; (iii) *σ-finite* if there is a sequence of mutually orthogonal elements of $\mathscr{L}$, $\{a_i\}_{i=1}^{\infty}$, such that $\bigvee_{i=1}^{\infty} a_i = 1$ and $m(a_i) < \infty$ for any $i \geqq 1$. In the sequel we shall use only measures with $m(1) \neq 0$. An observable

x is σ-finite with respect to m if there is a sequence $\{E_i\}_{i=1}^{\infty} \subset \mathscr{A}$ such that $E_i \wedge E_j = 0$ whenever $i \neq j$, $\bigvee_{i=1}^{\infty} E_i = 1$, and $m(x(E_i)) < \infty$, $i \geqq 1$.

We say that a system $\{\mathscr{A}_t: t \in T\}$ of Boolean sub-(σ-)algebras of a Boolean (σ-)algebra $\mathscr{A}$ is *independent* (*σ-independent*) if for any finite (countable) subset $\alpha \subset T$

$$\bigwedge_{t \in \alpha} A_t \neq 0 \tag{1.1}$$

for any $0 \neq A_t \in \mathscr{A}_t$ and any $t \in \alpha$.

For example, let $(X_t, \mathscr{S}_t)$ $t \in T$, be a measure space, that is, $\mathscr{S}_t$ is a (σ-) algebra of subsets of a set $X_t \neq \emptyset$. Denote by X the Cartesian product of all spaces X_t, i.e., the set of all $\omega = \{\omega_t: t \in T\}$, $\omega_t \in X_t$ for $t \in T$. Let π_t be the t-th projection function of X onto X_t, that is $\pi_t \omega = \omega_t$, $\omega \in X$. Let $\mathscr{S}_t^* := \{\pi_t^{-1}(A): A \in \mathscr{S}_t\}$, $t \in T$. Then $\mathscr{S}_t$ is (σ-)isomorphic to $\mathscr{S}_t^*$. The minimal sub-(σ-)algebra of X generated by all $\mathscr{S}_t^*$ is denoted by $\mathscr{S} = \prod_{t \in T} \mathscr{S}_t$, and the system $\{\mathscr{S}_t^*: t \in T\}$ of Boolean sub-(σ-)algebras of $\mathscr{S}$ is (σ-)independent [2].

Let $\{\mathscr{A}_t: t \in T\}$ be a system of (σ-)independent Boolean sub-(σ-)algebras of a Boolean (σ-)algebra $\mathscr{A}$. Denote by $\mathscr{D}$ the system of all Boolean rectangles $\bigwedge_{t \in \alpha} A_t$ defined for any $A_t \in \mathscr{A}_t$, $t \in \alpha$, and each finite $\alpha \subset T$. As in the Cartesian product of (σ-)algebras of subsets of X_t, one may verify that the minimal subalgebra $\mathscr{R}$ of $\mathscr{A}$, generated by all $\mathscr{A}_t$, $t \in T$, consists of all finite joins of orthogonal elements from $\mathscr{D}$. The minimal sub-σ-algebra of $\mathscr{A}$ generated by all sub-σ-algebras $\{\mathscr{A}_t: t \in T\}$ is denoted by $\prod_{t \in T} \mathscr{A}_t$.

2. JOINT DISTRIBUTION OF COMPATIBLE OBSERVABLES

One of important problems of the quantum logic theory is the determination of a joint distribution for noncompatible observables, as is indicated in [4, Problem VII]. Following Gudder [5] we give the following generalization of the notion of the joint distribution.

Definition. *Let m be a measure on a quantum logic $\mathscr{L}$. We say that* (i) *a finite system $x_1, \ldots, x_n$, where x_i is an $\mathscr{A}_i$-σ-observable of $\mathscr{L}$, $i = 1, \ldots, n$, and $\mathscr{A}_1, \ldots, \mathscr{A}_n$ are independent Boolean sub-σ-algebras of a Boolean σ-algebra $\mathscr{A}$, has a joint distribution in m if there is a measure μ on the minimal Boolean sub-σ algebra $\mathscr{A}_1 \times \ldots \times \mathscr{A}_n$ of $\mathscr{A}$ generated by $\mathscr{A}_1, \ldots, \mathscr{A}_n$ such that*

$$\mu(\bigwedge_{i=1}^{n} A_i) = m(\bigwedge_{i=1}^{n} x_i(A_i)) \tag{2.0}$$

for any $A_i \in \mathscr{A}_i$, $i = 1, \ldots, n$;

(ii) *an infinite system* $\{x_t: t \in T\}$, *where* x_t *is an* $\mathscr{A}_t$-σ-*observabls*, $t \in T$, *and* $\{\mathscr{A}_t: t \in T\}$ *are* σ-*independent Boolean sub*-σ-*algebras of a Boolean* σ-*algebra* $\mathscr{A}$, *has a joint distributitn in an* m *if* $\{x_t: t \in \alpha\}$ *has a joint distribution in* m *for any finite* $\alpha \subset T$.

S. P. Gudder introduced the notion of a joint distribution only for $\mathscr{B}(R^1)$-σ-observables and states. Necessary and sufficient conditions for the existence of a joint distribution for $\mathscr{B}(R^1)$-σ-observables in a given state may be found in [5–11]. The case of a σ-finite measure, including also a logic $\mathscr{L} = \mathscr{L}(H)$ of a separable Hilbert space H, is investigated in [12].

It is known [6, 7] that the existence of a joint distribution in a measure closely depends on mutually compatible σ-observables of some quantum logic. Therefore in this section we concentrate ourselves on the study of a joint distribution of mutually compatible observables.

Lemma 2.1. *If* x_i, $i \geqq 1$, *are mutually compatible* $\mathscr{B}(X)$-σ-*observables of a quantum logic* $\mathscr{L}$, *where* X *is a complete separable Banach space and* $\mathscr{B}(X)$ *is the Borel* σ-*algebra of subsets of* X, *then there is a unique* $\prod_{i=1}^{\infty} \mathscr{B}(X)$-$\sigma$-*observable* x of $\mathscr{L}$, *with*

$$x(\bigcap_{i\in\alpha} \pi_i^{-1}(E_i)) = \bigwedge_{i\in\alpha} x_i(E_i)$$

for any $E_i \in \mathscr{B}(X)$, $i \in \alpha$, *and any finite subset* α *of* $\{1, 2, \ldots\}$. *Here* π_i *denotes the* i-*th projection function from* $\prod_{i=1}^{\infty} X$ *onto* X.

Proof. According to P. Pták [13] there is a $\mathscr{B}(X)$-σ-observable z of $\mathscr{L}$ and Borel measurable functions f_n: $X \to X$ such that $x_n(A) = z(f^{-1}(A))$ for any $A \in \mathscr{B}(X)$. Define $f(t) = (f_1(t), f_2(t), \ldots)$: $X \to \prod_{i=1}^{\infty} X$. Then z: $B \to x(f^{-1}(B))$, $B \in \prod_{i=1}^{\infty} \mathscr{B}(X)$ is the desired σ-observable. Q.E.D.

Theorem 2.2. Let $\{\mathscr{A}_t: t \in T\}$ *be a system of* σ-*independent free Boolean sub*-σ-*algebras with countable generators of a Boolean* σ-*algebra* $\mathscr{A}$. *Let* x_t *be an* $\mathscr{A}_t$-σ-*observable of a logic* $\mathscr{L}$, $t \in T$. *If* $\{x_t : t \in T\}$ *are mutually compatible observables and at least one of them is* σ-*finite with respect to* m, *then* $\{x_t: t \in T\}$ *have a joint distribution in* m. *Moreover, there is a unique* σ-*finite measure* μ *on* $\prod_{t\in T} \mathscr{A}_t$ *with*

$$\mu(\bigwedge_{t\in\alpha} A_t) = m(\bigwedge_{t\in\alpha} x_t(A_t)) \tag{2.1}$$

for any $A_t \in \mathscr{A}_t$ *and any finite subset* $\emptyset \neq \alpha \subset T$.

Proof. (i) First of all we show that if x_t is an $\mathscr{A}_t$-observable of $\mathscr{L}$, $t \in T$, where $\{\mathscr{A}_t: t \in T\}$ is a system of independent Boolean subalgebras of a Boolean algebra $\mathscr{A}$, and $\{x_t: t \in T\}$ are mutually compatible, then there is a unique $\mathscr{R}$-observable x of $\mathscr{L}$ such that

$$x(\bigwedge_{t\in\alpha} A_t) = \bigwedge_{t\in\alpha} x_t(A_t) \tag{2.2}$$

for any $A_t \in \mathscr{A}_t$ and any finite subset $\alpha \subset T$. Here $\mathscr{R}$ denotes the minimal Boolean subalgebra of $\mathscr{A}$ containing all $\mathscr{A}_t$, $t \in T$.

Notice that any two Boolean rectangles $\bigwedge_{t\in\alpha} A_t$ and $\bigwedge_{s\in\beta} B_s$ can be assumed to have the same finite index set $\alpha \cup \beta$. Indeed, if we put $A_t^* = A_t$ if $t \in \alpha$, $A_t^* = 1$ if $t \in \beta - \alpha$, and $B_t^* = B_t$ if $t \in \beta - \alpha$, $B_t^* = 1$, $t \in \alpha$, then $\bigwedge_{t\in\alpha} A_t = \wedge \{A_t^*: t \in \alpha \cup \beta\}$, $\bigwedge_{s\in\beta} B_s = \wedge \{B_t^*: t \in \alpha \cup \beta\}$. Therefore (i) $\bigwedge_{t\in\alpha} A_t = 0$, $A_t \in \mathscr{A}_t$, $t \in \alpha$, iff at least one $A_t = 0$; (ii) $0 \neq \bigwedge_{t\in\alpha} A_t < \bigwedge_{t\in\alpha} B_t$ iff $A_t < B_t$ for any $t \in \alpha$; (iii) $0 \neq \bigwedge_{t\in\alpha} A_t = \bigwedge_{t\in\alpha} B_t$ iff $A_t = B_t$ for any $t \in \alpha$.

Hence, the map x defined via (2.2) is well defined on the set $\mathscr{D}$ of all Boolean rectangles. Using the remark on the form on the minimal subalgebra $\mathscr{R}$ containing all $\mathscr{A}_t$, $t \in T$, and the fact that there is a Boolean subalgebra of $\mathscr{L}$ containing all ranges $\mathscr{R}(x_t)$ [1], x may be uniquely extended to an $\mathscr{R}$-observable of $\mathscr{L}$. The uniqueness of x follows from (2.2).

(ii) Now we show that if x_t is an $\mathscr{A}_t$-σ-observable of a logic $\mathscr{L}$ and $\{\mathscr{A}_t: t \in T\}$ are σ-independent free Boolean sub-σ-algebras with countable generators of a Boolean σ-algebra $\mathscr{A}$ and $\{x_t: t \in T\}$ are mutually compatible σ-observables, then for an $\mathscr{R}$-observable x of $\mathscr{L}$ we have: if $A_n \in \mathscr{R}$, $n \geqq 1$, and $A = \bigvee_{n=1}^{\infty} A_n \in \mathscr{R}$, then

$$x(A) = \bigvee_{n=1}^{\infty} x(A_n)\,. \tag{2.3}$$

Since any free Boolean σ-algebra $\mathscr{A}_t$ with a countable generator is σ-isomorphic to $B(R_1)$ [16, p. 335], (2.3) follows from Lemma 2.1.

(iii) Let x be the $\mathscr{R}$-observable on $\mathscr{L}$ whose existence is guaranteed by the first part of the present proof. Let m be a measure on $\mathscr{L}$ fulfilling the conditions of Theorem 2.2. Then, due to (i) and (ii), $\mu(A) := m(x(A))$, $A \in \mathscr{R}$, is a σ-finite σ-additive measure on $\mathscr{R}$. Using the well known Carathéodory extension method concerning the extension of a σ-additive σ-finite set function defined on an algebra of subsets to a measure defined on the minimal σ-algebra generated by the algebra [17], we obtain the analogous result also for a Boolean subalgebra $\mathscr{R}$ and $\prod_{t\in T} \mathscr{A}_t$ [18]. It is clear that μ is the joint distribution of $\{x_t: t \in T\}$ in m, and the proof is complete.

Q.E.D.

The next result is a simple consequence of the Theorem 2.2 (see Preliminaries).

Corollary 2.2.1. *Let x_t be an $\mathscr{S}_t$-σ-observable of a quantum logic $\mathscr{L}$, $t \in T$, and let $x_t \leftrightarrow x_s$ for any $s, t \in T$, where $\mathscr{S}_t$ is Borel a σ-algebra of subsets of a complete* *separable metric space X_t. If at least one of x_t's is σ-finite with respect to m, then $\{x_t: t \in T\}$ has a joint distribution in m, and there is a unique σ-finite measure μ on $\prod_{t\in T} \mathscr{S}_t$ with*

$$\mu(\bigcap_{t\in\alpha} \pi_t^{-1}(E_t)) = m(\bigwedge_{t\in\alpha} x_t(E_t))$$

for any $E_t \in \mathscr{S}_t$, $t \in \alpha$, and any finite subset $\emptyset \neq \alpha \subset T$.

We note that for the $\mathscr{R}$-observable x of $\mathscr{L}$ with (2.2) there may exist no extension of x to a $\prod_{t\in T} \mathscr{A}_t$-$\sigma$-observable of $\mathscr{L}$. To establish this interesting fact, we need the following notions. Let $\mathscr{A}$ be a Boolean σ-algebra. A non-empty subset $\mathscr{J} \subset \mathscr{A}$ is said to be σ-ideal if (i) $A_n \in \mathscr{J}$, $n \geqq 1$, implies $\bigvee_{n=1}^{\infty} A_n \in \mathscr{J}$; (ii) if $A < B$ and $B \in \mathscr{J}$, then $A \in \mathscr{J}$. The factor σ-algebra, $\mathscr{A}/\mathscr{J}$, is the system of all $[A]_{\mathscr{J}} := \{B \in \mathscr{A}: B \wedge A^{\perp} \vee A \wedge B^{\perp} \in \mathscr{J}\}$, $A \in \mathscr{A}$. The Boolean operations in $\mathscr{A}/\mathscr{J}$ are defined via $[A]_{\mathscr{J}} \vee [B]_{\mathscr{J}} := [A \vee B]_{\mathscr{J}}$, $[A]_{\mathscr{J}}^{\perp} := [A^{\perp}]_{\mathscr{J}}$.

Example 1. There is a quantum logic $\mathscr{L}$ with a non-empty set of states (even with two-valued states), and with two compatible σ-observables $x_i: \mathscr{S}_i \to \mathscr{L}$, where $\mathscr{S}_i$ is a separable σ-algebra of subsets of a set X_i, $i = 1, 2$, such that there is no $\mathscr{S}_1 \times \mathscr{S}_2$-$\sigma$-observable x of $\mathscr{L}$ with

$$x(E \times F) = x_1(E) \wedge x_2(F), \quad E \in \mathscr{S}_1, F \in \mathscr{S}_2. \tag{2.4}$$

On the other hand, x_1 and x_2 have a joint distribution in any σ-finite measure m on $\mathscr{L}$.

Proof. Let C be some analytic subset of R^1 which is not a Borel set. Let $X_1 = R^1 - C$, $X_2 = R^1$ and $\mathscr{S}_1 := \mathscr{B}(R^1) \cap (R_1 - C) := \{B \cap C: B \in \mathscr{B}(R^1)\}$, $\mathscr{S}_2 := \mathscr{B}(R^1)$. It is clear that $\mathscr{S}_1$ and $\mathscr{S}_2$ are separable σ-algebras of subsets, i.e., they contain generators with countably many elements. Denote by $\mathscr{J}'$ the σ-ideal of the Borel σ-algebra $\mathscr{B}(R^2)$ of the real plane R^2 generated by all sets $B \times R^1$, where $B \in \mathscr{B}(R^1)$ and $B \subset C$. Let us put $\mathscr{L} = \mathscr{B}(R^2)/\mathscr{J}'$. The formulae

$$x_1(B \cap X_1) := [B \times R^1]_{\mathscr{J}'}, \quad B \in \mathscr{B}(R^1),$$
$$x_2(B) := [R^1 \times B]_{\mathscr{J}'}, \quad B \in \mathscr{B}(R^1),$$

determine two compatible σ-observables x_i: $\mathscr{S}_i \to \mathscr{L}$, $i = 1, 2$. Moreower, x_i is a σ-isomorphism of $\mathscr{S}_i$ into $\mathscr{L}$. As has been shown in [18, p. 17; 2, § 37, Example A], there is no $\mathscr{S}_1 \times \mathscr{S}_2$-$\sigma$-observable of $\mathscr{L}$ with (2.4).

Now we will prove the second part of the proposition. Define the σ-ideal $\mathscr{J}$ of $\mathscr{B}(R^2)$ as follows: $\mathscr{J} = \{A \in \mathscr{B}(R^2): A \subset C \times R^1\}$. It is obvious that $\mathscr{J}' \subset \mathscr{J}$. We show that $\mathscr{J}'$ is a proper subset of $\mathscr{J}$. If we had $\mathscr{J}' = \mathscr{J}$, then $\mathscr{B}(R^2)/\mathscr{J}$ would be σ-isomorphic to $\mathscr{S} = \mathscr{B}(R^2) \cap ((R^1 - C) \times R^1) := \{B \cap (R^2 - C) \times R^1: B \in \mathscr{B}(R^2)\}$ (a σ-isomorphism h of $\mathscr{B}(R^2)/\mathscr{J}$ onto $\mathscr{S}$ is defined by $h(B \cap ((R^1 - C) \times R^1)) = [B]_{\mathscr{J}}$ for any $B \in \mathscr{B}(R^2)$). Consequently, $\mathscr{L}$ possesses the strong σ-extension property (for definition see below or [2]) and, therefore, there is an x with (2.4) which contradicts the first part of the proof.

Now we define an $\mathscr{L}$-σ-observable z of a quantum logic $\mathscr{L}_1 := \mathscr{B}(R_2)/\mathscr{I}$ via

$$z([A]_{\mathscr{I}'}) = [A]_{\mathscr{I}}, \quad A \in \mathscr{B}(R^2).$$

The z is well defined because if $[A_1]_{\mathscr{I}'} = [A_2]_{\mathscr{I}'}$, then $A_1 \wedge A_2^\perp \vee A_2 \wedge A_1^\perp \in \mathscr{I}' \subset \subset \mathscr{I}$ and $[A_1]_{\mathscr{I}} = [A_2]_{\mathscr{I}}$.

Te logic $\mathscr{L}_1$ is σ-isomorphic to the σ-algebra of subsets, $\mathscr{B}(R^2) \cap ((R^1 - C) \times \times R^1)$, hence $\mathscr{L}_1$ has an order determining system of states (and also an order determining system of two-valued states). (We recall that a system $\mathscr{M}$ of states on a quantum logic is order determining if $m(a) \leqq m(b)$ for any $m \in \mathscr{M}$ iff $a < b$.)

Let m be a measure on $\mathscr{L}_1$, then $\bar{m}$: $a \to m(z(a))$, $a \in \mathscr{L}$, is a measure on $\mathscr{L}$.

Now let m be a σ-finite measure on $\mathscr{L}$ and let $\bigvee_{i=1}^{\infty} a_i = 1$, $a_i \perp a_j$ whenever $i \neq j$, $a_i \in \mathscr{L}$, $0 < m(a_i) < \infty$, $i \geqq 1$. Then $m_i(a) = m(a \wedge a_i)$, $a \in \mathscr{L}$, is a finite measure for any $i \geqq 1$. Using the result of Duchoň [15] we see that x_1, x_2 have a joint distribution in any m_i, $i \geqq 1$, and consequently, in $m = \sum_{i=1}^{\infty} m_i$.

Motivated by the above we say that mutually compatible σ-observables x_t: $\mathscr{A}_t \to \mathscr{L}$ of a quantum logic $\mathscr{L}$, $t \in T$, where $\{\mathscr{A}_t: t \in T\}$ is a system of σ-independent Boolean sub-σ-algebras of a Boolean σ-algebra $\mathscr{A}$, have a joint σ-observable if there is a $\prod_{t \in T} \mathscr{A}_t$ – σ-observable x of $\mathscr{L}$ with (2.2). Neither existence of joint-σ-observable nor joint distribution for compatible σ-observables in general case is not known to the author.

Lemma 2.1 determines an important class of compatible observables which has a joint σ-observable. According to [2], we say that a Boolean σ-algebra $\mathscr{A}'$ has the strong σ-extension property if, for every Boolean σ-algebra $\mathscr{A}$, every map f (from a set $\mathscr{G}$ σ-generating $\mathscr{A}$) into $\mathscr{A}'$ satisfying the implication

$$\text{(2.5)} \qquad \bigwedge_{i=1}^{\infty} E_i^{\varepsilon(i)} = 0\,, \quad \text{then} \quad \bigwedge_{i=1}^{\infty} f(E_i))^{\varepsilon(i)} = 0\,,$$

for every sequence $\{E_i\}_{i=1}^{\infty} \subset \mathscr{G}$ and for every function $\varepsilon(i) \in \{0, 1\}$ $i \geqq 1$, can be extended to a σ-homomorphism h from $\mathscr{A}$ into $\mathscr{A}'$; here $E^0 := E$, $E^1 := E$.

Theorem 2.3. *Let x_t: $\mathscr{A}_t \to \mathscr{L}$, $t \in T$, be compatible σ-observables of a quantum logic $\mathscr{L}$, where $\{\mathscr{A}_t: t \in T\}$ is a system of σ-independent Boolean sub-σ-algebras of a Boolean σ-algebra $\mathscr{A}$, and let the minimal sub-σ-algebra of $\mathscr{A}$ generated by all ranges $\mathscr{R}(x_t)$, $t \in T$, have the strong σ-extension property (in particular, it is σ-isomorphic to some σ-algebra of subsets). Then $\{x_t: t \in T\}$ has a joint σ-observable of $\mathscr{L}$.*

Proof. It follows immediately from [2, Theorem 37.1]). Q.E.D.

In the frame of the study of a joint σ-observable of compatible observables, in particular, in connection with Lemma 2.1, it may be interesting to note that P. Pták [13] found the example of a quantum logic with two compatible $\mathscr{B}(X)$-σ-observables x and y such that the equalities $x = z \circ f^{-1}$, $y = z \circ g^{-1}$ do not simultaneously hold for any two Borel mappings $f, g: X \to X$ and any $\mathscr{B}(X)$-σ-observable z of $\mathscr{L}$. Here X is a Banach space of non-measurable cardinality, $\mathscr{B}(X)$ is its Borel σ-algebra and $\mathscr{L} = \mathscr{B}(X) \times \mathscr{B}(X)$. However, in this case there is a joint σ-observable of x and y, because x and y are induced by point transformations $T_i: X \times X \to X$ such that $x = T_1^{-1}$, $y = T_2^{-1}$.

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Súhrn

O ZDRUŽENOM ROZDELENÍ V KVANTOVÝCH LOGIKÁCH I. KOMPATIBILNÉ POZOROVATEĽNÉ

Anatolij Dvurečenskij

Študuje sa združené rozdelenie v σ-konečných mierach pre pozorovateľné kvantovej logiky, definovaných na niektorých systémoch σ-nezávislých voľných Booleových pod-σ-algebrách so spočítateľnými generátormi Booleovej σ-algebry. V tejto prvej časti práce sa študuje združené rozdelenie kompatibilných pozorovateľných. Dokazuje sa, že ono môže existovať, hoci združená pozorovateľná kompatibilných pozorovateľných môže i neexistovať.

Резюме

О СОВМЕСТНОМ РАСПРЕДЕЛЕНИИ В КВАНТОВЫХ ЛОГИКАХ. I. КОМПАТИБИЛЬНЫЕ НАБЛЮДАЕМЫЕ

Anatolij Dvurečenskij

Изучается понятие совместного распределения в σ-конечных мерах для наблюдаемых квантовой логики, определенных на некоторой системе σ-независимых булевых σ-подалгебр булевой σ-алгебры. В настоящей первой части заметки мы изучаем совместное распределение компатибильных наблюдаемых. Показано, что оно может существовать, хотя совместная наблюдаемая может и не существовать.

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