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Aplikace matematiky, Vol. 32 (1987), No. 6, 436–450

Persistent URL: `http://dml.cz/dmlcz/104275`

32 (1987) APLIKACE MATEMATIKY No. 6, 436—450

ON JOINT DISTRIBUTION IN QUANTUM LOGICS II. NONCOMPATIBLE OBSERVABLES

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(Received February 24, 1986)

Summary. This paper is a continuation of the first part under the same title. The author studies a joint distribution in σ-finite measures for noncompatible observables of a quantum logic defined on some system of σ-independent Boolean sub-σ-algebras of a Boolean σ-algebra. We present some necessary and sufficient conditions for the existence of a joint distribution. In particular, it is shown that an arbitrary system of observables has a joint distribution in a measure iff it may be embedded into a system of compatible observables of some quantum logic. The methods used are different from those developed for finite measures. Finally, the author deals with the connection between the existence of a joint distribution and the existence of a commutator of observables, and the quanrum logic of a nonseparable Hilbert space is mentioned.

Keywords: Quantum logic, observable, measure, commutator, joint distribution.

AMS Classification: 03G12, 81B10.

This paper is a continuation of the first part under the same title, hereafter referred to as [I]. Sections, theorems and formulae are numbered consecutively, starting with Section 3. References [1–18] are listed at the end of [I].

3. JOINT DISTRIBUTION OF NONCOMPATIBLE OBSERVABLES

In the present section we concentrate ourselves on the main aim of this paper. We shall study the problem of existence of noncompatible σ-observables. We give new results which are valid also for measures on $\mathscr{L}$ with infinite values, and which generalize the known results for states. We note that the methods developed in [6–11, 19, 20] are not applicable to our case.

The existence of a joint distribution closely depends on the concept of a commutator Let $\mathscr{L}$ be an OML. For a finite subset $F = \{a_1, \ldots, a_n\}$ of $\mathscr{L}$ let us put, following L. Beran [21],

$$\text{com}\, F := \bigvee_{j_1 \ldots j_n = 0}^{1} \bigwedge_{i=1}^{n} a_i^{j_i} \tag{3.1}$$

where $a^0 := a^\perp$, $a^1 := a$. The element com F is called the *commutator* of a finite set $F \subset \mathscr{L}$. For a two-element set $F = \{a, b\}$, com F has been defined by Marsden [22]. We recall that independently of [21] the commutator of F has been used in [7] to show that the so-called question observables $q_{a_1}, \ldots, q_{a_n}$ have a joint distribution in a state m iff $m(\text{com } F) = 1$ (here q_a is a $\mathscr{B}(R^1)$-σ-observable with $q_a(\{0\}) = a^\perp(\{1\}) = \text{a})$. In [21], the elements $\text{com}^\perp F$ and com F are called the upper and the lower commutator, respectively, of F. It is clear that $a_1, \ldots, a_n \in \mathscr{L}$ are mutually compatible iff com $F = 1$, where $F = \{a_1, \ldots, a_n\}$.

Now, for any M, $M \subset \mathscr{L}$, put

$$\text{com } M = \bigwedge\{\text{com } F : F \text{ is a finite subset of } M\}\,, \tag{3.2}$$

if the element on the right-hand side of (3.2) exists in $\mathscr{L}$. By definition we put com $\emptyset = 1$. The commutator com M of M has been introduced for the first time in [6] for the study of joint distributions.

The following notion has been defined in [9]. We say that a subset M of $\mathscr{L}$ is partially compatible with respect to a, $a \in \mathscr{L}$, if (i) $a \leftrightarrow b$ for any $b \in M$; (ii) $\{b \wedge a: b \in M\}$ is a set of mutually compatible elements of $\mathscr{L}$. It is known [11] that if $a = \text{com } M$ exists, then M is partially compatible with respect to a.

Let $\{a_s: s \in S\}$ be an indexed set of elements of $\mathscr{L}$. The element $a \in \mathscr{L}$ is said to be *countably obtainable* over $\{a_s: s \in S\}$ [10, 8] if $a = \bigwedge_{s\in S} a_s$, and if there is a countable subset $S_0 \subset S$ with $a = \bigwedge_{s\in S_0} a_s$. From [10, Prop. 2.3] if follows that if there is an at most countable subset $N \subset M$ of a quantum logic $\mathscr{L}$ which generates the minimal sublogic $\mathscr{L}_0$ of $\mathscr{L}$ containing M, then com M exists and is countably obtainable over {com F: F is a finite subset of M}.

The following result, due to W. Puguntke [23], is of particular interest for the present study: There is an OML $\mathscr{L}$ and $M \subset \mathscr{L}$ for which the commutator of M does not exist in $\mathscr{L}$.

Let x_i be an $\mathscr{A}_i$-observable of $\mathscr{L}$, $i = 1, \ldots, n$. Define

$$a(E_1, \ldots, E_n) = \bigvee_{j_1 \ldots j_n = 0}^{1} \bigwedge_{i=1}^{n} x_i(E_i^{j_i})\,, \tag{3.3}$$

where $E_i \in \mathscr{A}_i$, $i = 1, \ldots, n$. Then, due to [10], for $M = \bigcup_{i=1}^{n} \mathscr{R}(x_i)$, where x_i is a $\mathscr{B}(R^1)$-σ-observable, $i = 1, \ldots, n \leqq \infty$, the commutator com M exists in the quantum logic $\mathscr{L}$ and it is countably obtainable over {com F: F finite subset of M}, and, moreover, com M is countably obtainable over $\{a(E_1, \ldots, E_n): E_i \in \mathscr{B}(R^1), i = 1, \ldots, n\}$. If there is $a_0 = \text{com}\,(\bigcup_{t\in T} \mathscr{R}(x_t))$, we call a_0 the *commutator of the* σ-*observables* $\{x_t: t \in T\}$.

The following result has been proved in [12].

Theorem 3.1. *Let $x_1, \ldots, x_n$ be $\mathscr{B}(R^1)$-σ-observables of a quantum logic $\mathscr{L}$ and* *let m be a measure on* $\mathscr{L}$. *Let us denote* $a_0 = \text{com}\,(\bigcup_{i=1}^{n} R(x_i))$. *If*

$$(3.4) \qquad m(a_0^{\perp}) = 0\,,$$

then there is a joint distribution in m. If at least one x_i *is* σ*-finite with respect to m, then the joint distribution is unique.*

If $x_1, \ldots, x_n$ *have a joint distribution in m and at least one* x_i *is* σ*-finite with respect to m, then* (3.4) *holds.*

Moreover, maps $x_{i0}\colon E \to x_i(E) \wedge a_0$, $E \in \mathscr{B}(R^1)$, are mutually compatible σ-observables of a quantum logic $\mathscr{L}(0, a_0) := \{b \in \mathscr{L}\colon b < a_0\}$ (here an orthocomplementation "$'$" is defined as $b' = b^{\perp} \wedge a_0$, $b < a_0$).

It is known [7, 8] that $\mathscr{B}(R^1)$-σ-observables $x_1, \ldots, x_n$ have a joint distribution in a state m (finite measure, too) iff

$$(3.5) \qquad m(a^{\perp}(E_1, \ldots, E_n)) = 0$$

for any $E_1, \ldots, E_n \in \mathscr{B}(R^1)$. This is equivalent to the condition [7, 19, 20]

$$(3.6) \qquad m(\bigwedge_{j=1}^{n} x_j(E_{j1} \cup E_{j2})) = \sum_{k_1 \ldots k_n = 1}^{2} m(\bigwedge_{j=1}^{n} x_j(E_{jk_j}))$$

for any $E_{j1} \cap E_{j2} = \emptyset$, $E_{j1}, E_{j2} \in \mathscr{B}(R^1)$, $j = 1, \ldots, n$.

For measures with infinite values this equivalence has been proved only in particular cases [12]: (3.5) for measures with carriers, and (3.6) only on a σ-continuous logic. Below we will prove the equivalence of (3.5) with (3.6) and with the existence of a joint distribution in measures attaining even infinite values.

In the following we shall suppose that $\mathscr{A}_1, \ldots, \mathscr{A}_n$ are independent Boolean sub-σ-algebras of a Boolean algebra $\mathscr{A}$ and x_i is an $\mathscr{A}_i$-σ-observable of a logic $\mathscr{L}$, $i = 1, \ldots, n$. A decomposition of 1 in a logic $\mathscr{L}$ is a system $\{a_i\} \subset \mathscr{L}$ such that $a_i \perp a_j$ whenever $i \neq j$, $\bigvee_i a_i = 1$.

Lemma 3.2. *Let* $h_i\colon \mathscr{B}_i \to \mathscr{A}_i$ *be a* σ*-homomorphism of a Boolean sub-*σ*-algebra* $\mathscr{B}_i$ *of* $\mathscr{A}_i$, $i = 1, \ldots, n$. *Then* $x_1, \ldots, x_n$ *have a joint distribution in a measure m iff* $x_1 \circ h_1, \ldots, x_n \circ h_n$ *have a joint distribution for any* σ*-homomorphism* h_i *and any Boolean sub-*σ*-algebra* $\mathscr{B}_i$ *of* $\mathscr{A}_i$, $i = 1, \ldots, n$.

Proof. It is evident. Q.E.D.

Lemma 3.3. *Let* $x_1, \ldots, x_x$ *have a joint distribution in a measure m. Then*

$$(3.7) \qquad m(\bigwedge_{i=1}^{n} x_i(E_i) \wedge \bigwedge_{k=1}^{K} a(E_1^k, \ldots, E_n^k)) = m(\bigwedge_{i=1}^{n} x_i(E_i))$$

for any $E_i, E_i^k \in \mathscr{A}_i$, $i = 1, \ldots, n$, $k = 1, \ldots, K$, *where K may be an integer or* $+\infty$.

Proof. It is the same as that of Lemma 2.2 in [12]. Q.E.D.

Theorem 3.4. *Let at least one of $x_1, \ldots, x_n$ be σ-finite with respect to m. Let the commutator a_0 of $x_1, \ldots, x_n$ exist and let it be countably obtainable. If $x_1, \ldots, x_n$ have a joint distribution in m, then*

$$m(a_0^\perp) = 0 \,. \tag{3.8}$$

The converse is true, e.g., if $\mathscr{A}$ are σ-isomorphic to $B(R^1)$.

Proof. From [6, 10] it follows that if a_0 is countably obtainable over $\{\text{com}\, F$: F is a finite subset of $\bigcup_{i=1}^{n} \mathscr{R}(x_i)\}$ then a_0 is countably obtained over $\{a(E_1, \ldots, E_n)$: $E_i \in \mathscr{A}_i$, $i = 1, \ldots, n\}$, and vice versa.

Let $x_1, \ldots, x_n$ have a joint distribution in m, and let x_1 be σ-finite with respect to m. Using (3.7) we can establish that if $m(x_1(E)) < \infty$ for some $E \in \mathscr{A}_1$, then $m(x_1(E) \wedge a_0^\perp) = 0$. Therefore, if $\{E_k\}_{k=1}^{\infty} \subset \mathscr{A}_1$ is a countable decomposition of 1 with $m(x_1(E_k)) < \infty$, $k \geqq 1$, then

$$m(a_0^\perp) = m(a_0^\perp \wedge x_1(\bigvee_{k=1}^{\infty} E_k)) = \sum_{k=1}^{\infty} m(a_0^\perp \wedge x_1(E_k)) = 0 \,,$$

where we use the property of the partial compatibility of a_0.

Let (3.8) hold. Then putting $\overline{m} := m \mid \mathscr{L}(0, a_0)$, we have

$$m(\bigwedge_{i=1}^{n} x_i(E_i)) = m(\bigwedge_{i=1}^{n} x_i(E_i) \wedge a_0) = \overline{m}(\bigwedge_{i=1}^{n} x_{i0}(E_i)) \,,$$

where $x_{i0}(E) := x_i(E) \wedge a_0$ are $\mathscr{A}_i$-σ-observables of $\mathscr{L}(0, a_0)$, $i = 1, \ldots, n$, which are mutually compatible. Refering to Theorem 2.2 of [I] we see that $x_1, \ldots, x_n$ have a joint distribution in m. Q.E.D.

We note that if x_t is an $\mathscr{A}_t$-σ-observable of the quantum logic $\mathscr{L}$, $t \in T$, where $\{\mathscr{A}_t: t \in T\}$ is a system of σ-independent Boolean sub-σ-algebras of a Boolean σ-algebra $\mathscr{A}$, and if $\{x_t: t \in T\}$ have a countably obtainable commutator a_0, then Theorem 3.4 is valid for $\{x_t: t \in T\}$, too.

Lemma 3.5. *Let $x_1, \ldots, x_n$ have a joint distribution in m and let x_1, say, be σ-finite with respect to m. Then*

$$\text{(i)} \quad m(a^\perp(E_1, \ldots, E_n)) = 0 \tag{3.9}$$

for any $E_i \in \mathscr{A}_i$, $i = 1, \ldots, n$;

$$\text{(ii)} \quad m(\text{com}^\perp F) = 0 \tag{3.10}$$

for any finite subset $F \subset \bigcup_{i=1}^{n} \mathscr{R}(x_i)$.

Proof. Any finite $\emptyset \neq F \subset \bigcup_{i=1}^{n} \mathscr{R}(x_i)$ generates a finite decomposition $\mathscr{K}_i$ of 1 in each $\mathscr{A}_i$ in the following manner. If there is no element of $\mathscr{A}_i$ in F, then we put $\mathscr{K}_i = \{0, 1\}$. If $\mathscr{A}_i \cap F = \{E_1, \ldots, E_k\}$, then we put $\mathscr{K}_i = \{E_1^{j_1} \wedge \ldots \wedge E_k^{j_k}: j_s \in \{0, 1\},$

$s = 1, \ldots, k\}$. Let $\{F_j\}_{j=1}^{\infty}$ be a countable decomposition of 1 in $\mathscr{A}_1$ with $m(x_1(F_j)) < < \infty$, $j \geqq 1$. Denote by $\mathscr{B}_1$ the minimal sub-σ-algebra of $\mathscr{A}_1$ containing $\{F_j \cap A: A \in \mathscr{K}_1, j \geqq 1\}$, and, for $2 \leqq i \leqq n$, denote by $\mathscr{B}_i$ the minimal sub-σ-algebra of $\mathscr{A}_i$ generated by $\mathscr{K}_i$. Then $\bar{x}_i := x_i \mid \mathscr{B}_i$ are a $\mathscr{B}_i$-σ-observables of $\mathscr{L}$ and, due to Lemma 3.2, they have a joint distribution in m. Since the subset $\mathscr{M} = \{\text{com } G{:}G$ is a finite subset of $\bigcup_{i=1}^{n} \mathscr{R}(\bar{x}_i)\}$ of $\mathscr{L}$ has at most countably many elements, there is a commutator a_0 of $\bar{x}_1, \ldots, \bar{x}_n$ and, moreover, a_0 is countably obtainable. Because $\bar{x}_1$ is σ-finite with respect to m, by Theorem 3.4 we have $m(a_0^\perp) = 0$. Since com $F \in \mathscr{M}$ and $a_0 < \text{com } F$, we obtain (3.10).

To prove (3.9) it is sufficient to put $F = \{x_1(E_1), \ldots, x_n(E_n)\}$.

The next three technical lemmas will be useful in the following. If $F = \{c_1, \ldots, c_i\} \subset \subset \mathscr{L}$, $\mathscr{L}$ is an OML, then we put $\text{com}\,(c_1, \ldots, c_i) := \text{com } F$.

Lemma 3.6. *Let $\mathscr{L}$ be an OML. If $a_1, \ldots, a_k$ are mutually orthogonal elements of $\mathscr{L}$, then for any $b_1, \ldots, b_n$,*

$$(3.11) \qquad \Big(\bigvee_{i=1}^{k} a_i\Big) \wedge \text{com}^\perp (a_1, \ldots, a_k, b_1, \ldots, b_n) = \bigvee_{i=1}^{k} (a_i \wedge \text{com}^\perp (a_i, b_1, \ldots, b_n)) .$$

Proof. Lemma 2.1 of [8] implies that

$$\text{com}\,(a_1, \ldots, a_k, b_1, \ldots, b_n) = \bigvee_{d \in D^n} (a_1 \wedge b^d \vee \ldots \vee a_k \wedge b^d \vee (a_1 \vee \ldots \vee a_k)^\perp \wedge b^d)$$

where $D = \{0, 1\}$, $b^d := b_1^{d_1} \wedge \ldots \wedge b_n^{d_n}$, $d = (d_1, \ldots, d_n) \in D^n$. Calculate

$$a := \Big(\bigvee_{i=1}^{k} a_i\Big) \wedge \text{com}^\perp (a_1, \ldots, a_k, b_1, \ldots, b_n) =$$

$$= \Big(\bigvee_{i=1}^{k} a_i\Big) \wedge \bigwedge_{d \in D^n} ((a_1^\perp \vee b^{d\perp}) \wedge \ldots \wedge (a_k^\perp \vee b^{d\perp}) \wedge (a_1 \vee \ldots \vee a_k \vee b^{d\perp}) .$$

For all $i, j = 1, \ldots, n$, and each $d \in D^n$, $a_i \leftrightarrow a_j^\perp \vee b^{d\perp}$, we have $a_i \leftrightarrow (a_1 \vee \ldots \ldots \vee a_k \vee b^{d\perp})$. Hence, according to [1, Lemma 6.10], we may apply the distributive law. Consequently,

$$a = \bigvee_{i=1}^{k} (a_i \wedge \bigwedge_{d \in D^n} (a_1^\perp \vee b^{d\perp}) \wedge \ldots \wedge (a_k^\perp \vee b^{d\perp})) = \bigvee_{i=1}^{k} (a_i \wedge \bigwedge_{d \in D^n} (a_i^\perp \vee b^{d\perp})) =$$

$$= \bigvee_{i=1}^{k} (a_i \wedge \text{com}^\perp (a_i, b_1, \ldots, b_n)) . \qquad \text{Q.E.D.}$$

Lemma 3.7. *Let $\mathscr{L}$ be an OML. If for $F = \{a, b_1, \ldots, b_n\}$ we have $m(\text{com}^\perp F) = 0$, then*

$$(3.12) \qquad m(a) = \sum_{d \in D^n} m(a \wedge b^d) = m(a \wedge \text{com } F) .$$

Proof. $m(a) = m(a \wedge \text{com } F) + m(a \wedge \text{com}^\perp F) = m(a \wedge \text{com } F) = = m(\bigvee_{d \in D^n} a \wedge b^d)$,

where we use the notation from Lemma 3.6. Q.E.D.

Lemma 3.8. *Let $\mathscr{L}$ be an OML.* Let $F = \{a_1, \ldots, a_k, b_1, \ldots, b_n\} \subset \mathscr{L}$, *where $a_1, \ldots, a_k$ are mutually orthogonal elements. If $m(\mathrm{com}^{\perp} F) = 0$, then*

$$\mathrm{m}((\bigvee_{i=1}^{k} a_i) \wedge \bigwedge_{j=1}^{n} b_j) = \sum_{i=1}^{k} m(a_i \wedge \bigwedge_{j=1}^{n} b_j) . \tag{3.13}$$

Proof. Using [8, Lemma 2.1] and the distributive law we obtain

$$m((\bigvee_{i=1}^{k} a_i) \wedge \bigwedge_{j=1}^{n} b_j) = m((\bigvee_{i=1}^{k} a_i) \wedge \bigwedge_{j=1}^{n} b_j \wedge \mathrm{com}\, F) =$$

$$= m((\bigvee_{i=1}^{k} a_i) \wedge \bigwedge_{j=1}^{n} b_j \wedge \bigvee_{d \in D^n} ((a_1 \wedge b^d) \vee \ldots \vee (a_k \wedge b^d) \vee$$

$$\vee (a_1 \vee \ldots \vee a_k)^{\perp} \wedge b^d)) = m(\bigvee_{i=1}^{k} (a_i \wedge \bigwedge_{j=1}^{n} b_j)) . \qquad \text{Q.E.D.}$$

The following notions are needed for the main result of this section. Let $\mathscr{L}$ be an OML. A non-empty subset $\mathscr{J} \subset \mathscr{L}$ is said to be a *p*-ideal [22, 3] if (i) if $a, b \in \mathscr{J}$, then $a \vee b \in \mathscr{J}$; (ii) if $b \in \mathscr{J}$, $a < b$, then $a \in \mathscr{J}$; (iii) $b \in \mathscr{J}$ implies $(b \vee a^{\perp}) \wedge \wedge a \in \mathscr{J}$ for all $a \in \mathscr{L}$. If instead of (i) we suppose (i)′ if $\{a_n\}_{n=1}^{\infty} \subset \mathscr{J}$, then $\bigvee_{n=1}^{\infty} a_n \in \mathscr{J}$, is called a *σ-p-ideal* of $\mathscr{L}$. If h is a σ-homomorphism of a quantum logic $\mathscr{L}$ into a quantum logic $\mathscr{L}_1$, then *Ker h* is a σ-*p*-ideal of $\mathscr{L}$. We write $a \sim b$ iff $(a \vee b) \wedge \wedge (a \wedge b)^{\perp} \in \mathscr{J}$, $a, b \in \mathscr{L}$. Then the relation "$\sim$" is an equivalence relation on $\mathscr{L}$, and (i) if $a \sim b$, then $a^{\perp} \sim b^{\perp}$; (ii) $a_1 \sim b_1$, $a_2 \sim b_2$ imply $a_1 \vee b_1 \sim a_2 \vee b_2$. Denote by $\mathscr{L}/\mathscr{J}$ the factor OML defined via $\mathscr{L}/\mathscr{J} = \{[a]_{\mathscr{J}}, a \in \mathscr{L}\}$, where $[a]_{\mathscr{J}} := := \{b \in \mathscr{L}: b \sim a\}$ and $[a]_{\mathscr{J}}^{\perp} := [a^{\perp}]_{\mathscr{J}}$, $[a]_{\mathscr{J}} \vee [b]_{\mathscr{J}} := [a \vee b]_{\mathscr{J}}$. The map $h_{\mathscr{J}}: \mathscr{L} \to \mathscr{L}/\mathscr{J}$ which assigns $[a]_{\mathscr{J}}$ to any $a \in \mathscr{U}$ is a homomorphism of $\mathscr{L}$ onto $\mathscr{L}/\mathscr{J}$.

Finally, we present a theorem which generalizes all the known conditions concerning the existence of a joint distribution in a measure in two main aspects; (i) measures may attain also the infinite values, (ii) the conditions do not depend on the existence of the commutator of a given system $\{x_t: t \in T\}$ of σ-observables.

Theorem 3.9. *Let $x_t: \mathscr{A}_t \to \mathscr{L}$ be a σ-observable of a quantum logic $\mathscr{L}$, $t \in T$, where $\{\mathscr{A}_t: t \in T\}$ is a system of σ-independent free Boolean sub-σ-algebras with countable generators of a Boolen σ-algebra $\mathscr{A}$. Let at least one σ-observable, say x_{t_0}, be σ-finite with respect to a measure m. Then the following conditions are equivalent*:

(i) *$\{x_t: t \in T\}$ have a joint distribution in m*;

(3.14) (ii) $m(\mathrm{com}^{\perp}(\{x_t(A_t): t \in \alpha\})) = 0$

for any $A_t \in \mathscr{A}_t$, $t \in \alpha$, and any finite $\emptyset \neq \alpha \subset T$;

(3.15) (iii) $m(\mathrm{com}^\perp F) = 0$

for any finite subset F of $\bigcup \{\mathscr{R}(x_t): t \in T\}$;

(3.16) (iv) $m(\bigwedge_{t\in\alpha} x_t(A_{1t} \vee A_{2t})) = \sum_{\substack{i_t=1\\ t\in\alpha}}^{2} m(\bigwedge_{t\in\alpha} x_t(A_{i_t t}))$

for any $A_{1t} \wedge A_{2t} = 0$, $A_{1t}, A_{2t} \in \mathscr{A}_t$, $t \in \alpha$, and any finite $\emptyset \neq \alpha \subset T$;

(3.17) (v) $m(\bigwedge_{t\in\alpha} x_t(\bigvee_{k=1}^{\infty} A_{kt})) = \sum_{\substack{k_t=1\\ t\in\alpha}}^{\infty} m(\bigwedge_{t\in\alpha} x_t(A_{k_t t}))$

for any $\{A_{kt}\}_{k=1}^{\infty} \subset \mathscr{A}_t$, $A_{it} \wedge A_{jt} = 0$, $i \neq j$, $t \in \alpha$, and any finite $\emptyset \neq \alpha \subset T$;

(vi) *there exists a Boolean σ-algebra $\mathscr{B}$, $\mathscr{B} \neq \{0\}$, and a σ-homomorphism h of the minimal sublogic $\mathscr{L}_0$ of $\mathscr{L}$ containing all $\mathscr{R}(x_t)$ onto $\mathscr{B}$ such that $m(a) = 0$ for all $a \in \mathrm{Ker}\, h$;*

(vii) *there is a quantum logic $\mathscr{L}_1 \neq \{0\}$ and a σ-homomorphism h of $\mathscr{L}_0$ onto $\mathscr{L}_1$ such that $\{h \circ x_t: t \in T\}$ are mutually compatible σ-observables of L_1 and $m(a) = 0$ for all $a \in \mathrm{Ker}\, h$;*

(viii) *there is a (unique) measure μ on $\prod_{t\in T} \mathscr{A}_t$ such that*

$$\mu(\bigwedge_{t\in\alpha} A_t) = m(\bigwedge_{t\in\alpha} x_t(A_t)) \tag{3.18}$$

for any $A_t \in \mathscr{A}_t$, $t \in \alpha$, and any finite $\emptyset \neq \alpha \subset T$.

Proof. We shall prove the following implications: (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ ⇒ (v) ⇒ (i), and (i) ⇒ (vi) ⇒ (vii) ⇒ (viii) ⇒ (i).

(i) ⇒ (ii). Let a non-empty finite subset $\alpha \subset T$ be given. If $t_0 \in \alpha$, then we apply Lemma 3.5 and (3.9). If $t_0 \notin \alpha$, we change α to $\alpha \cup \{t_0\}$ and use the monotonicity of m.

(ii) ⇒ (iii). Let a finite $F \subset \bigcup\{\mathscr{R}(x_t): t \in T\}$ be given. Then there is a finite subset $\alpha \subset T$ with diverse indices $t_1, \ldots, t_n$. Analogously as in the proof of Lemma 3.5, for each $i = 1, \ldots, n$, F generates a finite decomposition $\mathscr{K}_i = \{A_1^i, \ldots, A_{k_i}^i\} \subset \mathscr{A}_{t_i}$ of 1. Theorem 2.2 of [8] yields $\mathrm{com}\, F = \mathrm{com}\,(\bigcup_{i=1}^{n} \mathscr{K}_i)$ and a repeated application of (3.11) gives

$$0 \leqq m(\mathrm{com}^\perp F) = m(1 \wedge \mathrm{com}^\perp (\bigcup_{i=1}^{n} \mathscr{K}_i)) =$$

$$= m((\bigvee_{i_1=1}^{k_1} x_{t_1}(A_{i_1}^1)) \wedge \mathrm{com}^\perp (\bigcup_{i=1}^{n} \mathscr{K}_i)) = \sum_{i_1=1}^{k_1} m(x_{t_1}(A_{i_1}^1) \wedge \mathrm{com}^\perp(\{x_{t_1}(A_{i_1}^1)\} \cup \bigcup_{i=2}^{n} \mathscr{K}_i)) \leqq$$

$$\leqq \sum_{i_1=1}^{k_1} m(\mathrm{com}^\perp (\{x_{t_1}(A_{i_1}^1)\} \cup \bigcup_{i=2}^{n} \mathscr{K}_i)) \leqq \ldots \leqq \sum_{i_1=1}^{k_1} \ldots$$

$$\ldots \sum_{i_n=1}^{k_n} m(\mathrm{com}^\perp (x_{t_1}(A_{i_1}^1), \ldots, x_{t_n}(A_{i_n}^n))) = 0\,.$$

(iii) ⇒ (iv). Let a non-empty set $\alpha = \{t_1, \ldots, t_n\} \subset T$ be given. Let $A_{1t_i} \wedge A_{2t_i} = 0$, $A_{1t_i}, A_{2t_i} \in \mathscr{A}_{t_i}$, $i = 1, \ldots, n$. Define $a_k = x_{t_1}(A_{kt_1})$, $k = 1, 2$, $b_j = x_{t_j}(A_{1t_j} \vee A_{2t_j})$, $j = 2, \ldots, n$. Repeatedly applying Lemma 3.8 we see that (3.16) is true.

(iv) ⇒ (v). To prove (3.17) we limit ourselves to the following case. Let a non-empty subset $\alpha = \{t_1, \ldots, t_n\} \subset T$ be given. Let $\{A_{ki}\}_{k=1}^{\infty} \subset \mathscr{A}_{t_i}$, $A_{ki} \wedge A_{kj} = 0$ whenever $k \neq j$, be given for any $i = 1, \ldots, n$. There are two possible cases: (i) $t_0 \in \alpha$, then we put $t_1 = t_0$; (ii) $t_0 \notin \alpha$, then without loss of generality we can change α to $\alpha \cup \{t_0\}$ and we also put $t_1 = t_0$. There is a countable decomposition $\{E_v\}_{v=1}^{\infty} \subset \subset \mathscr{A}_{t_1}$ of 1 with $m(x_{t_1}(E_v)) < \infty$, $v \geqq 1$. Define $\mathscr{B}_1$ as the minimal Boolean sub-σ-algebra of $\mathscr{A}_{t_1}$ containing $\{E_v \wedge A_{k1},\ E_v \wedge A_1^{\perp}: v, k \geqq 1\}$, where $A_i = \bigvee_{k=1}^{\infty} A_{ki}$, $i = 1, \ldots, n$, and let $\mathscr{B}_i$, $i = 2, \ldots, n$, be the Boolean sub-σ-algebras of $\mathscr{A}_{t_i}$ generated by $\{A_i^{\perp}, A_k: k \geqq 1\}$.

A mapping $\bar{x}_i := x_i \mid \mathscr{B}_i$ is a $\mathscr{B}_i$-σ-observable of $\mathscr{L}$, $i = 1, \ldots, n$, and $\bar{x}_1$ is σ-finite with respect to m. It is clear that $\bar{x}_1, \ldots, \bar{x}_n$ have a countably obtainable commutator a_0 over $\{\text{com}\, F: F \text{ is a finite subset of } \bigcup_{i=1}^{n} \mathscr{R}(\bar{x}_i)\}$, because the last set has at most countably many elements.

We will show $m(a_0^{\perp}) = 0$. Let $a_0 = \bigwedge_{k=1}^{\infty} \text{com}\, F_k$. It is evident that if F and G are two finite subsets of $\mathscr{L}$ with $F \subset G$, then $\text{com}\, G < \text{com}\, F$. Therefore, we can choose F_k to be nondecreasing and containing $\bar{x}_1(E_v)$ for any fixed $v \geqq 1$. Indeed, put $G_k = \bigcup_{i=1}^{k} F_i \cup (\bar{x}_1(E_v)\}$, then

$$a_0 = \bigwedge_{k=1}^{\infty} \text{com}\, F_k > \bigwedge_{k=1}^{\infty} \text{com}\, G_k > a_0\,.$$

Using the continuity from above of m and the properties of the commutators we obtain

$$m(\bar{x}_1(E_v) \wedge a_0) = \lim_k m(\bar{x}_1(E_v) \wedge \text{com}\, F_k) = \lim_k m(\bar{x}_1(E_v)) = m(\bar{x}_1(E_v))\,.$$

In the previous step we used Lemma 3.7 and (3.12). This implies $m(\bar{x}_1(E_v) \wedge a_0^{\perp}) = 0$ for any $v \geqq 1$, and, consequently, $m(a_0^{\perp}) = \sum_{v=1}^{\infty} m(\bar{x}_1(E_v) \wedge a_0^{\perp}) = 0$. Theorem 3.4 entails that $\bar{x}_1, \ldots, \bar{x}_n$ have a joint distribution in m, so, in particular, (3.17) holds.

(v) ⇒ (i). Let $\emptyset \neq \alpha \subset T$, α finite, be given. Without loss of generality we may assume that $t_0 \in \alpha$. The property (v) and the Carathéodory method of measure extension applied to Boolean algebras, [17, 18], guarantees that there is a (unique) measure μ_α on $\prod_{t \in \alpha} \mathscr{A}_t$ such that

$$\mu_\alpha(\bigwedge_{t \in \alpha} A_t) = m(\bigwedge_{t \in \alpha} x_t(A_t)) \quad \text{for any} \quad A_t \in \mathscr{A}_t,\ t \in \alpha\,.$$

(i) $\Rightarrow$ (vi). Let $\mathscr{L}_0$ be the minimal sublogic of a logic $\mathscr{L}$ containing $M = \bigcup_{t\in T} \mathscr{R}(x_t)$.
Define

(3.19) $\quad \mathscr{J} = \{a \in \mathscr{L}_0 \colon a < \bigvee_{i=1}^{\infty} \operatorname{com}^{\perp} F_i,\ F_i \text{ is a finite subset of } M,\ i \geqq 1\}.$

It follows from [11] that $\mathscr{J}$ is a σ-p-ideal of $\mathscr{L}_0$. Theorem 5 of [22] says that the factor logic $\mathscr{L}_0/\mathscr{J}$ is a Boolean σ-algebra. Let us put $\mathscr{B} = \mathscr{L}_0/\mathscr{J}$. A map $h\colon \mathscr{L}_0 \to \mathscr{B}$ defined by $h(a) = [a]_{\mathscr{J}}$ is $\mathscr{G}$ a σ-homomorphism of $\mathscr{L}_0$ onto $\mathscr{B}$.

Now we show that $m(a) = 0$ whenever $h(a) = 0$. In other words, $m(a) = 0$ whenever $a \in \mathscr{J}$. From the definition of $\mathscr{J}$ it follows that there is a sequence $\{F_i\}_{i=1}^{\infty}$ of finite subsets of M such that $a < \bigvee_{i=1}^{\infty} \operatorname{com}^{\perp} F_i$. Let us put $G_i = F_i \cup \{x_{t_0}(E_i)\}$, $i \geqq 1$, where $\{E_i\}_{i=1}^{\infty}$ is a decomposition of 1 in $\mathscr{A}_{t_0}$ with $m(x_{t_0}(E_i)) < \infty$. For any G_i, there is a finite subset α_i of T such that for any $b \in G_i$ there is $t \in \alpha$ with $x_t(A) = b$ for some $A \in \mathscr{A}_t$. Put $G = \bigcup_{i=1}^{\infty} G_i$ and $\alpha = \bigcup_{i=1}^{\infty} \alpha_i$. We order the elements of α as follows: $\alpha = \{t_1, t_2, \ldots\}$.

Let $\mathscr{B}_i$ be the minimal Boolean sub-σ-algebra of $\mathscr{A}_{t_i}$ generated by $\mathscr{A}_{t_i} \cap G$. Then $\bar{x}_i := x_{t_i} \mid \mathscr{B}_i$ is a $\mathscr{B}_i$-σ-observable of $\mathscr{L}$, and $\{\bar{x}_i\}_{i=1}^{\infty}$ have a countably obtainable commutator a_0. $\{\mathscr{B}_i\colon i \geqq 1\}$ is a system of σ-independent Boolean sub-σ-algebras of a Boolean σ-algebra $\mathscr{A}$. Since at least one of $\{\bar{x}_i\}_{i=1}^{\infty}$ is σ-finite with respect to m, Lemma 3.2 and Theorem 3.4 yield $m(a_0^{\perp}) = 0$.

An easy calculation yields

$$a < \bigvee_{i=1}^{\infty} \operatorname{com}^{\perp} F_i < \bigvee_{i=1}^{\infty} \operatorname{com}^{\perp} G_i < a_0^{\perp}$$

so that $m(a) = 0$.

It remains to show that $\mathscr{B}$ is not a degenerate Boolean σ-algebra. In the opposite case $0 = 1$ in $\mathscr{B}$ and, therefore, $h(0) = h(1)$ so that $m(1) = 0$ which is a contradiction.

(vi) $\Rightarrow$ (vii). Let (vi) hold. Defining $\mathscr{L}_1 := \mathscr{B}$ and taking the σ-homomorphism h from (vi), the condition (vii) is proved.

(vii) $\Rightarrow$ (viii). Let (vii) hold. Define a measure $\bar{m}$ on $\mathscr{L}_1$ as follows: $\bar{m}(h(a)) = m(a)$. We show that $\bar{m}$ is well defined. Let $h(a) = h(b)$. Then $h(a \vee b) = h(a \wedge b)$ and $h((a \vee b) \wedge (a \wedge b)^{\perp}) = 0$. Therefore $m((a \vee b) \wedge (a \wedge b)^{\perp}) = 0$. Using the orthomodular law we have $m(a \vee b) = m(a \wedge b) + m((a \vee b) \wedge (a \wedge b)^{\perp}) = (m(a \wedge b)$. Consequently, $m(a) = m(b)$.

Clearly, $\bar{m}(0) = 0$. Let $\{h(a_i)\}_{i=1}^{\infty}$ be orthogonal elements in $\mathscr{L}_1$. In $\mathscr{L}_0$ we define $b_1 = a_1$, $b_n = a_n \wedge (\bigvee_{n=1}^{\infty} a_i)^{\perp}$, $n \geqq 2$. Then $\{b_n\}_{n=1}^{\infty}$ are orthogonal elements and $h(b_n) = h(a_n)$. An easy check shows

$$\bar{m}(\bigvee_{n=1}^{\infty} h(a_n)) = \bar{m}(\bigvee_{n=1}^{\infty} h(b_n)) = \bar{m}(h(\bigvee_{n=1}^{\infty} b_n)) = m(\bigvee_{n=1}^{\infty} b_n) =$$

$$= \sum_{n=1}^{\infty} m(b_n) = \sum_{n=1}^{\infty} \overline{m}(h(b_n)) = \sum_{n=1}^{\infty} \overline{m}(h(a_n)) .$$

Since at least one σ-observable of $\mathscr{L}_1$ from $\{h \circ x_t : t \in T\}$ is σ-finite with respect to $\overline{m}$, Theorem 2.2 of [I] entails that there is a unique measure μ on $\prod_{t \in T} \mathscr{A}_t$ such that

$$\mu(\bigwedge_{t \in \alpha} A_t) = \overline{m}(\bigwedge_{t \in \alpha} h \circ x_t(A_t))$$

for any $A_t \in \mathscr{A}_t$ and any finite $\emptyset \neq \alpha \subset T$. Using the definition of $\overline{m}$ we prove (3.18).

(viii) $\Rightarrow$ (i). This implication is evident.

Theorems 3.9 is completely proved. Q.E.D.

Remark 1. As an example of particular interest for the present study we give a proof of the implication (vii) $\Rightarrow$ (i) in which we do not apply Theorem 2.2.

So, let (vii) hold. First of all we show that $\mathscr{L}_1$ is a Boolean σ-algebra. For $b \in \mathscr{L}_0$, denote by $K(b)$ the set of all $a \in \mathscr{L}_0$ such that $h(a) \leftrightarrow h(b)$. If $b = x_t(A)$, where $t \in T$ and $A \in \mathscr{A}_t$ are arbitrary, then $K(b)$ is a sublogic of $\mathscr{A}_0$ containing $\bigcup\{\mathscr{R}(x_t): t \in T\}$. So, $x_t(A) \leftrightarrow a$ for each $a \in \mathscr{L}_0$. Now let $b \in \mathscr{L}_0$, then the same argument shows that $K(b) = \mathscr{L}_0$. Hence $h(a) \leftrightarrow h(b)$ for any $a, b \in \mathscr{L}_0$. In other words, $\mathscr{L}_1$ is a Boolean σ-algebra.

It is known that Ker h is a σ-p-ideal of $\mathscr{L}_0$. The factor logic $\mathscr{L}_0/\text{Ker } h$ is σ-isomorphic to $\mathscr{L}_0$ [24, p. 41], hence $\mathscr{L}_0/\text{Ker } h$ is a Boolean σ-algebra. A result of Marsden [22] shows that in this case Ker h contains the σ-p-ideal $\mathscr{J}$ from (3.19) as a subset. Hence $m(a) = 0$ for any $a \in \mathscr{J}$, in particular, $m(\text{com}^{\perp} F) = 0$ for any finite subset F of $\bigcup\{\mathscr{R}(x_t): t \in T\}$. In virtue of the condition (ii) of the last theorem, this is equivalent to (i).

Remark 2. (a) The implication (v) $\Rightarrow$ (i) has been proved by Gudder [5] for $\mathscr{B}(R^1)$-σ-observables and states.

(b) The implication (iv) $\Rightarrow$ (v) was proved by Pulmannová [1] for states and σ-observables defined on Borel σ-algebras of topological spaces equipped with a tight topology, by using the results of compact approximations on these spaces [25].

(c) The implication (ii) $\Rightarrow$ (iv) has been proved in [7] for states and σ-observables, where the main tool of the proof has been the following simple observation: if $t_i \leqq s_i$ $i \in \{1, 2, \ldots\}$ and $-\infty < \sum_i t_i = \sum_i s_i < \infty$, then $t_i = s_i$ for any i. However, when at least one of $t_i(s_i)$ is $+\infty$, then this is not true in general.

(d) A very elementary proof of (ii) $\Rightarrow$ (i) for $\mathscr{B}(R^1)$-σ-observables and states is presented in [19]. It is based on the properties of the distribution function $F(t_1, \ldots, t_n) := m(\bigwedge_{i=1}^{n} x_i((-\infty, t_i)),\ t_i \in R^1,\ i = 1, \ldots, n$. This approach is not applicable to the general cases.

(e) The equivalence between (i) and (vi) has been established in [11] for a system of $\mathscr{B}(R^1)$-σ-observables and states.

(f) The implications (i) → (ii) → (iii) → (iv), (v) → (i), (vii) → (i), (vii) → (i) are true under the more general conditions, too.

In the rest of this section we deal with some corollaries of Theorem 3.9.

Proposition 3.10. *Let the assumptions of Theorem* 3.9 *hold. If* (i) *of Theorem* 3.9 *is valid then, for any* $a \in \mathscr{L}_0$, $m_a(b) := m(a \wedge b)$, $b \in \mathscr{L}_0$ *is a σ-additive σ-finite measure on* $\mathscr{L}_0$.

Proof. If $m(a) = 0$, the proposition is evident. Let $m(a) > 0$, and let $b = \bigvee_{i=1}^{\infty} b_i$, $\{b_i\} \in \mathscr{L}_0$, $b_i \perp b_j$ if $i \neq j$. Due to (vi) of Theorem 3.9 there is a Boolean σ-algebra $\mathscr{B}$ and a σ-homomorphism h from $\mathscr{L}_0$ onto $\mathscr{B}$. Therefore $\overline{m}(h(a)) := m(a)$ is a σ-finite measure on $\mathscr{B}$. Then $m_a(\bigvee_{i=1}^{\infty} b_i) = m(a \wedge \bigvee_{i=1}^{\infty} b_i) = \overline{m}(h(a \wedge \bigvee_{i=1}^{\infty} b_i)) =$
$= \overline{m}(\bigvee_{i=1}^{\infty} (h(a \wedge b_i)) = \sum_{i=1}^{\infty} \overline{m}(h(a \wedge b_i)) = \sum_{i=1}^{\infty} m_a(b_i)$. Q.E.D.

Remark 3. If $a \in \mathscr{L}_0$ and $0 < m(a) < \infty$, then $m_a(b)/m(a)$, $b \in \mathscr{L}_0$, may be treated as a conditional probability on $\mathscr{L}_0$.

Proposition 3.11. *Let the assumptions of Theorem* 3.9 *hold. Then* $\{x_t: t \in T\}$ *have a joint distribution in a measure m iff, for any* $a \in \bigcup\{\mathscr{R}(x_t): t \in T\}$, *the function* $m_a(b) := m(a \wedge b,)$, $b \in \mathscr{L}_0$, *is additive on* $\mathscr{L}_0$, *that is,* $m_a(b_1 \vee b_2) = m_a(b_1) + m_a(b_2)$ *whenever* $b_1, b_2 \in \mathscr{L}_0$ *and* $b_1 \perp b_2$. *Moreover,* m_a *is always a σ-additive σ-finite measure on* $\mathscr{L}_0$, *and* $m_a(b \vee c) = m((a \wedge b) \vee (a \wedge c))$, $b, c \in \mathscr{L}_0$.

Proof. One part of the proposition follows from Proposition 3.10.

To prove the second part we show that (iv) of Theorem 3.9 holds. First of all let $\alpha = \{t_1, t_2\} \subset T$ and $A_{1i}, A_{2i} \in \mathscr{A}_{t_i}$, $A_{1i} \wedge A_{2i} = 0$, $i = 1,2$, be given. Then

$$m\ (\bigwedge_{i=1}^{2} x_{t_i}(A_{1i} \vee A_{2i})) = m_b(x_{t_1}(A_{11} \vee A_{21})) =$$

$$= m_1(x_{t_i}(A_{11})) + m_b(x_{t_2}(A_{21})) = m_{a_1}(x_{t_2}(A_{12} \vee A_{22})) + m_{a_2}(x_{t_2}(A_{12} \vee A_{22})) =$$

$$= \sum_{j_1,j_2=1}^{2} m(\bigwedge_{i=1}^{2} x_{t_i}(A_{j_i i}))\,,$$

where we use $b = x_{t_2}(A_{12} \vee A_{21})$, $a_j = x_{t_1}(A_{j1})$, $=$ ' 1, 2.

The general case of (iv) is obtained from the just established fact by using mathematical induction, which proves that $\{x_t: t \in T\}$ have a joint distribution in m.

The last assertion of Proposition 3.11 follows from Proposition 3.10. Q.E.D.

Corollary 3.11.1. *Under the hypotheses of Theorem* 3.9 *we have*: (i) *let* $a \in \mathscr{L}_0$, $m(a) > 0$; *if* $\{x_t: t \in T\}$ *have a joint distribution in m, then* $\{x_t: t \in T\}$, *as σ-observables of* $\mathscr{L}_0$, *have a joint distribution in* m_a; (ii) $\{x_t: t \in T\}$ *have a joint distribution in m iff* (3.10) *holds for any finite* $F \subset \mathscr{L}_0$.

Proof. (i) If $a \in \bigcup\{\mathscr{R}(x_t): t \in T\}$, then the assertion follows from Proposition 3.11. In the general case, according to (vi) of Theorem 3.9, there is a Boolean σ-algebra $\mathscr{B}$ and a σ-homomorphism h from $\mathscr{L}_0$ onto $\mathscr{B}$ such that $m(a) = 0$ whenever $h(a) = 0$. Hence, if $F \subset \bigcup\{\mathscr{R}(x_t): t \in T\}$ is a finite subset, then $h(a_1), \ldots, h(a_n)$ are compatible in $\mathscr{B}$, where $F = \{a_1, \ldots, a_n\}$. Therefore

$$m_a(\text{com}^\perp F) = \bar{m}(h(a) \wedge h(\text{com}^\perp F)) = 0\,,$$

where $\bar{m}(h(a)) := m(a)$, $a \in \mathscr{L}_0$, is a measure on $\mathscr{B}$.

(ii) Let $\{x_t: t \in T\}$ have a joint distribution in m. Analogously as in the first part we can prove that $h(\text{com}^\perp F) = 0$ whenever F is a finite subset of $\mathscr{L}_0$. Hence $m(\text{com}^\perp F) = 0$. Q.E.D.

We say that a measure m on a logic $\mathscr{L}$ has a *Jauch-Piron property* if $m(a) = m(b) = 0$ implies $m(a \vee b) = 0$.

Corollary 3.11.2. *Let the assumptions of Theorem* 3.9 *hold. If* $\{x_t: t \in T\}$ *have a joint distribution in a measure m, then* $m(\bigvee_{i=1}^{\infty} a_i) = 0$ *whenever* $m(a_i) = 0$, $a_i \in \mathscr{L}_0$, $i \geqq 1$.

Proof. This is a consequence of Corollary 3.11. 1 and the observation that for a measure $\bar{m}$ on $\mathscr{B}$ we have $\bar{m}(h(a) \vee h(b)) + \bar{m}(h(a) \wedge h(b)) = \bar{m}(h(a)) + \bar{m}(h(b))$, $a, b \in \mathscr{L}_0$ (this is a valuation property of $\bar{m}$ and m, respectively). Q.E.D.

4. JOINT DISTRIBUTIONS AND COMMUTATORS

We have seen that the cornerstone of the theory of a joint distribution of σ-observables in a measure is the commutator of observables. Although it need not exist in general, see for instance [23], and in Theorem 3.9 it does not appear, it is implicitly involved in partial steps of Theorem 3.9. In the present section we shall study some relationships between the existence of a joint distribution of observables and the existence of a commutator of observables.

First of all we notice that the following is true. Let x_t be an $\mathscr{A}_t$-σ-observable of a quantum logic $\mathscr{L}$, $t \in T$. Then

$$\begin{aligned}(4.1)\qquad &\bigwedge\{\text{com}\, F: F \text{ is a finite subset of } \bigcup\{\mathscr{R}(x_t): t \in T\}\} = \\ &= \bigwedge\{\text{com}\,(\{x_t(A_t): t \in \alpha\}): (\forall A_t \in \mathscr{A}_t), (\forall t \in \alpha), (\forall \alpha \text{ a finite subset of } T)\}\,.\end{aligned}$$

This is understood as follows: if one of the elements in (4.1) exists in $\mathscr{L}$, the the other one exists, too, and both are equal. This assertion may be proved similarly as Propositions 2.1 and 2.2 from [10].

Let $\emptyset \neq M \subset \mathscr{L}$. By $\mathscr{L}_0(M)$ we denote the minimal sublogic of $\mathscr{L}$ containing M.

Proposition 4.1. *Let* $\emptyset \neq M \subset \mathscr{L}$ *and let* $\mathscr{J} = \mathscr{J}(M)$ *be the* σ-*p-ideal of* $\mathscr{L}_0(M)$ *defined by* (3.19). *Then* (i)

$$(4.2)\qquad a_0^\perp = \bigvee\{x: x \in \mathscr{J}(M)\} \quad (\text{in } \mathscr{L})\,.$$

This means that if one the elements in (4.2) *exists in* $\mathscr{L}$, *then the other one also exists, and both are equal; here* a_0 *is the commutator of* M.

(ii) *The commutator* a_0 *of* M *is countably obtainable if and on if* $a_0^\perp \in \mathscr{J}(M)$.

Proof. (i) and (ii) follow immediately from the definitions of $\mathscr{J}(M)$ and a_0. Q.E.D.

Proposition 4.2. *Let there be* $a_0 = \text{com } M$ *and let* $a_0 \neq 0$. *Let* $\mathscr{L}_{a_0}$ *be the minimal sublogic of a logic* $\mathscr{L}(0, a_0)$ *containing* $\{a \wedge a_0 : a \in M\}$. *Then* $h_{a_0} : a \to a \wedge a_0$, $a \in \mathscr{L}_0(M)$, *is a* σ*-homomorphism of* $\mathscr{L}_0(M)$ *onto* $\mathscr{L}_{a_0}$, *and*

$$\text{Ker } h_{a_0} \subset \mathscr{J}(M) . \tag{4.3}$$

Proof. Since the set $K = \{a \in \mathscr{L}_0(M) : a \leftrightarrow a_0\}$ is a sublogic of $\mathscr{L}_0(M)$, the map h_{a_0} is well defined and is a σ-homomorphism. Now we show that it transforms $\mathscr{L}_0(M)$ onto $\mathscr{L}_{a_0}$. Denote $\mathscr{B} = \{a \in \mathscr{L}_{a_0}$: there is $c \in \mathscr{L}_0(M)$ with $c \wedge a_0 = a\}$. Then $\mathscr{B}$ is a sublogic of $\mathscr{L}_{a_0}$ containing $\{a \wedge a_0 : a \in M\}$.

Using the result of Marsden [22] we can establish (4.3), because $\mathscr{L}_{a_0}$ is a Boolean σ-algebra. (4.3) is also a consequence of the following simple observation: $\text{Ker } h_{a_0} = \{b \in \mathscr{L}_0(M) : b \perp a_0\}$. Q.E.D.

We say that an element a, $a \in \mathscr{L}$, is the *carrier* of a measure m if $m(b) = 0$ whenever $b \perp a$. It is clear that if a carrier exists, then it is unique.

Proposition 4.3. *Let the assumptions of Theorem* 3.9 *be fulfilled, and the commutator* a_0 *of* $\{x_t : t \in T\}$ *exist in* $\mathscr{L}$. *If* a *is the carrier of* m, *then the following conditions are equivalent*:

(i) $\{x_t : t \in T\}$ *have a joint distribution in* m;
(ii) $m(a_0^\perp) = 0$;
(iii) $a < a_0$.

Proof. Using the properties of the carrier and the commutator and applying (ii) or (iii) of Theorem 3.9, the equivalence can be proved.

This result may be applied to an important case of quantum logics – to the logic $\mathscr{L}(H)$ of all closed subspaces of a Hilbert space H whose dimension is a non-measurable cardinal. We recall that a set X has a *non-measurable cardinal* if there is no nontrivial finite measure ν on the power set 2^X such that $\nu(\{x\}) = 0$ for all $x \in X$.

Theorem 4.4. *Let* $\mathscr{L} = \mathscr{L}(H)$ *be a quantum logic of a real or complex Hilbert space whose dimension is a non-measurable cardinal* $\neq 2$. *Let the assumptions of Theorem* 3.9 *be fulfilled. Then the following conditions are equivalent*:

(i) $\{x_t : t \in T\}$ *have a joint distribution in* m;
(ii) $m(a_0^\perp) = 0$;
(iii) $x_{t_{i_1}}(E_{i_1}) \dots x_{t_{i_n}}(E_{i_n}) f = x_{t_1}(E_1) \dots x_{t_n}(E_n) f$

for any permutation $(i_1, \dots, i_n)$ *of* $(1, \dots, n)$, $n \geq 1$, *any* $E_i \in \mathscr{A}_{t_i}$, *any finite* $\emptyset \neq \alpha = \{t_1, \dots, t_n\} \subset T$ *and any vector* $f \in a$, *where* a *is the carrier of the measure* m.

Moreover, the Boolean σ-algebra in (vi) *of Theorem* 3.9 *may be chosen as a Boolean sub-σ-algebra of a quantum logic of some Hilbert space.*

Proof. Since $\mathscr{L}(H)$ is a complete lattice, the commutator a_0 of $\{x_t: t \in T\}$ always exists in $\mathscr{L}(H)$. According to [25], any σ-finite measure m on $\mathscr{L}(H)$ possesses a carrier which is a separable subspace of H. Proposition 4.3 yields the equivalence of (i) and (ii). The equivalence of (i) and (iii) is a simple modification of the results in [12, 25].

The last assertion follows from Proposition 4.2 and (4.3). Moreover, we note that $x_{t0}: E \mapsto x_t(E) \wedge a_0$ is an $\mathscr{A}_t$-σ-observable of $\mathscr{L}(a_0)$ and $\{x_{t0}: t \in T\}$ are mutually compatible. Q.E.D.

We conclude this section with the following remark. If the commutator a_0 of $\{x_t: t \in T\}$ exists and (3.8) holds, then $\{x_t: t \in T\}$ have a joint distribution in m. The converse implication is known only in special cases, for example, if a_0 is countably obtainable or m has a carrier or $a_0 = 1$. Therefore it would be of interest to establish conditions when (i) and (ii) of Proposition 4.3 are equivalent.

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Súhrn

O ZDRUŽENOM ROZDELENÍ V KVANTOVÝCH LOGIKÁCH II. NEKOMPATIBILNÉ POZOROVATEĽNÉ

ANATOLIJ DVUREČENSKIJ

Predložená práca je pokračovanie prvej časti s rovnakým názvom. Študuje sa združené rozdelenie v σ-konečných mierach pre nekompatibilné pozorovateľné kvantovej logiky, definované na niektorom systéme σ-nezávislých Booleovych pod-σ-algebier Booleovej σ-algebry. Dané sú nutné a postačujúce podmienky pre existenciu združených rozdelení. Ako dôsledok je ukázané,

že ľubovoľný systém pozorovateľných má združené rozdelenie v miere vtedy a len vtedy, keď pozorovateľné môžu byť vnorené do systému kompatibilných pozorovateľných niektorej kvantovej logiky s dodatočnými vlastnosťami. Použité metódy sú odlišné od metód, známych pre konečné miery. Nakoniec sa pojednáva o vzťahu medzi existenciou združeného rozdelenia a existenciou komutátora pozorovateľných, a taktiež sa spomína kvantová logika neseparabilného Hilbertovho priestoru.

Резюме

О СОВМЕСТНОМ РАСПРЕДЕЛЕНИИ В КВАНТОВЫХ ЛОГИКАХ II. НЕКОМПАТИБИЛЬНЫЕ НАБЛЮДАЕМЫЕ

ANATOLIJ DVUREČENSKIJ

Предлагаемая работа является продолжением первой части работы с тем же самым названием. Изучаются совместные распределения в σ-конечных мерах для некомпатибильных наблюдаемых квантовой логики, определенных на некоторой системе σ-независимых булевых σ-подалгебр булевой σ-алгебры. Предложены некоторые необходимые и достаточные условия для существования совместного распределения. В частности показано, что любая система наблюдаемых имеет совместное распределение тогда и только тогда, когда она может быть внедрена в систему компатибильных наблюдаемых некоторой квантовой логики. Использованные методы отличаются от методов, известных для конечных мер. В конце работы исследуется соотношение между существованием совместного распределения и существованием коммутатора наблюдаемых, а также упоминается квантовая логика несепарабельнного гильбертова пространства.

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