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SMALL TIME-PERIODIC SOLUTIONS TO A NONLINEAR EQUATION OF A VIBRATING STRING

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Summary. In this paper, the system consisting of two nonlinear equations is studied. The former is hyperbolic with a dissipative term and the latter is elliptic. In a special case, the system reduces to the approximate model for the damped transversal vibrations of a string proposed by G. F. Carrier and R. Narasimha. Taking advantage of accelerated convergence methods, the existence of at least one time-periodic solution is stated on condition that the right-hand side of the system is sufficiently small.

Keywords: Nonlinear string equation, Accelerated convergence, Time-periodic solution.

AMS Classification: 35L70, 35B10.

The problem we shall be concerned with is to demonstrate the existence of a couple of functions $\boldsymbol{u} = [u_1, u_2]$ of $x \in (0, l)$, $t \in R$ satisfying the equations

$$(\mathrm{E}_1) \qquad \frac{\partial^2 u_1}{\partial t^2} + \sigma_1\left(\frac{\partial u_1}{\partial x}, \frac{\partial u_2}{\partial x}, \frac{\partial^2 u_1}{\partial x^2}, \frac{\partial u_1}{\partial t}\right) = f_1,$$

$$(\mathrm{E}_2) \qquad -\frac{\partial^2 u_2}{\partial x^2} + \sigma_2\left(\frac{\partial u_1}{\partial x}, \frac{\partial^2 u_1}{\partial x^2}\right) = f_2$$

together with the conditions

$$(\mathrm{B}) \qquad u_i(0, t) = u_i(l, t) = 0 \quad \text{for all} \quad t \in R, \quad i = 1, 2,$$

$$(\mathrm{P}) \quad u_i(x, t + \omega) = u_i(x, t) \quad \text{for all} \quad x \in (0, l), \quad t \in R, \quad i = 1, 2.$$

As to the functions $\sigma_1 = \sigma_1(v_1, v_2, v_3, v_4)$, $\sigma_2 = \sigma_2(v_1, v_2)$, we assume that

$$(0.1) \qquad \sigma_i(\boldsymbol{0}) = 0, \quad \frac{\partial \sigma_i}{\partial v_j}(\boldsymbol{0}) = 0 \quad \text{for} \quad i, j = 1, 2,$$

$$(0.2) \qquad \frac{\partial \sigma_1}{\partial v_3}(\boldsymbol{0}) = -a, \quad \frac{\partial \sigma_1}{\partial v_4}(\boldsymbol{0}) = d \quad \text{where} \quad a, d > 0,$$

σ_i being defined and smooth on some open neighborhood of the point $\mathbf{0} \in \mathbb{R}^{6-2i}$, $i = 1, 2$. These requirements seem to be the sine qua non condition we are unable to remove. In particular, the problem with dissipation is studied here since the constant d is strictly positive.

In case we set $\sigma_1(v_1, v_2, v_3, v_4) = \varrho(v_3, v_2 + \frac{1}{2}v_1^2) + dv_4$, $\sigma_2(v_1, v_2) = -v_1 v_2$, $f_2 = 0$, the system (E_1), (E_2) together with (B) reduces to an integrodifferential equation of the form

$$\frac{\partial^2 u_1}{\partial t^2} + d \frac{\partial u_1}{\partial t} + \varrho\left(\frac{\partial^2 u_1}{\partial x^2}, \frac{1}{2l}\int_0^l \left(\frac{\partial u_1}{\partial x}\right)^2 \mathrm{d}x\right) = f_1 ,$$

which is the approximate model for the damped transversal vibrations of a string proposed by G. F. Carrier and R. Narasimha (see [1]).

Observe that the problem above possesses a solution, namely the function $\boldsymbol{u} = [0, 0]$, provided we have $\boldsymbol{f} = [f_1, f_2] = [0, 0]$. We intend to find at least one solution to (E_1), (E_2), (B), (P) provided the function $\boldsymbol{f}$ has been chosen small and smooth enough. Since the corresponding linearized operator "loses" derivatives, no method seems to be available but that one suggested by J. Nash and J. Moser taking advantage of the accelerated convergence of approximate solutions.

Similarly as in [2], [3], we prefer to use the Newton iteration scheme directly rather than refering to some abstract results of Moser (see Section 2). This rapidly convergent process requires both the existence of smoothing operators and the invertibility of the linearized operator in a full neighborhood of the function $\boldsymbol{u} = \mathbf{0}$ (Sections 3, 4). Working with substitution operators, we will need some estimates proved essentially by J. Moser in [4] (Section 4).

Throughout the paper, the symbol c denotes all strictly positive constants. Using the symbol $c(L)$, we want to emphasize that this constant depends essentially on the quantity L only.

1. STATEMENT OF THE MAIN RESULT

To agree upon notation, let us consider functionals (norms)

$$|\varphi|_L = \left(\sum_{K+J \leq L} \int_0^l \int_0^\omega \left|\frac{\partial^{K+J}}{\partial x^K \partial t^J}\right|^2 \mathrm{d}x\, \mathrm{d}t\right)^{1/2},$$

$$\|\varphi\|_L = \sum_{K+J \leq L} \sup\left\{\left|\frac{\partial^{K+J}\varphi}{\partial x^K \partial t^J}\right| \mid x \in [0, l] , \ t \in \mathbb{R}\right\}$$

for $L = 0, 1 \ldots$ The symbols H^L and H_0^L stand for the completion of all (real) functions being both smooth and satisfying (P) and (B), (P), respectively, with respect to the norm $|\ |_L$. Analogously, we define the spaces C^L, C_0^L using $\|\ \|_L$.

Taking into account the known embedding theorems (see [5, Chapter I]), one obtains $H^{L+2} \hookrightarrow C^L$ and

$$\|\varphi\|_L \leqq c(L)\,|\varphi|_{L+2} \quad \text{for} \quad \varphi \in H^{L+2}\,, \quad L = 0, 1 \ldots \tag{1.1}$$

Finally, we set $E_L = H_0^L \times H_0^L$, $F_L = H^L \times H^L$. For a vector function $\boldsymbol{w} = [w_1, w_2]$ belonging to E_L or F_L, the norm is defined as the sum of norms of its components w_1, w_2. For the sake of simplicity, the symbol $|\ |_L$ will denote this norm as well. The norms on C^L are defined in the same way.

We are on the point of formulating our main existence theorem.

Theorem 1.1. *Let an integer $M \geqq 11$ be given. Suppose that the numbers d, a are strictly positive and the conditions* (0.1), (0.2) *are fulfilled. Then there is a number $\delta_1(M) > 0$ such that the system* (E_1), (E_2), (B), (P) *admits a classical solution $\boldsymbol{u} = [u_1, u_2]$ belonging to the space $E_{[M/2]}$* whenever $\boldsymbol{f} \in F_M$ and

$$|\boldsymbol{f}|_M < \delta_1\,. \tag{1.2}$$

Remark. Actually, all we need concerning the smoothness of the functions σ_i is that σ_i are $(M + 2)$-times continuously differentiable on their domains, $i = 1, 2$.

The next section is devoted to the proof of Theorem 1.1.

2. ITERATION SCHEME

We start with an existence result concerning smoothing operators on the scale $\{F_L\}$, $L = 0, 1 \ldots.$

Proposition 2.1. *Choose a real number $A > 1$ and a positive integer M.*

Then there exists a sequence $\{S_n\}$, $n = 0,1 \ldots$ of linear operators such that the following conditions are valid:

$$S_n \colon F_0 \to \bigcap_{K \geqq 0} F_K \quad \textit{for} \quad n = 0, 1 \ldots, \tag{2.1}$$

$$|S_n \boldsymbol{w}|_{L+K} \leqq c(K)\, A^{Kn} |\boldsymbol{w}|_L \quad \textit{for} \quad K, n = 0, 1 \ldots, \tag{2.2}$$

$$|(\mathrm{Id} - S_n)\, \boldsymbol{w}|_K \leqq c(K)\, A^{(K-L)n} |\boldsymbol{w}|_L \quad \textit{for} \quad K = 0, \ldots, L, \quad n = 0, 1 \ldots, \tag{2.3}$$

and for $L = 0, 1 \ldots, M$, $\boldsymbol{w} \in F_L$.

In [3], the smoothing operators were constructed on the space H^L. Clearly, this technique applies to our situation as well.

To simplify notation, we set

$$D_1 \boldsymbol{w} = \left(\frac{\partial w_1}{\partial x}, \frac{\partial w_2}{\partial x}, \frac{\partial^2 w_1}{\partial x^2}, \frac{\partial w_1}{\partial t}\right), \quad D_2 \boldsymbol{w} = \left(\frac{\partial w_1}{\partial x}, \frac{\partial^2 w_1}{\partial x^2}\right).$$

Consider a mapping $\mathbf{G}(\mathbf{w}) = [G_1(\mathbf{w}), G_2(\mathbf{w})]$,

$$G_1(\mathbf{w}) = \frac{\partial^2 w_1}{\partial t^2} + \sigma_1(D_1 \mathbf{w}),$$

$$G_2(\mathbf{w}) = -\frac{\partial^2 w_2}{\partial x^2} + \sigma_2(D_2 \mathbf{w}).$$

Similarly, we define $\mathbf{G}'(\mathbf{w})\,\mathbf{y}$ as

$$G_1'(\mathbf{w})\,\mathbf{y} = \frac{\partial^2 y_1}{\partial t^2} + \frac{\partial \sigma_1}{\partial v_1}(D_1 \mathbf{w})\frac{\partial y_1}{\partial x} + \frac{\partial \sigma_1}{\partial v_2}(D_1 \mathbf{w})\frac{\partial y_2}{\partial x} +$$

$$+ \frac{\partial \sigma_1}{\partial v_3}(D_1 \mathbf{w})\frac{\partial^2 y_1}{\partial x^2} + \frac{\partial \sigma_1}{\partial v_4}(D_1 \mathbf{w})\frac{\partial y_1}{\partial t}.$$

$$G_2'(\mathbf{w})\,\mathbf{y} = -\frac{\partial^2 y_2}{\partial x^2} + \frac{\partial \sigma_2}{\partial v_1}(D_2 \mathbf{w})\frac{\partial y_1}{\partial x} + \frac{\partial \sigma_2}{\partial v_2}(D_2 \mathbf{w})\frac{\partial^2 y_1}{\partial x^2}.$$

We solve the sequence of linear equations

$$(2.4)_{-1} \qquad \mathbf{G}'(\mathbf{0})\,\mathbf{u}^0 = \mathbf{f},$$

$$(2.4)_0 \qquad \mathbf{G}'(S_0 \mathbf{u}^0)\,\mathbf{y}^0 = \mathbf{h}^0 \quad \text{where} \quad \mathbf{h}^0 = S_0 \mathbf{e}^0, \quad \mathbf{e}^0 = -\mathbf{G}(\mathbf{u}^0) + \mathbf{G}'(\mathbf{0})\,\mathbf{u}^0.$$

Next, for $n = 1, 2, \ldots$, we set

$$\mathbf{u}^n = \mathbf{u}^0 + \sum_{k=0}^{n-1} \mathbf{y}^k, \quad \mathbf{h}^n = S_n \mathbf{e}^n + (S_n - S_{n-1}) \sum_{k=0}^{n-1} \mathbf{e}^k.$$

The quantities $\mathbf{e}^k$ are given as sums $\mathbf{e}^k = {}^1\mathbf{e}^k + {}^2\mathbf{e}^k$ where

$${}^1\mathbf{e}^k = -\mathbf{G}(\mathbf{u}^k) + \mathbf{G}(\mathbf{u}^{k-1}) + \mathbf{G}'(\mathbf{u}^{k-1})\,\mathbf{y}^{k-1},$$

$${}^2\mathbf{e}^k = (\mathbf{G}'(S_{k-1}\,\mathbf{u}^{k-1}) - (\mathbf{G}'(\mathbf{u}^{k-1}))\,\mathbf{y}^{k-1}.$$

The functions $\mathbf{y}^n$ are determined successively as the solutions of the equations

$$(2.4)_n \qquad \mathbf{G}'(S_n \mathbf{u}^n)\,\mathbf{y}^n = \mathbf{h}^n, \quad n = 1, 2 \ldots$$

The iteration scheme just described is analogous to that which was used in [2], [3]. It is a matter of routine to check that

$$(2.5) \qquad \mathbf{G}(\mathbf{u}^{n+1}) = \mathbf{f} - \mathbf{e}^{n+1} - (\mathrm{Id} - S_n) \sum_{k=0}^{n} \mathbf{e}^k$$

holds for each $n = 0, 1 \ldots$

Let us pause to list two important results. The proof of the first one represents the bulk of the paper and is carried out in Sections 3,4.

Proposition 2.2. *Let an integer $M \geqq 4$ be given. Then we are able to find a number $\delta_2(M) > 0$ such that the equation*

$$G'(w)\, y = h$$

possesses a unique solution $y \in E_M$ whenever $h \in F_M$, $w \in F_{M+5}$ and

$$|w|_5 < \delta_2 . \tag{2.6}$$

The above conditions being satisfied, we have the estimates

$$|y|_L \leqq c(L)\,(|h|_L + |w|_{L+5}\,|h|_0) \quad \text{for} \quad L = 0, \ldots, M . \tag{2.7}$$

Proposition 2.3. *Consider an integer $N \geqq 1$. Take u^n, $y^n \in E_{N+4}$ small enough so that the quantities e^{n+1} may be well defined.*

Then the relations

$$|e^0|_L \leqq c(L) \sum_{L_1+L_2+L_3=L} (1 + |u^0|_{L_1+4})\,|u^0|_{L_2+4}\,|u^0|_{L_3+2}\,, \tag{2.8}$$

$$|{}^1e^{n+1}|_L \leqq c(L) \sum_{L_1+L_2+L_3=L} (1 + |u^n|_{L_1+4} + |y^n|_{L_1+4})\,|y^n|_{L_2+4}\,|y^n|_{L_3+2}\,, \tag{2.9}$$

$$|{}^2e^{n+1}|_L \leqq c(L) \sum_{L_1+L_2+L_3=L} (1 + |u^n|_{L_1+4})\,|y^n|_{L_2+4}\,|(\mathrm{Id} - S_n)\,u^n|_{L_3+2} \tag{2.10}$$

hold for each $n = 0, 1, \ldots, L = 0, 1, \ldots, N$.

We postpone the proof to Section 4.

Now we are going to solve the iteration equations (cf. [3]). Consider the number $M \geqq 11$ which appears in Theorem 1.1. We are able to choose a number D in such a way that $2D \in (M + 1, M + 2)$. Finally, let the function $f \in F_M$ satisfy

$$|f|_M < \delta_3 \tag{2.11}$$

where δ_3 is, for the present, an arbitrary strictly positive number.

As to the equation $(2.4)_{-1}$, we claim, in view of Proposition 2.2, that there is a (unique) solution u^0, the norm of which is estimated by

$$|u^0|_L \leqq c(L)\,|f|_L \quad \text{for each} \quad L = 0, \ldots, M . \tag{2.12}$$

In order to solve $(2.4)_0$, with (2.2) in mind, it is necessary that u^0 be small in E_5. To achieve this, it suffices to pick out the number δ_3 in (2.11) small enough. In this way, taking into account (1.1), we simultaneously obtain that the term e^0 is well defined. We conclude that there exists y^0,

$$|y^0|_L \leqq c(L)\,(|h^0|_L + |S_0\, u^0|_{L+5}\,|h^0|_0) \quad \text{for} \quad L = 0, \ldots, M .$$

Combining (2.2), (2.8) with (2.12), we deduce

$$|y^0|_L < \varepsilon \quad \text{for all} \quad L = 0, \ldots, M \tag*{$(2.13)_0$}$$

where the number $\varepsilon > 0$ can be arbitrarily chosen provided the number δ_3 is appropriately small.

Having chosen $\varepsilon > 0$ small enough, our aim is to solve (2.4) in each step of the iteration and to establish the estimate

$$(2.13)_m \qquad |\mathbf{y}^m|_L \leqq \varepsilon A^{(L-D)m} \quad \text{for all} \quad L = 0, \ldots, M .$$

For this purpose, we assume that the existence of such $\mathbf{y}^m$ has been already established for all integers $m \leqq n$.

In accordance with (2.12), (2.13), we get

$$|\mathbf{u}^{m+1}|_L \leqq |\mathbf{u}^0|_L + \sum_{k=0}^{m} |\mathbf{y}^k|_L \leqq \varepsilon\, c(L)\, \Big(1 + \sum_{k=0}^{m} A^{(L-D)k}\Big) .$$

Using the standard summation rule, one obtains

$$(2.14) \qquad |\mathbf{u}^{m+1}|_L \leqq \varepsilon\, c(L)\, \big(1 + A^{(L-D)(m+1)}\big), \quad L = 0, \ldots, M, \quad m \leqq n .$$

In view of (2.3), we get the estimate

$$|(\mathrm{Id} - S_{m+1})\, \mathbf{u}^{m+1}|_L \leqq c(L)\, A^{(L-M)(m+1)} |\mathbf{u}^{m+1}|_M \quad \text{for} \quad L = 0, \ldots, M .$$

Since $M > D$, we can deduce from (2.14) the inequality

$$(2.15) \qquad |(\mathrm{Id} - S_{m+1})\, \mathbf{u}^{m+1}|_L \leqq \varepsilon\, c(L)\, A^{(L-D)(m+1)}, \quad L = 0, \ldots, M, \quad m \leqq n .$$

Now, the terms $\mathbf{e}^{m+1}$ will be treated. First of all, we are able to set $\varepsilon > 0$ in (2.14) so small that all quantities in Proposition 2.3 may be well defined since $D \geqq 6$. Taking into account (2.9), (2.10), (2.13)–(2.15), we summarize that

$$|\mathbf{e}^{m+1}|_L \leqq \varepsilon^2\, c(L) \sum_{L_1 + L_2 + L_3 = L} \big(1 + A^{(L_1 + 4 - D)(m+1)}\big)\, A^{(L_2 + L_3 + 6 - 2D)(m+1)}$$

when $L = 0, \ldots, M - 4$ and $m \leqq n$. Finally, we get

$$(2.16) \qquad |\mathbf{e}^{m+1}|_L \leqq \varepsilon^2\, c(L)\, A^{(L-2D+6)(m+1)} \quad \text{for} \quad L = 0, \ldots, M - 4, \quad m \leqq n .$$

The following estimate is analogous to (2.14):

$$(2.17) \qquad \Big|\sum_{k=0}^{m} \mathbf{e}^k\Big|_L \leqq \varepsilon^2\, c(L)\, \big(1 + A^{(L-2D+6)(m+1)}\big), \quad L = 0, \ldots, M - 4, \quad m \leqq n .$$

Seeing that $M + 2 - 2D > 0$, we are able to prove the following, using an analogous argument as in (2.15):

$$(2.18) \qquad \Big|(\mathrm{Id} - S_j) \sum_{k=0}^{n} \mathbf{e}^k\Big|_L \leqq \varepsilon^2\, c(L)\, A^{(L-2D+6)(n+1)}$$

for each $L = 0, \ldots, M - 4$, and $j = n, n + 1$.

Eventually, we are to derive estimates related to $\mathbf{h}^{n+1}$. According to (2.2), (2.16), we obtain

$$(2.19) \qquad |S_{n+1} \mathbf{e}^{n+1}|_L \leqq c(L)\, A^{L(n+1)} |\mathbf{e}^{n+1}|_0 \leqq \varepsilon^2\, c(L)\, A^{(L-2D+6)(n+1)} \quad \text{for all} \quad L = 0, 1, \ldots$$

Setting $S_{n+1} - S_n = (S_{n+1} - \mathrm{Id}) - (S_n - \mathrm{Id})$, we obtain

$$(2.20) \qquad |(S_{n+1} - S_n)\sum_{k=0}^{n} \mathbf{e}^k|_L \leqq \varepsilon^2\, c(L)\, A^{(L-2D+6)(n+1)}$$

for each $L = 0, 1, \ldots, M - 4$, (2.18) being taken into account.

On the other hand, it is possible to write

$$|(S_{n+1} - S_n)\sum_{k=0}^{n} \mathbf{e}^k|_L \leqq |S_{n+1}\sum_{k=0}^{n} \mathbf{e}^k|_L + |S_n\sum_{k=0}^{n} \mathbf{e}^k|_L \leqq$$

$$\leqq c(L)\, A^{(L-M+4)(n+1)}|\sum_{k=0}^{n} \mathbf{e}^k|_{M-4} \quad \text{provided} \quad L > M - 4\,.$$

Combining the above relation with (2.17), (2.19) and (2.20), we conclude that

$$(2.21) \qquad |\boldsymbol{h}^{n+1}|_L \leqq \varepsilon^2\, c(L)\, A^{(L-2D+6)(n+1)} \quad \text{for all} \quad L = 0, 1 \ldots$$

Having derived all necessary estimates, we focus our effort on solving $(2.4)_{n+1}$. In order that all terms appearing in the equation may be defined and the condition (2.6) be fulfilled, we must be able to keep $|\boldsymbol{u}^{n+1}|_5$ small enough. To comply with this requirement, we set $\varepsilon > 0$ in (2.13) sufficiently small. Observe that this can be done for all steps of the iteration simultaneously since the constant appearing in (2.14) is independent of n.

By virtue of Proposition 2.2, we obtain the solution $\boldsymbol{y}^{n+1}$ of $(2.4)_{n+1}$ satisfying

$$(2.22) \qquad |\boldsymbol{y}^{n+1}|_L \leqq c(L)\,(|\boldsymbol{h}^{n+1}|_L + |S_{n+1}\boldsymbol{u}^{n+1}|_{L+5}\,|\boldsymbol{h}_n{}^{+1}|_0) \leqq$$

(according to (2.14), (2.21))

$$\leqq \varepsilon^2\, c(L)\, A^{(L-2D+6)(n+1)} \quad \text{for} \quad L = 0, 1, \ldots, M\,.$$

We can choose $\varepsilon > 0$ in (2.22) in such a way that the inequality $\varepsilon\, c(L) \leqq 1$ holds for all $L = 0, 1, \ldots, M$. Noticing that $D \geqq 6$, we see that $(2.13)_{n+1}$ follows.

Repeating the procedure just described, we get, by induction on m, the existence of a sequence $\{\boldsymbol{y}^n\}$ satisfying $(2.13)_n$. In particular, the sequence $\{\boldsymbol{u}^n\}$ admits a limit $\boldsymbol{u}$ in E_{L_0}, $L_0 < D$ being chosen arbitrarily. As a consequence of the inequality $D \geqq 6$ and by continuity of $\boldsymbol{G}$, we obtain

$$(2.23) \qquad \|\boldsymbol{G}(\boldsymbol{u}^n) - \boldsymbol{G}(\boldsymbol{u})\|_0 \to 0\,.$$

On the other hand, employing (2.5), we get

$$\|\boldsymbol{G}(\boldsymbol{u}^{n+1}) - \boldsymbol{f}\|_0 \leqq c(|\mathbf{e}^{n+1}|_2 + |(\mathrm{Id} - S_n)\sum_{k=0}^{n} \mathbf{e}^k|_2) \leqq \varepsilon^2 c A^{(8-2D)(n+1)}.$$

Consequently, $\boldsymbol{G}(\boldsymbol{u}) = \boldsymbol{f}$ by (2.23), which completes the proof of Theorem 1.1.

In this section, our main goal is to show some existence and regularity results related to the system of linear equations

$$(\mathrm{L}_1)\qquad \frac{\partial^2 y_1}{\partial t^2} + d\,\frac{\partial y_1}{\partial t} - a\,\frac{\partial^2 y_1}{\partial x^2} + b^1\,\frac{\partial y_1}{\partial x} + b^2\,\frac{\partial y_2}{\partial x} + b^3\,\frac{\partial^2 y_1}{\partial x^2} + {}$$

$$+ b^4\,\frac{\partial y_1}{\partial t} = h_1\,,$$

$$(\mathrm{L}_2)\qquad -\frac{\partial^2 y_2}{\partial x^2} + b^5\,\frac{\partial y_1}{\partial x} + b^6\,\frac{\partial^2 y_1}{\partial x^2} = h_2$$

where the unknown functions y_1, y_2 of $x \in (0, l)$, $t \in \mathbb{R}$ are required to satisfy (B), (P). As is easily seen, such results will be a suitable tool for proving Proposition 2.2. We have used the symbols b^j, $j = 1, \ldots, 6$ for functions $b^j \in C^{M+1}$, the symbols h_i, $i = 1, 2$ stand for functions belonging to the space H^M, M appearing in Proposition 2.2.

We start with the inequality

$$(3.1)\qquad [w_1]_J \{w_2\}_K \leqq c(J, K)\left([w_1]_I \{w_2\}_{J+K-I} + [w_1]_{J+K-I} \{w_2\}_I\right),$$

which is an easy consequence of the Nirenberg interpolation inequality and of the well known relation $a \,.\, b \leqq a^r \,.\, r^{-1} + b^s \,.\, s^{-1}$, $r^{-1} + s^{-1} = 1$. In (3.1), we suppose $0 \leqq I \leqq J, K$ and the symbols $[\;]$, $\{\;\}$ can be replaced both by $|\;|$ and by $\|\;\|$.

Besides, we have the Poincaré inequality

$$(3.2)\qquad \int_0^\omega \int_0^l \left|\frac{\partial w}{\partial x}\right|^2 \mathrm{d}x\, \mathrm{d}t \geqq c(\omega, l) \int_0^\omega \int_0^l |w|^2\, \mathrm{d}x\, \mathrm{d}t\,, \quad w \in H_0^1\,.$$

As a rule, all we need for solving linear problems are suitable a priori estimates. Let us multiply (L_1) and (L_2) by the expressions

$$(-1)^L\left(d\,\frac{\partial^{2L} y_1}{\partial t^{2L}} + 2\,\frac{\partial^{2L+1} y_1}{\partial t^{2L+1}}\right) \quad\text{and}\quad (-1)^L \frac{\partial^{2L} y_2}{\partial t^{2L}}\,,$$

respectively. Summing the results, integrating by parts and using (3.2), we get

$$(3.3)\qquad \Xi^2(\mathbf{y}, L) \leqq c(L)\, \Xi(\mathbf{y}, L)\left(\sum_{K=0}^{L} \sum_{j=1}^{6} \|b^j\|_{L+1-K}\, \Xi(\mathbf{y}, K) + |\mathbf{h}|_L\right),$$

$$L = 0, 1, \ldots, M\,.$$

We have denoted

$$\Xi(\mathbf{y}, L) = \left|\frac{\partial^L y_1}{\partial t^L}\right|_1 + \left|\frac{\partial^{L+1} y_2}{\partial x\, \partial t^L}\right|_0 + \left|\frac{\partial^L y_2}{\partial t^L}\right|_0\,.$$

We only sketch the way how to treat the most difficult term:

$$(-1)^L \int_0^\omega \int_0^l b^3 \frac{\partial^2 y_1}{\partial x^2} \frac{\partial^{2L+1} y_1}{\partial t^{2L+1}} \, dx \, dt = -\int_0^\omega \int_0^l \frac{1}{2} \frac{\partial b^3}{\partial t} \left(\frac{\partial^{L+1} y_1}{\partial x \, \partial t^L} \right)^2 dx \, dt +$$

$$+ \int_0^\omega \int_0^l \sum_{K=0}^{L} \left\{ \binom{L+1}{K} \frac{\partial^{L+1-K} b^3}{\partial t^{L+1-K}} \frac{\partial^{K+1} y_1}{\partial x \, \partial t^K} \frac{\partial^{L+1} y_1}{\partial x \, \partial t^L} - \right.$$

$$\left. - \binom{L}{K} \frac{\partial^{L+1-K} b^3}{\partial x \, \partial t^{L-K}} \frac{\partial^{K+1} y_1}{\partial x \, \partial t^K} \frac{\partial^{L+1} y_1}{\partial t^{L+1}} \right\} dx \, dt$$

Using (3.1), (3.3), the estimates

$$\Xi(\mathbf{y}, L) \leqq c(L) \left(\sum_{j=1}^{6} \|b^j\|_{L+1} |\mathbf{h}|_0 + |\mathbf{h}|_L \right) \tag{3.4}$$

can be derived for $L = 0, 1, \ldots, M$ by induction, since

$$\sum_{j=1}^{6} \|b^j\|_1 < \delta_4 \, . \tag{3.5}$$

The number $\delta_4 > 0$ has to be chosen small enough.

Having obtained a priori estimates (3.4), we are able to employ the Galerkin approximate method based on the system $\{e_{k,j}\} = \{\sin(\pi k l^{-1} x) \sin(2\pi j \omega^{-1} t), \sin(\pi k l^{-1} x) \cos(2\pi j \omega^{-1} t) \mid k = 1,/2, \ldots, j = 0, 1, \ldots\}$ so that the existence of a (unique) pair $\mathbf{y} = [y_1, y_2]$ may be stated, which satisfies

$$\int_0^\omega \int_0^l \left(\frac{\partial^2 y_1}{\partial t^2} + d \frac{\partial y_1}{\partial t} + b^1 \frac{\partial y_1}{\partial x} + b^2 \frac{\partial y_2}{\partial x} + b^4 \frac{\partial y_1}{\partial t} \right) \varphi + \tag{3.6}$$

$$+ a \frac{\partial y_1}{\partial x} \frac{\partial \varphi}{\partial x} - \frac{\partial y_1}{\partial x} \frac{\partial (b^3 \varphi)}{\partial x} \, dx \, dt = \int_0^\omega \int_0^l h_1 \varphi \, dx \, dt \, ,$$

$$\int_0^\omega \int_0^l \frac{\partial y_2}{\partial x} \frac{\partial \varphi}{\partial x} + b^5 \frac{\partial y_1}{\partial x} \varphi - \frac{\partial y_1}{\partial x} \frac{\partial (b^5 \varphi)}{\partial x} \, dx \, dt = \int_0^\omega \int_0^l h_2 \varphi \, dx \, dt \tag{3.7}$$

for all smooth function φ satisfying (B), (P). Obviously, the estimates (3.4) remain true.

Now, from (3.6), (3.7) we can deduce

$$\left| \frac{\partial^2 y_1}{\partial x^2} \right|_0 + \left| \frac{\partial^2 y_2}{\partial x^2} \right|_0 \leqq c \left(\sum_{j=1}^{6} \|b^j\|_2 |\mathbf{h}|_0 + |\mathbf{h}|_1 \right) . \tag{3.8}$$

The generalized derivatives of $\mathbf{y}$ satisfy

$$a \frac{\partial^2 y_1}{\partial x^2} = \frac{\partial^2 y_1}{\partial t^2} + d \frac{\partial y_1}{\partial t} + b^1 \frac{\partial y_1}{\partial x} + b^2 \frac{\partial y_2}{\partial x} + b^3 \frac{\partial^2 y_1}{\partial x^2} + \tag{3.9}$$

$$+ b^4 \frac{\partial y_1}{\partial t} - h_1 \, ,$$

$$(3.10) \qquad \frac{\partial^2 y_2}{\partial x^2} = b^5 \frac{\partial y_1}{\partial x} + b^6 \frac{\partial^2 y_1}{\partial x^2} - h_2 .$$

For $K = 0, 1, \ldots, M - 2$ and $J = 0, 1, \ldots, M - (K + 2)$ we apply the differential operator $\partial^{K+J}/\partial x^K \partial t^J$ successively to both sides of (3.9), (3.10). Keeping (3.1) in mind, we get by induction

$$(3.11) \qquad \left|\frac{\partial^{K+J+2} y_i}{\partial x^{K+2} \partial t^J}\right|_0 \leqq c(K, J) \left(\sum_{j=1}^{6} \|b^j\|_{K+J+2} |\boldsymbol{h}|_0 + |\boldsymbol{h}|_{K+J+1}\right)$$

Combining (3.4), (3.11) with (3.6), (3.7), one finally obtains

Lemma 3.1. *Let $M \geqq 4$, $b^j \in C^{M+1}$ for $j = 1, \ldots, 6$. Then we can find a number $\delta_5(M) > 0$ such that the condition*

$$(3.12) \qquad \sum_{j=1}^{6} \|b^j\|_1 < \delta_5$$

guarantees the existence of a unique classical solution $\boldsymbol{y} = [y_1, y_2]$ of (L_1), (L_2), (B), (P) *for arbitrary $\boldsymbol{h} \in F_M$. Moreover, $\boldsymbol{y} \in E_M$ and for each $L = 0, \ldots, M$ we have the estimate*

$$(3.13) \qquad |\boldsymbol{y}|_L \leqq c(L) \left(\sum_{j=1}^{6} \|b^j\|_{L+1} |\boldsymbol{h}|_0 + |\boldsymbol{h}|_L\right).$$

Remark. Actually, we have obtained somewhat better estimates concerning y_1 than in (3.13).

4. PROOFS OF AUXILIARY LEMMAS

The last section is devoted to completing the proofs of Propositions 2.2, 2.3. To this end, some estimates connected with a substitution operator are needed. According to Moser ([4]), we have

Lemma 4.1. *Let Φ: $\mathbb{R}^n \to \mathbb{R}$ be a mapping Φ being N-times continuously differentiable on some neighborhood of the point $\boldsymbol{0} \in \mathbb{R}^n$. Suppose that all derivatives of Φ up to the order N are bounded by a constant $B > 0$.*

Then for functions $w_i \in C^N$, $i = 1, \ldots, n$ we have the estimate

$$(4.1) \qquad \|\Phi(w_1, \ldots, w_n)\|_N \leqq c(N)\, B\left(1 + \sum_{i=1}^{n} \|w_i\|_N\right)$$

provided $(w_1, \ldots, w_n)$ ranges in the domain of Φ.

With help of Lemma 4.1, it is not difficult to complete the proof of Proposition 2.2, the results of Section 3 being taken into account.

As to Proposition 2.3, let us treat the quantity ${}^1e_2^{n+1}$, for instance. In view of the Taylor expansion formula, we have

$${}^1e_2^{n+1} = -\int_0^1 (1 - s) \frac{\partial^2 \sigma_2}{\partial v_1^2} (D_2(\boldsymbol{u}^n + s\boldsymbol{y}^n)) \left(\frac{\partial y_1}{\partial x}\right)^2 +$$

$$+ 2 \frac{\partial^2 \sigma_2}{\partial v_1 \partial v_2} \left(D_2(\mathbf{u}^n + s\mathbf{y}^n)\right) \frac{\partial y_1}{\partial x} \frac{\partial^2 y_1}{\partial x^2} +$$

$$+ \frac{\partial^2 \sigma_2}{\partial v_2^2} \left(D_2(\mathbf{u}^n + s\mathbf{y}^n)\right) \left(\frac{\partial^2 y_1}{\partial x^2}\right)^2 ds .$$

Combining Lemma 4.1, the estimates (1.1) and the Hölder inequality, we get a relation analogous to (2.9). The other terms are treated in a similar way.

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Souhrn

MALÁ PERIODICKÁ ŘEŠENÍ NELINEÁRNÍ ROVNICE STRUNY

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V článku je studována soustava nelineárních rovnic, z nichž první je hyperbolického typu s disipačním členem a druhá eliptická. Ve speciálním případě rovnice představují model pro tlumené kmity struny navržený G. F. Carrierem a R. Narasimhou. Užitím metod urychlené konvergence je dokázána existence periodického řešení za předpokladu, že pravá strana soustavy je dostatečně malá.

Резюме

МАЛЫЕ ПЕРИОДИЧЕСКИЕ РЕШЕНИЯ НЕЛИНЕЙНОГО УРАВНЕНИЯ СТРУНЫ

EDUARD FEIREISL

В статье изучается система нелинейных уравнений, в которой первое уравнение является уравнением гиперболического типа с диссипацей и второе является уравнением эллиптического типа. В частном случае эта система представляет модель для поперечных колебаний струны. Пользуясь методом ускоренной сходимости, автор доказывает существование по крайней мере одного периодического решения с влучае достаточно малой функции в правой части уравнения.

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