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ON THE EXISTENCE OF FREE VIBRATIONS FOR A BEAM EQUATION WHEN THE PERIOD IS AN IRRATIONAL MULTIPLE OF THE LENGTH

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Summary. The author examined non-zero T-periodic (in time) solutions for a semilinear beam equation under the condition that the period T is an irrational multiple of the length. It is shown that for a.e. $T \in R^1$ (in the sense of the Lebesgue measure on R^1) the solutions do exist provided the right-hand side of the equation is sublinear.

Keywords: Semilinear equation, periodic solution, irrational periods, dual variational method.

AMS classification: 35L70, 35B10.

I. INTRODUCTION

We shall investigate the problem

{P}

(E)
$$u_{xx}(x, t) + u_{xxxx}(x, t) + f(x, u(x, t)) = 0$$
$$x \in (0, \pi)\,, \quad t \in R^1$$

where the unknown function u satisfies the boundary conditions

(B)
$$u(0, t) = u(\pi, t) = 0$$
$$u_{xx}(0, t) = u_{xx}(\pi, t) = 0 \quad \text{for all} \quad t \in R^1\,.$$

Further u is to be periodic in the t-variable with the period $T > 0$, i.e.

(PE)
$$u(x, t + T) = u(x, t) \quad \text{for all} \quad x \in (0, \pi)\,, \quad t \in R^1\,.$$

The function f is supposed to satisfy the following conditions:

(F1)
$$f \quad \text{is continuous on} \quad [0, \pi] \times R^1\,,$$

(F2)
$$f(x, 0) \equiv 0 \quad \text{for all} \quad x \in [0, \pi]\,,$$

the function $f(x, u) + u$ is increasing in the variable u for all $x \in [0, \pi]$,

$$\text{(F3)} \qquad \lim_{u \to \pm\infty} \frac{f(x, u)}{u} = 0 ,$$

$$\text{(F4)} \qquad \liminf_{u \to 0} \frac{f(x, u)}{u} \geqq a_0 ,$$

where a_0 is a fixed positive real number.

All limits are assumed to hold uniformly with respect to x.

We say that the solution u of the problem $\{P\}$ is trivial if u is independent of the variable t. The solution u_1 is a translation of u_2 if there exists $\tau \in R^1$ such that $u_1(x, t) = u_2(x, t + \tau)$ holds for all x, t.

Our main goal is the proof of the following theorem.

Theorem 1. *Let the function f satisfy the assumptions* (F1)–(F4). *Then for an arbitrary positive integer n there exists a real constant $T_0 > 0$ such that for almost every $T \in (T_0, +\infty)$ (in the sense of the Lebesgue measure on R^1) there exist n different nontrivial solutions of the problem $\{P\}$ which are not translation of one another.*

Note that we have the existence of nontrivial solutions for almost all sufficiently large periods instead of rational multiples of the number π only. Moreover, we do not require monotonicity of the function f. Eventually we do not use any symmetry of f regarding the x-variable (as in [1]). Unfortunately, the approach presented depends essentially upon the spectrum of the "beam" operator and is not applicable for example in the case of the wave equation.

II. VARIATIONAL FORMULATION OF THE PROBLEM $\{P\}$

Let us consider the problem $\{P'\}$ given by

$$\text{(1)} \qquad \frac{1}{T^2} u_{tt}(x, t) + u_{xxxx}(x, t) + f(x, u(x, t)) = 0$$

with the boundary conditions (B). Clearly it suffices to find 2π-periodic solutions of the equation (1).

If v is such a solution, then the function $u(x, t) = v(x, T^{-1}t)$ is a solution of the problem $\{P\}$ with the period $2\pi T$.

Let us introduce the system of functions

$$\text{(2)} \qquad e_{kj}(x, t) = \begin{cases} \sqrt{(2)}\, \pi^{-1} \sin(kx) \sin(jt) & \text{for} \quad k \in N ,\ j \in N , \\ \pi^{-1} \sin(kx) & \text{for} \quad k \in N ,\ j = 0 , \\ \sqrt{(2)}\, \pi^{-1} \sin(kx) \cos(jt) & \text{for} \quad k \in N ,\ -j \in N , \end{cases}$$

$$x \in [0, \pi] , \quad t \in R^1 , \quad k \in N , \quad j \in Z ,$$

where the symbols N, Z denote the set of positive integers and the set of integers, respectively. The basic space we shall use in the following is the space H which arises as a complete real linear hull of the system $\{e_{kj}\}$ with regard to the inner product

$$\langle u, v \rangle = \int_0^{2\pi} \int_0^{\pi} u(x, t)\, v(x, t)\, \mathrm{d}x\, \mathrm{d}t\,, \tag{3}$$

H is a Hilbert space with the norm

$$\|u\| = \langle u, u \rangle^{1/2}\,. \tag{4}$$

Further we consider the linear operator

$$L'_T v = \frac{1}{T^2} v_{tt} + v_{xxxx} \tag{5}$$

defined for sufficiently smooth functions which are 2π-periodic and satisfy the boundary conditions (B). L'_T has a self-adjoint extension L_T on H with the spectral resolution

$$L_T v = \sum_{\substack{k \in N \\ j \in Z}} \left(k^4 - \frac{1}{T^2} j^2 \right) a_{kj}(v)\, e_{kj}\,, \tag{6}$$

where $a_{kj}(v)$ are the Fourier coefficients with regard to the basis $\{e_{kj}\}$.

Definition. *The function u is called the solution of the problem* $\{P'\}$ *if $u \in H$ and*

$$\langle u, L'_T \varphi \rangle + \langle f(\cdot, u), \varphi \rangle = 0 \tag{7}$$

for all functions φ which are smooth, 2π-periodic in t and satisfy the conditions (B).

Remark. $f(\cdot, u)$ denotes the function – an element of the space H having the value $f(x, u(x, t))$ at the point (x, t).

We are going to prove an easy modification of the well known Chinčin theorem (see also [4]).

Lemma 1. *There exists a set $D \subset (0, +\infty)$ of irrational numbers, $\mu((0, +\infty) \setminus D) = 0$ (μ is the Lebesgue measure on R^1), such that for an arbitrary element $d \in D$ there exists a positive constant $c(d)$ satisfying*

$$\left| k^4 - \frac{1}{d^2} j^2 \right| \geqq c(d) \frac{k}{\lg^2 k} \tag{8}$$

for all $j \in Z$, $k \in N$, $k \geqq 2$.

Proof. Let us consider the interval $(0, a)$. Denote by S_k the set of all numbers $b \in (0, a)$ satisfying

$$\left| k^4 - \frac{1}{b^2} j^2 \right| < \frac{k}{\lg^2 k} \tag{9}$$

for an appropriately chosen $j \in Z$. For such b we have

$$\left| b^2 - \frac{j^2}{k^4} \right| < \frac{a^2}{k^3 \lg^2 k},$$

hence we obtain

$$\mu(S_k) \leqq c_1(a) \frac{1}{k \lg^2 k}, \tag{10}$$

$$c_1(a) > 0 .$$

Now let S be the set of all $b \in (0, a)$ such that (9) holds for infinitely many $k \in N$, $j \in Z$, $k \geqq 2$. Obviously

$$S = \bigcap_{k=2}^{\infty} \bigcup_{m=k}^{\infty} S_m .$$

As a consequence of the summability of $\sum_{k=2}^{\infty} 1/(k \lg^2 k)$ we obtain $\mu(S) = 0$. ■

As an easy consequence we have

Lemma 2. *For every $T \in D$ the spectrum $\sigma(L_T)$ of the operator L_T consists of isolated eigenvalues with no accumulation point on R^1, Moreover, $0 \notin \sigma(L_T)$ and all eigenspaces are of finite dimensions.*

In what follows we shall suppose that $T \in D$. Let us consider the linear operator

$$K_T v = L_T v - v \tag{11}$$

and let us denote by $\mathcal{N}$ the null space of K_T. Set $V = \mathcal{N}^{\perp}$ in the sense of H. We define the operator

$$M_T = K_T^{-1} \tag{12}$$

on the space V. Observe that Lemma 2 implies that the operator M_T is well defined on the whole space V and is a compact linear operator.

Further, let us set

$$F(x, u) = \int_0^u f(x, s) \, ds + \tfrac{1}{2} u^2 . \tag{13}$$

Recall that F is strictly convex in u via (F2) and $F(x, 0) \equiv 0$. Further, there exists a continuous partial derivative

$$\frac{\partial}{\partial u} F(x, u) = f(x, u) + u \tag{14}$$

Moreover, for arbitrary $\varepsilon > 0$ we have the estimates

$$(1 - \varepsilon) \frac{u^2}{2} - c_2(\varepsilon) \leqq F(x, u) \leqq (1 + \varepsilon) \frac{u^2}{2} + c_2(\varepsilon) , \tag{15}$$

$$c_2(\varepsilon) > 0$$

due to the assumption (F3).

Now we consider the conjugate function in the sense of convex analysis (see [3])

(16) $$F^*(x, v) = \sup_{u \in R^1} \{uv - F(x, u)\} .$$

Since (15) holds, we have a possibility of defining the dual action functional

(17) $$\Phi_T(v) = \tfrac{1}{2}\langle M_T v, v\rangle + \int_0^\pi \int_0^{2\pi} F^*(x, v(x, t)) \, \mathrm{d}t \, \mathrm{d}x$$

on the space V. The functional Φ_T is of the class $C^1(V, R^1)$ with the Frèchet differential

(18) $$\langle D\Phi_T v, w\rangle = \langle M_T v, w\rangle + \left\langle \frac{\partial}{\partial v} F^*(\cdot, v), w \right\rangle \quad \text{for all} \quad v, w \in V .$$

Lemma 3. *Let $v \in V$ be a critical point of the functional Φ_T. Then the function u defined by*

(19) $$u(x, t) = \frac{\partial}{\partial v} F^*(x, v(x, t))$$

is a solution of the problem {P'}.

Proof. The equality

$$\langle M_T v, w\rangle + \frac{\partial}{\partial v} F^*(\cdot, v), w\rangle = 0$$

holds for all $w \in V$. Thus we get the existence of $h \in \mathcal{N}$ satisfying

$$M_T v + \frac{\partial}{\partial v} F^*(\cdot, v) = h .$$

Now we can apply K_T to the both sides of our equality and we have

$$v = -L_T u + u ,$$

by virtue of the duality

$$v(x, t) = \frac{\partial}{\partial u} F(x, u(x, t)) = f(x, u(x, t)) + u(x, t) \quad \blacksquare$$

III. EXISTENCE OF CRITICAL POINTS OF Φ_T

Our technique is almost identical with that used by Costa and Willem in [2]. We refer to [2] for details.

Let us consider the unitary representation U of the group $S^1 = [0, 2\pi]/\{0, 2\pi\}$ on V, i.e.

(20) $$U(\alpha)[v](x, t) = v(x, t + \alpha) \quad \text{for} \quad \alpha \in S^1 .$$

Let us denote the set of fixed points of U by

(21) $$\mathscr{F}(S^1) = \{u \in V \mid u \quad \text{does not depend on} \quad t\} .$$

We define the orbit of an element v as the set

$$o(v) = \{u \in V \mid u = U(\varphi)\, v,\ \varphi \in S^1\} .$$

Now we easily check that the functional Φ_T is S^1-invariant, i.e. Φ_T is constant on all orbits. We shall use the following abstract theorem.

Theorem 2. *Let $J \in C^1(V, R^1)$ be an S^1-invariant functional satisfying the following condition (Palais-Smale):*

(PS) *If $J(v_m)$ is bounded and $J'(v_m) \to 0$ for a sequence $\{v_m\}_{m=1}^{\infty} \subset V$, then $\{v_m\}_{m=1}^{\infty}$ contains a convergent subsequence in V.*

Further, let Y, Z be closed S^1-invariant subspaces of V satisfying

(22) $$\dim(Z) < +\infty\ ,\ \operatorname{codim}(Y) < +\infty\ ,$$

(23) $$\dim(Z) > \operatorname{codim}(Y)\ ,$$

(24) $$\mathscr{F}(S^1) \subset Y,\quad Z \cap \mathscr{F}(S^1) = \{0\}\ ,$$

(25) $$J \text{ is bounded from below on } Y\ ,$$

(26) $$\text{there exists } r > 0 \text{ such that } J(v) < 0 \text{ for all } u \in Z,\ \|u\| = r\ ,$$

(27) $$\text{if } v \in \mathscr{F}(S^1) \text{ and } J'(v) = 0,\ \text{then } J(v) \geqq 0\ .$$

Then there exist at least $\frac{1}{2}(\dim(Z) - \operatorname{codim}(Y))$ *orbits of critical points of J outside $\mathscr{F}(S^1)$.*

Proof. The proof is based on the concept of cohomological index and is contained in [2]. ■

We are going to verify the assumptions of Theorem 2 in the case $J = \Phi_T$.

1. *Validity of the condition* (PS)

Assume $\Phi_T'(v_m) \to 0$. Let us denote by P the orthogonal projection on the space $\mathscr{N}$. Recall that P is compact due to the finite dimension of $\mathscr{N}$. Thus we have

(28) $$M_T v_m + \frac{\partial}{\partial v} F^*(\cdot, v_m) = h_m + P \frac{\partial}{\partial v} F^*(\cdot, v_m)$$

where $h_m \to 0$ in V. Now we set

(29) $$u_m = P \frac{\partial}{\partial v} F^*(\cdot, v_m) - M_T v_m\ .$$

By duality we obtain

(30) $$v_m = f(\cdot, u_m + h_m) + u_m + h_m\ .$$

On the other hand, we can apply the operator K_T to both sides of (29) obtaining

(31) $$L_T u_m - u_m = -v_m .$$

Combing (30), (31), we get

(32) $$L_T u_m + f(\cdot, u_m + h_m) = -h_m .$$

As a consequence of $0 \notin \sigma(L_T)$ (see Lemma 2) and the growth condition (F3) we get in a standard way that

(33) $$\{u_m\}_{m=1}^{\infty} \text{ is bounded on } H .$$

From (30) and (F3) we get the existence of a subsequence $\{v_n\}_{n=1}^{\infty}$ which is weakly convergent in V and $P(\partial/\partial v) F^*(\cdot, v_n)$ converges strongly due to the compactness of P. Since M_T is compact and (28) holds, we have the strong convergence of the corresponding subsequence $\{u_n\}_{n=1}^{\infty}$ in H. Combining it with (30) we obtain the desired result.

■

2. *Verification of the condition* (27)

According to (18) we have

$$\sum_{k=2}^{\infty} \frac{1}{k^4 - 1} a_{k0}^2(v) + \int_0^{\pi}\int_0^{2\pi} \frac{\partial}{\partial v} F^*(x, v(x, t))\, v(x, t)\, dt\, dx = 0 .$$

Now $(\partial/\partial v) F^*$ is increasing in v due to the convexity of F, and $(\partial/\partial v) F^*(x, 0) = 0$. Hence we have $v \equiv 0$ since $a_{10}(v) = 0$ ($V = \mathcal{N}^{\perp}$).

3. *Choice of the space* Y

According to (15) we have an estimate

(34) $$F^*(x, v) \geqq \frac{1}{1 + \varepsilon} \frac{v^2}{2} + c_2(\varepsilon) .$$

Now we can set

(35) $$Y_1 = \mathrm{lin}\left\{e_{kj} \,\Big|\, \left(k^4 - \frac{j^2}{T^2} - 1\right) \in (-\infty, -1] \cup [0, +\infty)\right\},$$

$$Y = Y_1 \cap V .$$

Using (34) we easily check the validity of (25), (24) and (22) (by Lemma 2).

4. *Choice of the space* Z

It follows from (15) that

(36) $$F^*(x, v) \leqq \frac{1}{1 - \varepsilon} \frac{v^2}{2} + c_2(\varepsilon) .$$

Now, by (F4) and by duality we have

$$F^*(x, v) \leqq \frac{1 + \varepsilon}{a_0 + 1} \frac{v^2}{2} \quad \text{for all} \quad v \in R^1, \tag{37}$$

$|v| \leqq r$, r sufficiently small, $\varepsilon > 0$ arbitrary. We can set

$$Z_1 = \text{lin}\left\{e_{kj} \mid \left(k^4 - \frac{j^2}{T^2} - 1\right) \in (-1 - a_0, -1)\right\}, \tag{38}$$

$$Z = Z_1 \oplus Y^\perp .$$

Clearly (22), (24) hold. Using (37) and the equivalence of the L_∞ and L_2 norms on Z ($\dim(Z) < +\infty$), we get (26).

Now we are able to apply Theorem 2. Let $T \in D$ and let us denote by n, $n \geqq 0$ the number of eigenvalues of the operator L_T contained in the interval $(-a_0, 0)$. With regard to the fact that the corresponding eigenspaces have a dimension $2m$, $m \geqq 1$ we conclude that

(39) *there exist at least n distinct nontrivial solutions of the problem* $\{P\}$ *with the period* $2\pi T$.

In order to complete our proof of Theorem 1, we have only to show the following assertion:

Lemma 4. *Let ε be an arbitrary real number, $\varepsilon > 0$. Then for arbitrary $n \in N$ there exists $T_0 > 0$ such that the estimate*

$$\frac{j^2}{T^2} - k^4 \in (0, \varepsilon) \quad \textit{for all} \quad T > T_0 \tag{40}$$

holds for at least 2n *distinct pairs* (k, j), $k \in N$, $j \in Z$.

Proof. Let us set

$$j = [T] + l, \tag{41}$$

$$l = 1, \ldots, n,$$

where $[T]$ denotes the greatest integer which is less than or equal to T. Let further $k = 1$. Then

$$\frac{j^2}{T^2} - k^4 > 0$$

and

$$\frac{([T] + l)^2}{T^2} - 1 \leqq \frac{[T]^2 + 2[T]\, n + n^2}{[T]^2} - 1 .$$

Now it is easy to see that for T being sufficiently large (40) holds. Using the symmetry $j \sim -j$ in (40) we get the desired result. ∎

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Souhrn

EXISTENCE VOLNÝCH VIBRACÍ PRO ROVNICI TYČE ZA PŘEDPOKLADU, ŽE PERIODA JE IRACIONÁLNÍM NÁSOBKEM DÉLKY

EDUARD FEIREISL

Autor vyšetřuje nenulová T-periodická řešení semilineární rovnice tyče v případě, že časová perioda T je iracionálním násobkem délky tyče. Pro sublineární pravou stranu rovnice je dokázána existence řešení pro s. v. $T \in R^1$ (ve smyslu Lebesgueovy míry).

Резюме

СУЩЕСТВОВАНИЕ СВОБОДНЫХ КОЛЕБАНИЙ ДЛЯ УРАВНЕНИЯ СТЕРЖНЯ В СЛУЧАЕ, КОГДА ПЕРИОД ЯВЛЯЕТСЯ ИРРАЦИОНАЛЬНЫМ КРАТНЫМ ДЛИНЫ

EDUARD FEIREISL

В статье изучаются ненулевые T-периодические решения полулинейного уравнения стержня при предположении, что период времени T является иррациональным кратным длины стержня. Для сублинейной правой части уравнения доказывается существование таких решений для почти всех (в смысле меры Лебега) $T \in R^1$.

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