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INEQUALITIES OF KORN'S TYPE, UNIFORM WITH RESPECT TO A CLASS OF DOMAINS

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Summary. Inequalities of Korn's type involve a positive constant, which depends on the domain, in general. A question arises, whether the constants possess a positive infimum, if a class of bounded two-dimensional domains with Lipschitz boundary is considered. The proof of a positive answer to this question is shown for several types of boundary conditions and for two classes of domains.

Keywords: domain optimization, Korn's inequality, generalized Friedrichs inequality.

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INTRODUCTION

In the domain optimization for elliptic problems we encounter the following question: does a positive constant of the ellipticity condition exist, which is common for the whole class of admissible domains? The positive answer is crucial in proving the existence of an optimal domain. The present paper is devoted to the above mentioned question in two-dimensional problems.

We formulate the problem for a general elliptic system of equations in Section 1 and prove a general theorem, using some ideas of Haslinger, Neittaanmäki and Tiihonen [1]. In Section 2 we show some applications to equations of the second order: generalized Friedrichs inequality and the Korn's inequality under different kinds of boundary conditions.

1. INEQUALITY OF KORN'S TYPE FOR A CLASS OF BOUNDARY CONDITIONS AND VARIABLE DOMAINS

Let us define the following class of domains

$$\Omega(v) = \{(x_1, x_2) \mid 0 < x_1 < v(x_2),\ 0 < x_2 < 1\},$$

where $v \in \mathcal{U}_{ad}$ and

$$\mathcal{U}_{ad} = \left\{ v \in C^{(0),1}([0,1]) \text{ (i.e. Lipschitz) } \alpha \leqq v \leqq \beta, \left| \frac{dv}{dx_2} \right| \leqq C_1 \text{ a.e.} \right\}.$$

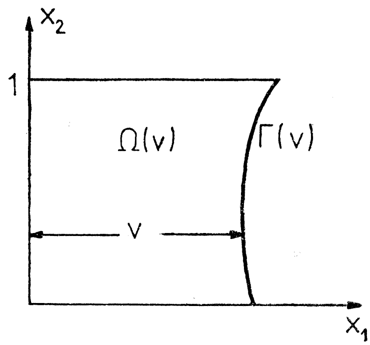

Fig. 1.

Here α, β and C_1 are given positive constants. Henceforth $\Gamma(v)$ will denote the graph of the function v.

In this section we shall deal with a general inequality of Korn's type (cf. [2] – Section 11.3.1 or [3]). Let us consider the following Cartesian product of Sobolev spaces on the domain $\Omega(v)$:

$$W(\Omega(v)) = \prod_{s=1}^{r} H^{\varkappa_s}(\Omega(v)) \quad (H^{\varkappa_s} \equiv W^{\varkappa_s,2}),$$

where $\varkappa_s \geqq 1$, $s = 1, 2, \ldots, r$.

We define the following system of differential operators for $\boldsymbol{u} = (u_1, \ldots, u_r) \in W(\Omega(v))$:

$$N_i(\boldsymbol{u}) = \sum_{s=1}^{r} \sum_{|\alpha| \leqq \varkappa_s} n_{is\alpha} D^\alpha u_s, \quad i = 1, 2, \ldots, k,$$

where $n_{is\alpha} \in \mathbb{R}$.

($H1$) We assume that there exists a constant $\tilde{c} > 0$, independent of $v \in \mathcal{U}_{ad}$ and such that

$$\sum_{i=1}^{k} \|N_i(\boldsymbol{u})\|_{0,\Omega(v)}^2 + \|\boldsymbol{u}\|_{0,\Omega(v)}^2 \geqq \tilde{c} \|\boldsymbol{u}\|_{W(\Omega(v))}^2$$

holds for any $\boldsymbol{u} \in W(\Omega(v))$.

(Henceforth the subscript $0, \Omega(v)$ denotes the norm in $L^2(\Omega(v))$ and in $[L^2(\Omega(v))]^r$, respectively. The subscript $W(\Omega(v))$ denotes the standard norm in $W(\Omega(v))$.)

Let $V_p(\Omega(v))$ and $V_p^0(\Omega(v))$ be (closed) subspaces of $W(\Omega(v))$. Let us introduce the following set

$$\mathcal{U}_{ad}^E = \{v_\varepsilon \mid v_\varepsilon(x_2) = v(x_2) - \varepsilon,\ v \in \mathcal{U}_{ad},\ \varepsilon \in [0, \alpha/2)\} .$$

$(H2)$ We assume that an inequality of Korn's type holds on $V_p^0(\Omega(v))$ for any $v \in \mathcal{U}_{ad}^E$, i.e., there exists a constant $c(v) > 0$ for any $v \in \mathcal{U}_{ad}^E$, such that

$$\sum_{i=1}^{k} \|N_i(\boldsymbol{u})\|_{0,\Omega(v)}^2 \geqq c(v) \|\boldsymbol{u}\|_{W(\Omega(v))}^2 \quad \forall \boldsymbol{u} \in V_p^0(\Omega(v)) .$$

(H3) Assume finally the following relation between V_p and V_p^0: there exists $\varepsilon_0 > 0$ such that:

$$\{\boldsymbol{u} \in V_p(\Omega(v)),\ v \in \mathcal{U}_{ad},\ w \in \mathcal{U}_{ad}^E,\ 0 < v(x_2) - w(x_2) < \varepsilon_0\ \forall x_2 \in [0, 1]\} \Rightarrow$$
$$\Rightarrow \boldsymbol{u}|_{\Omega(w)} \in V_p^0(\Omega(w)) .$$

Theorem 1. *Assume that the conditions $(H1)$, $(H2)$ and $(H3)$ are satisfied.*
Then there exists a positive constant c, independent of $v \in \mathcal{U}_{ad}$ and such that

$$\sum_{i=1}^{k} \|N_i(\boldsymbol{u})\|_{0,\Omega(v)}^2 \geqq c\|\boldsymbol{u}\|_{W(\Omega(v))}^2 \tag{1}$$

holds for all $\boldsymbol{u} \in V_p(\Omega(v))$ and $v \in \mathcal{U}_{ad}$.

Proof. Let (1) be not true. Then there exist sequences $\{v_n\}$ and $\{\boldsymbol{u}_n\}$, $n = 1, 2, \ldots$, $\boldsymbol{u}_n \in V_p(\Omega_n)$, $v_n \in \mathcal{U}_{ad}$ (where $\Omega_n \equiv \Omega(v_n)$) such that

$$\sum_{i=1}^{k} \|N_i(\boldsymbol{u}_n)\|_{0,\Omega_n}^2 < \frac{1}{n} \|\boldsymbol{u}_n\|_{W(\Omega_n)}^2 . \tag{2}$$

Without any loss of generality we can set

$$\|\boldsymbol{u}_n\|_{W(\Omega_n)} = 1 \quad \forall n . \tag{3}$$

Since the set $\mathcal{U}_{ad}$ is compact in $C([0, 1])$, we can find a subsequence (and denote it by the same symbol) of $\{v_n\}$, such that

$$v_n \to v \quad \text{in} \quad C([0, 1]) , \quad v \in \mathcal{U}_{ad} .$$

Then

$$\lim_{n\to\infty} \sum_{i=1}^{k} \|N_i(\boldsymbol{u}_n)\|_{0,\Omega_n}^2 = 0 \tag{4}$$

follows from (2), (3).

On the other hand, (3), (4) and $(H1)$ imply that

$$\|\boldsymbol{u}_n\|_{0,\Omega_n}^2 \geqq \tfrac{1}{2}\tilde{c} \tag{5}$$

holds for n sufficiently large.

Let us denote $\Omega(v_{1/m}) = G_m$. Since $G_m \subset \Omega_n$ for $n > n_0(m)$, we have

$$\|\boldsymbol{u}_n\|^2_{W(G_m)} \leqq \|\boldsymbol{u}_n\|^2_{W(\Omega_n)} = 1$$

and there exists a subsequence of $\{\boldsymbol{u}_n\}$ such that

$$\boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \quad \text{(weakly)} \quad \text{in} \quad W(G_m)\,. \tag{6}$$

By assumption $V_p^0(G_m)$ is weakly closed, so that $\boldsymbol{u} \in V_p^0(G_m)$ follows from (6) and $(H3)$, if m is large enough.

Let m be fixed. Since the functional

$$\boldsymbol{u} \to \sum_{i=1}^{k} \|N_i(\boldsymbol{u})\|^2_{0,G_m}$$

is differentiable and convex on $W(G_m)$, we have

$$\sum_{i=1}^{k} \|N_i(\boldsymbol{u})\|^2_{0,G_m} \leqq \liminf_{n\to\infty} \sum_{i=1}^{k} \|N_i(\boldsymbol{u}_n)\|^2_{0,G_m} = 0\,,$$

where (4) has been used in the end.

Since $v_{1/m} \in \mathcal{U}^E_{\text{ad}}$ for m great enough, (H2) implies

$$\boldsymbol{u} = 0 \quad \text{on} \quad G_m\,.$$

Consequently,

$$\boldsymbol{u}_n \to 0 \quad \text{in} \quad [L^2(G_m)]^r \tag{7}$$

follows from the weak convergence (6) and the compact embedding $H^1(G_m) \subset$ $\subset L^2(G_m)$.

On the other hand,

$$\|\boldsymbol{u}_n\|^2_{0,\Omega_n} = \|\boldsymbol{u}_n\|^2_{0,G_m} + \|\boldsymbol{u}_n\|^2_{0,\Omega_n - G_m} \tag{8}$$

holds for $n > n_0(m)$. We can derive the estimate

$$\|\boldsymbol{u}_n\|^2_{0,\Omega_n - G_m} \leqq c \max_{x_2 \in [0,1]} |v(x_2) - 1/m - v_n(x_2)|$$

(see [1] – Appendix), with c independent of n, m, v. Consequently,

$$\|\boldsymbol{u}_n\|^2_{0,\Omega_n - G_m} \leqq \tfrac{1}{4}\tilde{c} \tag{9}$$

holds for m and n sufficiently large, $n > n_0(m)$. Combining (8), (9) and (5), we obtain

$$\|\boldsymbol{u}_n\|^2_{0,G_m} \geqq \tilde{c}/4$$

for $n > n_0(m)$, m sufficiently large. Thus we arrive at a contradiction with (7).

2. APPLICATIONS TO SECOND ORDER EQUATIONS

In this section we present several applications of Theorem 1 to second order differential equations.

2.1. The generalized Friedrichs inequality

Let us choose $r = 1$, $k = 2$, $N_i(u) = \partial u/\partial x_i$, $i = 1, 2$, $W(\Omega(v)) = H^1(\Omega(v))$. Then the assumption $(H1)$ is obviously satisfied with $\tilde{c} = 1$. In what follows, we shall distinguish two classes of boundary conditions: (i) with Dirichlet condition on a part of the boundary, (ii) without Dirichlet conditions.

(i) Let us define

$$V_p(\Omega(v)) = \{u \in H^1(\Omega(v)) \mid u = 0 \text{ on } \Gamma_1(v)\}$$

where $\Gamma_1(v) \subset \partial\Omega(v) - \Gamma(v)$, $\operatorname{mes} \Gamma_1(v) \geqq a > 0$ for all $v \in \mathscr{U}_{\mathrm{ad}}$, with $a \in \mathbb{R}$ independent of v;

$$V_p^0(\Omega(v)) = \{u \in H^1(\Omega(v)) \mid u = 0 \text{ on } \Gamma_0(v)\}\,,$$

where $\Gamma_0(v) \subset \partial\Omega(v)$, $\operatorname{mes} \Gamma_0(v) > 0$.

Then $(H2)$ coincides with the well-known generalized Friedrichs inequality

$$\|\nabla u\|_{0,\Omega(v)}^2 \geqq c(v)\,\|u\|_{1,\Omega(v)}^2 \quad \forall u \in V_p^0(\Omega(v))\,, \tag{10}$$

which is true for all $v \in \mathscr{U}_{\mathrm{ad}}^E$. (Here the norm of $H^1(\Omega(v))$ stands in the right-hand side).

Obviously, the assumption $(H3)$ is satisfied, as well.

(ii) Let $p_v\colon H^1(\Omega(v)) \to \mathbb{R}$ be linear continuous functionals defined for $v \in \mathscr{U}_{\mathrm{ad}}^E$, such that

$$\{p_v(c) = 0,\ c \in \mathbb{R}\} \Rightarrow c = 0\,,$$

$$\{p_v(u) = 0,\ v \in \mathscr{U}_{\mathrm{ad}},\ u \in H^1(\Omega(v))\} \Rightarrow p_w(u|_{\Omega(w)}) = 0$$

$$\forall w \in \mathscr{U}_{\mathrm{ad}}^E,\ 0 < v - w < \varepsilon_0 \text{ on } [0, 1]\,.$$

For example, we can choose

$$p_v(u) = \textstyle\int_0^1 u(0, x_2)\,\mathrm{d}x_2\,.$$

Defining

$$V_p(\Omega(v)) = V_p^0(\Omega(v)) = \{u \in H^1(\Omega(v)) \mid p_v(u) = 0\}\,,$$

we easily verify $(H3)$. The satisfaction of $(H2)$ is well-known (see e.g. [2] – 11.3.1).

In both cases (i) and (ii), Theorem 1 yields the existence of a positive constant c, independent of v and such that

$$\|\nabla u\|_{0,\Omega(v)}^2 \geqq c\|u\|_{1,\Omega(v)}^2 \quad \forall u \in V_p(\Omega(v))\,, \quad \forall v \in \mathscr{U}_{\mathrm{ad}}\,. \tag{11}$$

2.2. The Korn's inequality in two-dimensional elasticity

Let us choose $r = 2$, $k = 3$, $N_1(\mathbf{u}) = \partial u_1/\partial x_1 = \varepsilon_{11}(\mathbf{u})$, $N_2(\mathbf{u}) = 1/\sqrt{2}(\partial u_1/\partial x_2 + \partial u_2/\partial x_1) = \sqrt{2}\,\varepsilon_{12}(\mathbf{u})$, $N_3(\mathbf{u}) = \partial u_2/\partial x_2 = \varepsilon_{22}(\mathbf{u})$, $W(\Omega(v)) = [H^1(\Omega(v))]^2$.

The condition $(H1)$ is a consequence of the "uniform" second Korn's inequality

(12) $$\int_{\Omega(v)} \sum_{i,j=1}^{2} \varepsilon_{ij}^2(\boldsymbol{u})\,dx + \sum_{i=1}^{2} \int_{\Omega(v)} u_i^2\,dx \geqq \tilde{c}_1 \int_{\Omega(v)} \sum_{i,j=1}^{2} \left(\frac{\partial u_i}{\partial x_j}\right)^2 dx\,.$$

The existence of a positive constant $\tilde{c}_1$, independent of $v \in \mathscr{U}_{ad}$, follows from the results of Nitsche (see [4], Section 3, Remark 5). (We use the fact, that there exists a covering and a partition of unity common for all $\Omega(v)$, $v \in \mathscr{U}_{ad}$.)

We shall distinguish three classes of boundary conditions:

(i) with Dirichlet condition on a part of the boundary,

(ii) with tractions given on the whole boundary (i.e. a "free body"),

(iii) with a "bilateral" contact on the x_2-axis.

Case (i). Let us define

$$V_p(\Omega(v)) = \{\boldsymbol{u} \in [H^1(\Omega(v))]^2 |\ \boldsymbol{u} = 0 \text{ on } \Gamma_1(v)\}\,,$$

where

$$\Gamma_1(v) \subset \partial\Omega(v) - \Gamma(v)\,, \quad \text{mes}\,\Gamma_1(v) \geqq a > 0$$

for all $v \in \mathscr{U}_{ad}$, with $a \in \mathbb{R}$ independent of v;

$$V_p^0(\Omega(v)) = \{\boldsymbol{u} \in [H^1(\Omega(v))]^2 |\ \boldsymbol{u} = 0 \text{ on } \Gamma_0(v)\}\,,$$

where $\Gamma_0(v) \subset \partial\Omega(v)$, mes $\Gamma_0(v) > 0$.

Then $(H2)$ coincides with the (first) Korn's inequality on the domain $\Omega(v)$, $v \in \mathscr{U}_{ad}^E$. The latter is guaranteed by the positive length of $\Gamma_0(v)$ (see e.g. [2], Lemma 3.2 in chapter 11.3.1). It is easy to see that the condition $(H3)$ is satisfied.

Case (ii). Let us define the set of rigid body displacements

$$\mathscr{P} = \{\boldsymbol{z} = (z_1, z_2)|\ z_1 = a_1 - bx_2,\ z_2 = a_2 + bx_1,\ a_1, a_2, b \in \mathbb{R}\}$$

and three linear continuous functionals

$$p_v^{(i)}: [H^1(\Omega(v)]^2 \to \mathbb{R}\,, \quad i = 1, 2, 3\,,$$

such that

(13) $$\{p_v^{(i)}(\boldsymbol{z}) = 0,\ i = 1, 2, 3,\ \boldsymbol{z} \in \mathscr{P}\} \Rightarrow \boldsymbol{z} = 0\,.$$

Moreover, let for $i = 1, 2, 3$ and any $v \in \mathscr{U}_{ad}$

(14) $$p_v^{(i)}(\boldsymbol{u}) = 0 \Rightarrow p_w^{(i)}(\boldsymbol{u}|_{\Omega(w)}) = 0$$

hold for all $\boldsymbol{u} \in [H^1(\Omega(v)]^2$, $w \in \mathscr{U}_{ad}^E$, $0 < v - w < \varepsilon_0$ on $[0, 1]$. For example, we can choose

$$p_v^{(i)}(\boldsymbol{u}) = \int_0^1 u_i(0, x_2)\,dx_2\,, \quad i = 1, 2\,,$$

$$p_v^{(3)}(\boldsymbol{u}) = \int_0^1 x_2 u_1(0, x_2)\,dx_2\,.$$

Let us define

$$V_p(\Omega(v)) = V_p^0(\Omega(v)) = \{\boldsymbol{u} \in [H^1(\Omega(v))]^2 |\ p_v^{(i)}(\boldsymbol{u}) = 0,\ i = 1, 2, 3\}\,.$$

Then the condition $(H2)$ is satisfied (see [3]). The condition $(H3)$ follows from (14).

Case (iii). Let us define

$$\mathscr{P} = \{\mathbf{z} = (0, a) | \, a \in R\} ,$$

and a linear continuous functional

$$p_v \colon [H^1(\Omega(v))]^2 \to R$$

such that

$$\{p_v(\mathbf{z}) = 0, \ \mathbf{z} \in \mathscr{P}\} \Rightarrow \mathbf{z} = 0 .$$

Moreover, let (14) hold for p_v. For example, we may choose

$$p_v(\mathbf{u}) = \int_0^1 u_2(0, x_2) \, dx_2 .$$

We define

$$V_p(\Omega(v)) = V_p^0(\Omega(v)) =$$
$$= \{\mathbf{u} \in [H^1(\Omega(v))]^2 | \, p_v(\mathbf{u}) = 0, \ u_1(0, x_2) = 0 \ \text{for} \ x_2 \in (0, 1)\} .$$

Then $(H2)$ and $(H3)$ are satisfied. Note that the condition

$$u_1(0, x_2) = u_n(0, x_2) = 0 , \quad x_2 \in (0, 1)$$

of the "bilateral contact" corresponds with the bodies symmetric with respect to the x_2-axis.

In all 3 cases considered above, Theorem 1 yields the uniform "first" Korn's inequality

$$(15) \qquad \sum_{i,j=1}^{2} \|\varepsilon_{ij}(\mathbf{u})\|_{0,\Omega(v)}^2 \geqq c\|\mathbf{u}\|_{W(\Omega(v))}^2, \ \forall \mathbf{u} \in V_p(\Omega(v)) ,$$

where the constant c is independent of $v \in \mathscr{U}_{\text{ad}}$.

Remark 2.1. If the displacements vanish on the variable part of the boundary, the following subspace has to be considered

$$(16) \qquad V_p = V(\Omega(v)) = \{\mathbf{u} \in [H^1(\Omega(v))]^2 | \, \mathbf{u} = 0 \ \text{on} \ \Gamma(v)\} .$$

We cannot apply Theorem 1, since the condition $(H3)$ is violated. There is, however, a simple proof of the uniform Korn's inequality and the result is even more general in a certain sense, as we can enlarge the set $\mathscr{U}_{\text{ad}}$ slightly.

Let us define the set

$$\mathscr{U}_{\text{ad}}^0 = \{v \in C^{(0),1}([0, 1]) | \, \alpha \leqq v \leqq \beta\}$$

with some positive constants α, β. Then the Korn's inequality (15) holds for all $\mathbf{u} \in V(\Omega(v))$ and $v \in \mathscr{U}_{\text{ad}}^0$, with a constant c independent of $v \in \mathscr{U}_{\text{ad}}^0$.

In fact, we may extend $\mathbf{u} \in V(\Omega(v))$ by zero to a fixed rectangular domain $\Omega_\delta = (0, \delta) \times (0, 1)$, where δ is any number greater than β. Obviously, the extension $\hat{\mathbf{u}}$

belongs to the subspace $V(\Omega_\delta)$ and we have the Korn's inequality

$$\sum_{i,j=1}^{2} \|\varepsilon_{ij}(\hat{\boldsymbol{u}})\|_{0,\Omega_\delta}^2 \geqq c_\delta \|\hat{\boldsymbol{u}}\|_{W(\Omega_\delta)}^2 \quad \forall \hat{\boldsymbol{u}} \in V(\Omega_\delta) .$$

Since $\hat{\boldsymbol{u}} \equiv 0$ outside $\Omega(v)$, the uniform Korn's inequality with $c = c_\delta$ follows.

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Souhrn

NEROVNOSTI KORNOVA TYPU, STEJNOMĚRNÉ VZHLEDEM KE TŘÍDĚ OBLASTÍ

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Nerovnosti Kornova typu obsahují kladnou konstantu, závislou obecně na oblasti. Vzniká otázka, zda-li tyto konstanty mají kladné infimum, když uvažujeme celou třídu omezených rovinných oblastí s lipschitzovskou hranicí. Dokazuje se kladná odpověď na tuto otázku pro několik typů okrajových podmínek a pro dvě třídy oblastí.

Резюме

НЕРАВЕНСТВА ТИПА КОРНА, РАВНОМЕРНЫЕ ПО ОТНОШЕНИЮ К ДАННОМУ КЛАССУ ОБЛАСТЕЙ

IVAN HLAVÁČEK

Неравенства типа Корна содержат положительную постояную, зависящую от области. Возникает вопрос, имеют ли эти постоянные положительную нижнюю грань, если рассматривать некоторый класс ограниченных областей в плоскости. Доказывается положительный ответ на этот вопрос для нескольких типов краевых условий и для двух классов областей.

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