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CLASSIFICATION OF PROJECTIVE SPACE MOTIONS WITH ONLY PLANE TRAJECTORIES

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Summary. The paper contains the solution of the classification problem for all motions in the complex projective space, which have only plane trajectories. It is shown that each such motion is a submanifold of a maximal motion with the same property. Maximal projective space motions with only plane trajectories are determined by special linear submanifolds of dimensions 2, 3, 5, 8 in $GL(4, C)$, they are denoted as R, $E_1, \ldots, E_6$, S_1, S_2 and given by explicit expressions.

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I. INTRODUCTION

The paper presents a solution of the open problem of classification of all projective motions in the projective space, which have only plane trajectories. First we show that any 1-parametric motion with the above mentioned property lies (locally) on an s-parametric motion with plane trajectories, $s > 1$. Then we find all such motions with maximal number of parameters. As the problem is of algebraic nature, we extend it to the complex domain, which enables us to use known facts from algebra and algebraic geometry. The list of all maximal motions with plane trajectories is given at the end of the paper.

II. REAL PROJECTIVE SPACE MOTIONS WITH PLANE TRAJECTORIES

Let $\bar{P}_3$, P_3 be two copies of the 3-dimensional real projective space, points of which are considered as 1-dimensional subspaces of real 4-dimensional vector spaces $\bar{V}_4$, V_4, respectively. Let us fix bases $\bar{R}_0 = \{\bar{\mathbf{e}}_i\}$, $R_0 = \{\mathbf{e}_i\}$, $i = 1, \ldots, 4$, in $\bar{V}_4$, V_4, respectively. A projective motion $g(t)$ in P_3 is given as a sufficiently differentiable 1-parameter family of projective maps from $\bar{P}_3$ into P_3, which can be represented as a curve on the Lie group $SL(4, \mathbf{R})$. Because there is a canonical homomorphism from $GL(4, \mathbf{R})$ onto $SL(4, \mathbf{R})$, we may consider $g(t)$ as a curve on $GL(4, \mathbf{R})$ as well. The matrix of the motion is then given up to a multiple of the unit matrix E.

Similarly as in [3] we call a projective space motion $g(t)$ a F_r motion, if all trajectories of the points of $\bar{P}_3$ lie in subspaces of dimension r and not all trajectories lie in subspaces of dimension $r - 1$. We call a projective motion $g(t)$ a D_k motion, if it satisfies a differential equation of the form $g^{(k+1)}(t) = \sum_{i=0}^{k} \alpha_i(t)\, g^{(i)}(t)$, where k is the least number with this property. Obviously, each D_k motion is an F_k motion, the converse being not always true.

All trajectories of a D_k motion are projective images of a fixed curve in the projective space P_k – this can be proved similarly as the corresponding affine statement in [3]. Especially it means that the D_2 motions are characterized as those F_2 motions which have projectively equivalent trajectories (for details see [3]).

By an s-parametric projective space motion we mean an immersion of an open set $M \subset \mathbf{R}^s$ in $SL(4, \mathbf{R})$. The definition of an F_2 motion extends immediately to s-parametric motions.

Definition 1. *An s-parametric F_2 motion g defined on M is called maximal, if there is no open subset N of M such that the restriction of g to N is a submanifold of an $(s + 1)$-parametric F_2 motion.*

Lemma 1. *There are no 1-parametric maximal F_2 projective space motions.*

Proof. Let $g(t)$ be defined on an open interval $\mathbf{I}$. Let us consider the following two cases:

a) There is $t_0 \in \mathbf{I}$ and $X_0 \in \bar{V}_4$ such that $g(t_0) X_0$, $g'(t_0) X_0$, $g''(t_0) X_0$ are linearly independent vectors. Then there is a neighbourhood $U(X_0)$ of X_0 in $\bar{V}_4$ such that $g(t_0) X$, $g'(t_0) X$, $g''(t_0) X$ are independent. Let us denote by $\xi(X)$ the plane which contains the trajectory of X. $\xi(X)$ is a linear form in the dual V_4^* of V_4; the pairing $V_4 \times V_4^* \to \mathbf{R}$ will be denoted by $\langle , \rangle$.

We have $\langle g(t) X, \xi(X) \rangle = 0$ for all $X \in \bar{V}_4$. Differentiation at t_0 gives $\langle g(t_0) X, \xi(X) \rangle = \langle g'(t_0) X, \xi(X) \rangle = \langle g''(t_0) X, \xi(X) \rangle = 0$ for all $X \in U(X_0)$. This shows that $g(t) X$, $g(t_0) X$, $g'(t_0) X$, $g''(t_0) X$ lie in the kernel of $\xi(X)$ for all $X \in U(X_0)$ and so the determinant

$$(1) \qquad |g(t_0) X, g'(t_0) X, g''(t_0) X, g(t) X| = 0$$

for all $X \in U(X_0)$.

As (1) is algebraic, it must be satisfied by all $X \in \bar{V}_4$. The solution of (1) it as least a 2-parametric motion, as it contains the 2-parametric motion $g(\lambda, \mu, \nu) = \lambda g(t_0) + \mu g'(t_0) + \nu g''(t_0)$.

b) $g(t) X$, $g'(t) X$, $g''(t) X$ are linearly dependent on $\mathbf{I}$. The trajectory $X(t) = g(t) \bar{X}$ of the point $\bar{X} \in \bar{V}_4$ therefore satisfies an equation of the form $\alpha X'' + \beta X' + \gamma X = 0$, where α, β, γ depend on t as well as on $\bar{X}$. (The bar which denotes that a point belongs to the moving space $\bar{P}_3$ was omitted in the above text to simplify the notation. This will be done in the sequel as well.)

We see that the trajectory $X(\mathrm{t})$ is locally a solution of a linear differential equation of the second order. It has the general solution of the form $X(\mathrm{t}) = \mu(\mathrm{t})\, Y + \nu(\mathrm{t})\, Z$, where Y and Z are constants, so the trajectory lies on a straight line and the motion is an F_1 motion.

Lemma 1 shows that each 1-parametric F_2 motion is locally a curve on an at least 2-parametric F_2 motion. This means that if we want to find all 1-parametric F_2 motions, it is enough to find all maximal F_2 motions, 1-parametric F_2 motions are then composed from curves which lie on maximal F_2 motions.

In what follows we shall describe all maximal projective F_2 motions in the projective space.

Lemma 2. *Any maximal F_2 motion $g(\mathrm{u}_i)$, $i = 1, \ldots, s$, satisfies an equation of the form*

$$|X, AX, BX, g(\mathrm{u}_i)\, X| = 0 \tag{2}$$

for all $X \in \overline{V}_4$, A, B are constant matrices, vertical bars denote the determinant.

Proof. The motion $g(\mathrm{u}_i)$ has only plane trajectories iff for each $X \in \overline{V}_4$ there exists $\xi \in V_4^*$ such that $\langle g(\mathrm{u}_i)\, X, \xi(X)\rangle = 0$ for all u_i. Now we proceed similarly as in the proof of Lemma 1.

a) Let us suppose that there exist $X_0 \in \overline{V}_4$ and u_i^0 such that $g(\mathrm{u}_i^0)\, X_0$, $\partial g/\partial \mathrm{u}_1\, (\mathrm{u}_i^0)\, X_0$, $\partial g/\partial \mathrm{u}_2\, (\mathrm{u}_i^0)\, X_0$ are linearly independent. Then there exists neighbourhood $U(X_0)$ of X_0 such that $g(\mathrm{u}_i^0)\, X$, $\partial g/\partial \mathrm{u}_1\, (\mathrm{u}_i^0)\, X$, $\partial g/\partial \mathrm{u}_2\, (\mathrm{u}_i^0)\, X$ are independent for all $X \in U(X_0)$. As $g(\mathrm{u}_i^0)$ is a regular matrix, we may suppose that $g(\mathrm{u}_i^0) = E$ by a change of the basis in $\overline{V}_4$. Let us denote $\partial g/\partial \mathrm{u}_1\, (\mathrm{u}_i^0) = A$, $\partial g/\partial \mathrm{u}_2\, (\mathrm{u}_i^0) = B$. Then X, AX, BX are independent for $X \in U(X_0)$ and we have $\langle g(\mathrm{u}_i)\, X, \xi(X)\rangle = 0$. Differentiation at u_i^0 yields $\langle X, \xi(X)\rangle = 0$, $\langle AX, \xi(X)\rangle = 0$, $\langle BX, \xi(X)\rangle = 0$. As X, AX, BX generate the kernel of $\xi(X)$, we must have (2) on $U(X_0)$. As (2) is algebraic, it must be satisfied for all $X \in \overline{V}_4$.

b) Let the dimension of the linear space generated by $g(\mathrm{u}_i)\, X$, $\partial g/\partial \mathrm{u}_j\, (\mathrm{u}_i)\, X$ be equal to 2 for any $X \in \overline{V}_4$ and any u_i. Then there is a point u_i^0 such that $g(\mathrm{u}_i^0)\, X_0$, $\partial g/\partial \mathrm{u}_1\, (\mathrm{u}_i^0)\, X_0$ are independent for some X_0 (we permute the parameters if necessary). This means that there are neighbourhoods $U(X_0)$ and $U'(\mathrm{u}_i^0)$ such that $g(\mathrm{u}_i)\, X$ and $\partial g/\partial \mathrm{u}_1\, (\mathrm{u}_i)\, X$ are independent. This shows that

$$\frac{\partial g}{\partial \mathrm{u}_j} X = \alpha_j(X, \mathrm{u}_i)\, g(\mathrm{u}_i)\, X + \beta_j(X, \mathrm{u}_i) \frac{\partial g}{\partial \mathrm{u}_1} (\mathrm{u}_i)\, X$$

for suitable functions α and β. For each X and u_i we now choose (in a smooth way) two elements $\xi_\alpha \in V_4^*$, $\alpha = 1, 2$, such that

$$\langle g(\mathrm{u}_i)\, X, \xi_\alpha(X, \mathrm{u}_i)\rangle = 0\,, \quad \left\langle \frac{\partial g}{\partial \mathrm{u}_1} (\mathrm{u}_i)\, X, \xi_\alpha(X, \mathrm{u}_i)\right\rangle = 0\,.$$

Partial differentiation of the first equation with respect to u_j yields

$$\left\langle g(u_i)\,X,\ \frac{\partial \xi_\alpha}{\partial u_j}(X, u_i)\right\rangle + \left\langle \frac{\partial g}{\partial u_j}(u_i)\,X,\ \xi_\alpha(X, u_i)\right\rangle = \left\langle g(u_i)\,X,\ \frac{\partial \xi_\alpha}{\partial u_j}(X, u_i)\right\rangle = 0\,.$$

As we may suppose that ξ_α are linearly independent, we get $\partial\xi_\alpha/\partial u_j = a_{j\alpha\beta}(X, u_i)\,\xi_\beta$, $\alpha, \beta = 1, 2$.

We now apply Lemma 3 and so $\xi_\alpha(X, u_i) = a_{\alpha\beta}(u_i)\,\varkappa_\beta(X)$. This yields $\langle g(u_i)\,X, \varkappa_\alpha(X)\rangle = 0$ for $\alpha = 1, 2$ and for all X and u_i on a neighbourhood and the motion is an F_1 motion.

Lemma 3. *Let* $\mathbf{e}_\alpha(u_i)$, $\alpha = 1, \ldots, m$, $i = 1, \ldots, s$, *be vector functions in* $\mathbf{R}^n$ *such that* $\partial\mathbf{e}_\alpha/\partial u_i = a_{i\alpha\beta}(u_j)\,\mathbf{e}_\beta$. *Then* $\mathbf{e}_\alpha = m_{\alpha\beta}(u_i)\,\mathbf{C}_\beta$ *for suitable constant vectors* $\mathbf{C}_\beta$ *and functions* $m_{\alpha\beta}(u_i)$, $\beta = 1, \ldots, m$.

Proof. For simplicity let us suppose that we have only two variables u and v and let us write symbolically $\partial\mathbf{e}/\partial u = A\mathbf{e}$, $\partial\mathbf{e}/\partial v = B\mathbf{e}$, where $\mathbf{e}$ denotes the column of vectors $\mathbf{e}_1, \ldots, \mathbf{e}_s$. For the k-th coordinate $\mathbf{e}^k$ we now have $\partial\mathbf{e}^k/\partial u = A\mathbf{e}^k$, $\partial\mathbf{e}^k/\partial v = B\mathbf{e}^k$. If we integrate this system with respect to u, we get for each v_0: $\mathbf{e}^k(u, v_0) = M(u, v_0)\,\mathbf{C}^k(v_0)$, where $M(u, v_0)$ is a fundamental solution of the system $dy(u)/du = A(u, v_0)\,y$. This shows that we can write $\mathbf{e}(u, v) = M(u, v)\,\mathbf{C}(v)$. Proceeding similarly with respect to v we have $\mathbf{e}(u, v) = N(u, v)\,\mathbf{D}(u)$. A comparison now yields $M(u, v)\,\mathbf{C}(v) = N(u, v)\,\mathbf{D}(u)$, $\mathbf{D}(u) = N^{-1}(u, v)\,M(u, v)\,\mathbf{C}(v) = N^{-1}(u, v_0)\,M(u, v_0)\,\mathbf{C}(v_0)$ and so $\mathbf{e}(u, v) = N(u, v)\,N^{-1}(u, v_0)\,M(u, v_0)\,\mathbf{C}(v_0)$. This proves that the vectors $\mathbf{e}_\alpha$ are linear combinations of fixed m vectors $\mathbf{C}(v_0)$. An extension to the case of more than two variables is obvious.

Theorem 1. *A maximal s parametric* F_2 *motion is an immersion of an open set of the projective space* P_s *into* $SL(4, \mathbf{R})$.

Proof. According to Lemma 2 any maximal F_2 motion must satisfy (2). Any maximal solution of (2) must be a linear space, as $|X, AX, BX, C_1X| = 0$ and $|X, AX, BX, C_2X| = 0$ implies $|X, AX, BX, (\alpha C_1 + \beta C_2)\,X| = 0$. As the set of all regular matrices in a linear subspace of matrices is open, the statement follows.

Theorem 1 shows that any maximal F_2 motion can be expressed as $g(u_i) = \sum_{i=1}^{s} u_i A_i$, where A_i are constant matrices. Further on, we see that any real maximal F_2 motion can be extended to a maximal complex F_2 motion in the complex projective space $P_3(\mathbf{C})$ by taking $u_i \in \mathbf{C}$. Therefore we have

Theorem 2. *Any real* 1*-parametric* F_2 *motion is a real curve on a complex maximal* F_2 *motion in* $P_3(\mathbf{C})$.

III. SOLUTION OF THE EQUATION $|X, AX, BX, CX| = 0$

To classify all maximal complex F_2 motions in $P_3(\mathbf{C})$ we have to solve the equation

$$(3) \qquad |X, AX, BX, CX| = 0$$

for all $X \in V_4(\mathbf{C})$, where A, B, C are arbitrary 4×4 complex matrices.

Remark. In the course of the classification process we have to use some facts which are true only for an algebraically closed base field. Therefore the classification of complex F_2 motions appears to be more natural and easier than the same problem over **R**.

(3) has an obvious solution $C = \lambda E + \mu A + \nu B$. Let us call this solution regular and let us denote it by R. Further on, (3) has a solution such that $\operatorname{Im}(A) \supset \operatorname{Im}(B) \supset \operatorname{Im}(C)$, $\dim \operatorname{Im}(A) = 2$, where $\operatorname{Im}(A)$ denotes the image of the map $\mathbf{C}^4 \to \mathbf{C}^4$ which is determined by the matrix A. Let us call this solution a singular solution of rank 2 and denote it by S_2.

Similarly, if $\operatorname{Im}(A) = \operatorname{Im}(B)$, $\dim \operatorname{Im}(A) = 1$, C is arbitrary, we obtain also a solution of (3). Let us call it a singular solution of rank 1 and denote it by S_1. In what follows we have to determine all the other solutions of (3); we shall call them exceptional solutions. We start with several preliminary lemmas.

Lemma 4. *Let $\mathscr{F}(X) . \mathscr{G}(X) = 0$ for all $X \in V_4(\mathbf{C})$, where $\mathscr{F}$ and $\mathscr{G}$ are homogeneous polynomials in coordinates of X. Then either $\mathscr{F}(X) = 0$ for all X or $\mathscr{G}(X) = 0$ for all X.*

Lemma 5. *Let $A = (a_{\alpha i})$, $B = (b_{\alpha i})$, $\alpha = 1, 2$, $i = 1, \ldots, 4$. Then $|AX, BX| = 0$ for all X iff A and B are linearly dependent or* rank $(A \mid B) = 1$, *where $A \mid B$ means that we put matrices A and B next to each other.*

Remark. To simplify the notation, let us denote in what follows the i-th row of the matrix AX by α_i and similarly for B and C using β and γ.

Proof. We may suppose $a_{11} \neq 0$. Then we may change A and B in such a way that $a_{21} = 0$, $b_{11} = 0$ by adding a combination of the 1-st row to the 2-nd and adding a multiple of A to B. Then necessarily $\beta_2 = 0$. Now $\alpha_2 . \beta_1 = 0$ and according to Lemma 4 we have $\alpha_2 = 0$ or $\beta_1 = 0$.

Lemma 6. *Let E, A, B be linearly independent* 3×3 *complex matrices, E the unit matrix. Then the equation $|X, AX, BX| = 0$ for all X has only the following solution: There exist $\lambda, \mu \in \mathbf{C}$ such that* $\operatorname{rank}(A - \lambda E) = 1$, $\operatorname{Im}(A - \lambda E) = \operatorname{Im}(B - \mu E)$.

Proof. Let us denote $m_0 = \text{Min rank}(\lambda E + \mu A + \nu B)$ for all $\lambda, \mu, \nu \in \mathbf{C}$, $|\lambda| + |\mu| + |\nu| \neq 0$. Then $0 < m_0 < 3$. For A we can take the matrix with the smallest rank. Then we have two possibilities:

a) $m_0 = 1$. In a suitable basis we have $\alpha_1 = \alpha_2 = 0$. As $\alpha_3 \neq 0$, we get

$$\begin{vmatrix} x, & \beta_1 \\ y, & \beta_2 \end{vmatrix} = 0 .$$

As the rank of the matrix $(E \mid b_{\alpha i})$, $\alpha = 1, 2$, $i = 1, 2, 3$ is equal to 2, we have $b_{11} = b_{22}$, $b_{\alpha i} = 0$ for $\alpha \neq i$ and this gives the solution from the statement.

b) $m_0 = 2$. We may suppose $\alpha_1 = 0$. Let $x = 0$. Then we obtain $\beta_1(y\alpha_3 - z\alpha_2) = 0$ with $x = 0$.

1) $\beta_1 \neq 0$ with $x = 0$. Then $a_{22} = a_{33}$, $a_{23} = a_{32} = 0$, and adding a multiple of E we obtain a matrix of rank 1, which is impossible.

2) $\beta_1 = 0$ with $x = 0$. Then $\alpha_2\beta_3 - \alpha_3\beta_2 = 0$. As rank $A = 2$, B must be a multiple of A, which is a contradiction.

Remark. For the sake of simplicity we denote the coordinates in $V_4(\mathbf{C})$ by x, y, z, t.

Lemma 7. *The group* Int $GL(4, \mathbf{C})$ *takes solutions of* (3) *into solutions.*

Proof. Let $g \in GL(4, \mathbf{C})$. Then $|X, AX, BX, CX| = 0$ implies $|gX, gAg^{-1} . gX, gBg^{-1} . gX, gCg^{-1} . gX| = 0$ and we write $gX = Y$.

Now we are ready to start with (3). We may suppose that E, A, B, C are linearly independent matrices (as vectors in the matrix algebra of 4×4 matrices). Let us denote by $\pi_3(A, B, C)$ the projective space generated by matrices E, A, B, C: $\pi_3(A, B, C) = \{$the set of all nontrivial linear combinations of E, A, B, C with identified multiples$\}$. Further, let us denote $\mathrm{m}(\pi_3) = \min \text{rank } D$, where $D \in \pi_3(A, B, C)$.

Lemma 8. $1 \leqq \mathrm{m}(\pi_3) \leqq 3$.

Proof. The equation $|A - \lambda E| = 0$ has at least one solution

Lemma 9. *Let* $\mathrm{m}(\pi_3) = 3$. *Then* (3) *has no solution.*

Proof. In π_3 there exists a matrix which has 0 as a characteristic root of multiplicity 3. To see this, consider the equation $|\lambda E + \mu A + \nu B + \varrho C - \xi E| = 0$. If we expand this equation with respect to ξ, we get $\xi^4 + f_1\xi^3 + f_2\xi^2 + f_3\xi + f_4 = 0$, where f_i is a form of degree i in the variables $\lambda, \mu, \nu, \varrho$. If we put $f_2 = f_3 = f_4 = 0$, we get 3 algebraic surfaces in π_3, which have at least one common point according to the classical theorem of Bézout.

We take such a matrix for the matrix A and put it into the canonical Jordan form according to Lemma 7. As A has rank 3, we have only two possibilities:

$$1)\quad A = \begin{pmatrix} 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 1 \end{pmatrix} \qquad 2)\quad A = \begin{pmatrix} 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \\ 0, & 0, & 0, & 0 \end{pmatrix}.$$

First we consider the case 1). Let us write

$$B = \begin{pmatrix} B_1, & B_2 \\ B_3, & B_4 \end{pmatrix},$$

where B_1 is a 3×3 matrix. The group

$$G_1 = \begin{pmatrix} g, & 0 \\ 0, & 1 \end{pmatrix}, \quad \text{where} \quad g = \begin{pmatrix} 1, & \alpha, & \beta \\ 0, & 1, & \alpha \\ 0, & 0, & 1 \end{pmatrix},$$

preserves the Jordan normal form of A. For the change of B by G_1 we have $\tilde{B}_3 = (\tilde{b}_{41}, \tilde{b}_{42}, \tilde{b}_{43}) = B_3 \cdot g = (b_{41}, b_{41}\alpha + b_{42}, b_{41}\beta + b_{42}\alpha + b_{43})$.

The same result is obtained in the case 2) with the group

$$G_2 = \begin{pmatrix} g, & t \\ 0, & 1 \end{pmatrix}, \quad \text{where} \quad t = (\gamma, \beta, \alpha)^{\mathrm{T}}.$$

Now we combine the both cases.

Let $\varepsilon_1, \varepsilon_2$ be 1 or 0 with $\varepsilon_1^2 + \varepsilon_2^2 = 1$. Then (3) can be written as follows:

$$\begin{vmatrix} x, & y, & b_{13}z + b_{14}t, & c_{13}z + c_{14}t \\ y, & z, & \beta_2, & \gamma_2 \\ z, & \varepsilon_1 t, & \beta_3, & \gamma_3 \\ t, & \varepsilon_2 t, & \beta_4, & \gamma_4 \end{vmatrix} = 0.$$

a) Let $|b_{41}| + |c_{41}| \neq 0$. We may suppose $b_{41} \neq 0$, using the group G_1 or G_2 we may change B to $b_{42} = b_{43} = 0$, adding a multiple of B to C we obtain $c_{41} = 0$.

ai) $|c_{42}| + |c_{43}| \neq 0$. For $x = t = 0$ we obtain

$$\begin{vmatrix} 0, & y, & b_{13}z \\ y, & z, & b_{22}y + b_{23}z \\ z, & 0, & b_{32}y + b_{33}z \end{vmatrix} = 0.$$

This yields $b_{13} = b_{23} = b_{32} = 0$, $b_{22} = b_{33}$. We add a multiple of E to B and obtain a matrix of rank 2, which is a contradiction.

aii) $c_{42} = c_{43} = 0$. For $t = 0$ we have

$$\begin{vmatrix} x, & y, & c_{13}z \\ y, & z, & c_{21}x + c_{22}y + c_{23}z \\ z, & 0, & c_{31}x + c_{32}y + c_{33}z \end{vmatrix} = 0. \tag{4}$$

Using Lemma 6 we get rank $C \leqq 1$, which is a contradiction.

b) $b_{41} = c_{41} = 0$, $|b_{42}| + |c_{42}| \neq 0$, let $b_{42} \neq 0$. We may suppose $b_{43} = c_{42} = 0$ similarly as above.

bi) Let $c_{43} \neq 0$. Then $y = t = 0$ yields $b_{31} = b_{13} = b_{33} = 0$, $z = t = 0$ yields $b_{32} = 0$ and $t = 0$ now gives $c_{43} = 0$, a contradiction.

bii) Let $c_{43} = 0$. Then $t = 0$ gives (4) and rank $C \leqq 1$.

c) Let $b_{41} = b_{42} = c_{41} = c_{42} = 0$, $b_{43} \neq 0$, $c_{43} = 0$. We get a contradiction similarly as above.

d) $b_{4i} = c_{4i} = 0$, $i = 1, 2, 3$. We add a suitable multiple of the first column to the other columns to have zero in the last row. Then

$$\begin{vmatrix} y - \varepsilon_2 x, & \beta_1, & \gamma_1 \\ z - \varepsilon_2 y, & \beta_2, & \gamma_2 \\ \varepsilon_1 t - \varepsilon_2 z, & \beta_3, & \gamma_3 \end{vmatrix} = 0 . \tag{5}$$

If $\varepsilon_1 = 1$, $\varepsilon_2 = 0$, we put $x = 0$ and use Lemma 6 to see that the rank of B is at most 2. If $\varepsilon_1 = 0$, $\varepsilon_2 = 1$, we put $t = 0$. The first matrix in (5) is regular; denote it by D. We multiply (5) by D^{-1} from the left and use Lemma 6 to see that rank $B \leqq 2$. This completes the proof.

Lemma 10. *If* $\mathrm{m}(\pi_3) = 1$, *then any solution of* (3) *is either* S_1 *or a special case of* S_2.

Proof. We may suppose $\alpha_1 = \alpha_2 = \alpha_3 = 0$. As $\alpha_4 \neq 0$, we have

$$\begin{vmatrix} x, & \beta_1, & \gamma_1 \\ y, & \beta_2, & \gamma_2 \\ z, & \beta_3, & \gamma_3 \end{vmatrix} = 0 .$$

We put $t = 0$ and use Lemma 6.

a) $c_{ij} = \lambda \delta_{ij} + \mu b_{ij}$, $i, j = 1, 2, 3$. We may suppose $c_{ij} = 0$ by adding $-(\lambda E + \mu B)$ to C. What remains is

$$\begin{vmatrix} x, & \beta_1, & c_{14} \\ y, & \beta_2, & c_{24} \\ z, & \beta_3, & c_{34} \end{vmatrix} = 0 ,$$

If $c_{i4} = 0$ for $i = 1, 2, 3$, we have S_1. If one of c_{i4} is nonzero, we may change the basis to have $c_{34} \neq 0$, $c_{14} = c_{24} = 0$. Then $x\beta_2 - y\beta_1 = 0$ and by Lemma 5 we obtain that the solution is a special case of S_2.

b) $\mathrm{Im}(\tilde{B}) = \mathrm{Im}(\tilde{C})$, $\dim \mathrm{Im}(\tilde{B}) = 1$, where $\tilde{B}$ and $\tilde{C}$ are restrictions of B and C to indices 1, 2, 3. In a suitable basis we may write

$$\begin{vmatrix} x, & b_{14}t, & c_{14}t \\ y, & b_{24}t, & c_{24}t \\ z, & \beta_3, & \gamma_3 \end{vmatrix} = 0 .$$

Let all $b_{i4}, c_{i4}, i = 1, 2$ be zero. Then we have a special case of S_2. So, let one of them be nonzero. We may suppose that $b_{14} \neq 0$, $c_{14} = 0$. Then $c_{24} = 0$, $\gamma_3 = 0$ and we have a special case of S_1.

Lemma 11. *Let* $\mathrm{m}(\pi_3) = 2$. *Then any solution of* (3) *is either* S_2 *or one of the exceptional solutions* $E_1, \ldots, E_6$ *which are listed in the proof.*

Proof. We have to use similar arguments as in the proof of the preceding two lemmas. Having in mind that computations of this kind are not very interesting, we omit the details whenever possible.

Let us suppose $\alpha_1 = \alpha_2 = 0$ and let us write B in the form

$$B = \begin{pmatrix} K, & L \\ M, & N \end{pmatrix}, \quad \text{where } K, L, M, N \text{ are } 2 \times 2 \text{ matrices}.$$

The group

$$G = \begin{pmatrix} g_1, & 0 \\ g_2, & g_3 \end{pmatrix},$$

where g_i are 2×2 matrices, g_1 and g_3 regular, preserves the equation $\alpha_1 = \alpha_2 = 0$. For the change of B by elements of G we have

$$(6) \qquad \tilde{K} = \tilde{g}_1 K g_1 + \tilde{g}_1 L g_2, \quad \tilde{L} = \tilde{g}_1 L g_3, \quad \tilde{M} = (\tilde{g}_2 K + \tilde{g}_3 M) g_1 + (\tilde{g}_2 L + \tilde{g}_3 N) g_2, \quad \tilde{N} = \tilde{g}_2 L g_3 + \tilde{g}_3 N g_3,$$

where $\tilde{g}_i$ denotes the corresponding matrix of the inverse element in G.

Let us denote

$$P = \begin{pmatrix} c_{13}, & c_{14} \\ c_{23}, & c_{24} \end{pmatrix}, \quad Q = \begin{pmatrix} c_{11}, & c_{12} \\ c_{21}, & c_{22} \end{pmatrix}.$$

I. Let $L = P = 0$. We may suppose that K is in the normal Jordan form, so $b_{21} = b_{12} b_{22} = b_{11} = c_{11} = 0$. Computation shows that we get a solution only if $b_{12} \neq 0$. Then $b_{22} = c_{12} = 0$. For $x = 0$ we have

$$\begin{vmatrix} a_{32}y + a_{33}z + a_{34}t, & c_{32}y + (c_{33} - c_{22})z + c_{34}t \\ a_{42}y + a_{43}z + a_{44}t, & c_{42}y + c_{43}z + (c_{44} - c_{22})t \end{vmatrix} = 0.$$

The first matrix is not zero. If the second is a multiple of the first, then rank $C \leqq 1$. So let the 3-rd row be a multiple of the fourth. Then

$$\begin{vmatrix} x, & 0, & b_{12}y, & c_{11}x \\ y, & 0, & 0, & c_{21}x \\ z + \alpha t, & a_{31}x, & \beta_3, & c_{31}x \\ t, & \alpha_4, & \beta_4, & \gamma_4 \end{vmatrix} = 0.$$

As $a_{31} \neq 0$, we may suppose $b_{31} = c_{31} = 0$.

a) Let $c_{21} \neq 0$. Then for $y = 0$ we have

$$\begin{vmatrix} a_{31}x, & b_{33}z + b_{34}t \\ a_{41}x + a_{43}z + a_{44}t, & b_{41}x + b_{43}z + b_{44}t \end{vmatrix} = 0 .$$

If the second matrix is a multiple of the first, we have rank $B \leqq 1$, so the fourth row is a multiple of the third and we have the solution

(E₁)
$$\begin{vmatrix} x, & 0, & a_{42}y, & c_{11}x \\ y, & 0, & 0, & x \\ z, & x, & -t, & 0 \\ t, & a_{42}y, & 0, & c_{11}t - a_{42}z \end{vmatrix} = 0 .$$

b) Let $c_{21} = 0$. Similarly as above we obtain

(E₂)
$$\begin{vmatrix} x, & 0, & -y, & x \\ y, & 0, & 0, & 0 \\ z, & x, & \beta_3, & 0 \\ t, & a_{42}y, & 0, & a_{42}\beta_3 \end{vmatrix} = 0 .$$

Now we have to consider the case when not both L and P are zero. We have two possibilities:

a) There exists a matrix P_1 such that $P_1 = \lambda L + \mu P$ and $\det P_1 = 0$.

b) There is not such a matrix P_1. Then L and P are linearly dependent and we may suppose $L = 0$. P must be regular.

II. Let P be regular, $L = 0$. For the action of G on K, P, Q we obtain $\tilde{K} = \tilde{g}_1 K g_1$, $\tilde{Q} = g_1(Qg_1 + Pg_2)$, $\tilde{P} = \tilde{g}_1 P g_3$.

We change K to the Jordan normal form and fix g_1. As P is regular, we can change Q to 0 by g_2 and choose g_3 to have $\tilde{P} = E$. Now (3) yields

$$\begin{vmatrix} x, & 0, & b_{12}y, & z \\ y, & 0, & b_{22}y, & t \\ z, & \alpha_3, & \beta_3, & \gamma_3 \\ t, & \alpha_4, & \beta_4, & \gamma_4 \end{vmatrix} = 0 , \quad \text{where} \quad b_{12} . b_{22} = 0 .$$

a) $b_{12} \neq 0$, $b_{22} = 0$. After some computation we obtain the solution

(E₃)
$$\begin{vmatrix} x, & 0, & y, & z \\ y, & 0, & 0, & t \\ z, & x, & b_{32}y - t, & b_{32}z \\ t, & y, & 0, & b_{32}t \end{vmatrix} = 0 .$$

b) $b_{12} = 0$ leads to no solution.

III. Let L be singular and different from 0. Using the group G we may change B in such a way that $b_{13} = 1$, $b_{12} = b_{14} = b_{23} = b_{24} = b_{43} = 0$. For $x = y = 0$ we obtain

$$(c_{23}z + c_{24}t) \begin{vmatrix} z, & a_{33}z + a_{34}t \\ t, & a_{43}z + a_{44}t \end{vmatrix} = 0 .$$

1) $|c_{23}| + |c_{24}| \neq 0$. Let us suppose $a_{33} = 1$.

α) $|c_{12}| + |c_{14}| \neq 0$. This case yields the solution S_2.

β) $c_{12} = c_{14} = 0$. Then $b_{21} \neq 0$ yields S_2, so $b_{21} = 0$.

i) $c_{24} \neq 0$. Here we obtain the following solution:

$$(\mathrm{E}_4) \qquad \begin{vmatrix} x, & x, & z + b_{22}x, & 0 \\ y, & y, & b_{22}y, & t \\ z, & 0, & 0, & b_{42}x \\ t, & 0, & b_{42}y, & 0 \end{vmatrix} = 0 .$$

ii) $c_{24} = 0$ yields the solution $(b_{32} \neq 0)$

$$(\mathrm{E}_5) \qquad \begin{vmatrix} x, & x, & b_{33}x + a_{32}y + z, & 0 \\ y, & y, & b_{22}y, & (b_{33} - b_{22})x + a_{32}y + z \\ z, & 0, & b_{32}y, & -b_{32}x \\ t, & 0, & b_{42}y, & -b_{42}x \end{vmatrix} = 0 .$$

$b_{32} = 0$ leads to a special case of E_5 or to S_2.
$a_{33} = 0$ yields no solution at all.

2) $c_{24} = c_{23} = 0$, $a_{43} = 1$. Then $b_{21} \neq 0$ leads to no solution, so $b_{21} = 0$. We have

$$\begin{vmatrix} 0, & z - b_{22}x, & c_{12}y + c_{14}t - c_{22}x \\ a_{31}x + a_{32}y + a_{34}t, & \beta_3, & \gamma_3 \\ a_{41}x + a_{42}y + z + a_{44}t, & b_{41}x + b_{42}y + b_{44}t, & c_{41}x + c_{42}y + c_{44}t \end{vmatrix} = 0 ,$$

We add a multiple of the first row to the second to have $b_{33} = 0$. Using the coefficients at z^2 and z^3, we obtain $\gamma_3 = 0$. $z = b_{22}x$ yields

$$\begin{vmatrix} a_{31}x + a_{32}y + a_{34}t, & b_{31}x + b_{32}y + b_{34}t \\ (a_{41} + b_{22})x + a_{42}y + a_{44}t, & b_{41}x + b_{42}y + b_{44}t \end{vmatrix} = 0 .$$

If the second matrix is a multiple of the first, we obtain a special case of S_2, if the second row is a multiple of the first, we obtain

$$(\mathrm{E}_6) \qquad \begin{vmatrix} x, & 0, & \alpha_4, & \alpha_3 \\ y, & 0, & 0, & 0 \\ z, & \alpha_3, & \beta_3, & 0 \\ t, & \alpha_4, & 0, & -\beta_3 \end{vmatrix} = 0 .$$

3) $c_{23} = c_{24} = a_{43} = 0$. A rather long computation shows that we get no new solutions. This completes the proof.

IV. MAXIMAL F_2 MOTIONS

For each solution of (3) we now construct a maximal F_2 motion. Let us discuss the individual cases separately.

1) The regular solution R: let A, B be arbitrary. Then $g(\lambda, \mu, \nu) = \lambda E + \mu A + \nu B$ is in general a maximal 2-dimensional F_2 motion. Let us denote by $\mathrm{D}(A, B)$ the subset of $P_2(\mathbf{C})$ given by the equation $\det g(\lambda, \mu, \nu) = 0$. Then $\mathrm{D}(A, B)$ is a curve of degree four in $P_2(\mathbf{C})$ and $g(\lambda, \mu, \nu)$ is an immersion of the open set $P_2 - \mathrm{D}(A, B)$ in $SL(4, \mathbf{C})$. A similar remark applies to the other cases as well.

2) S_1: the corresponding maximal F_2 motion is

$$g(\lambda, \mu, \nu_1, \ldots, \nu_4) = \lambda E + \mu A + \begin{pmatrix} 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ \nu_1, & \nu_2, & \nu_3, & \nu_4 \end{pmatrix},$$

where A is arbitrary; we get a 5-dimensional motion.

3) S_2: we obtain an 8-dimensional maximal F_2 motion

$$g(\lambda, \mu_1, \ldots, \mu_8) = \lambda E + \begin{pmatrix} 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ \mu_1, & \mu_2, & \mu_3, & \mu_4 \\ \mu_5, & \mu_6, & \mu_7, & \mu_8 \end{pmatrix}.$$

4) E_1: let us have the 3-dimensional motion $g(\lambda, \mu, \nu, \varrho) = \lambda E + \mu A + \nu B + \varrho C$ for given constants a_{42} and c_{11}. Let us suppose that g is immersed in a motion g_1 which has greater dimension, then g has. Let g contain an element $D = \mu_1 A_1 + \nu_1 B_1 + \varrho_1 C_1$, where A_1, B_1, C_1 are similarly defined by constants a'_{42}, c'_{11}. Then we must have $|X, AX, BX, DX| = 0$ for all X and so we may suppose $\mu_1 = \nu_1 = 0$, $a'_{42} = a_{42}$. From the equation $|X, AX, CX, C_1X| = 0$ we obtain $c'_{11} = c_{11}$. This shows that g is maximal.

5) $E_2, \ldots, E_6$: in these cases C is uniquely determined by A and B, so we have a 3-dimensional maximal F_2 motion $g = \lambda E + \mu A + \nu B + \varrho C$.

Theorem 2. *The maximal F_2 motions in $P_3(\mathbf{C})$ are: the 2-dimensional motion R, 3-dimensional motions $E_1, \ldots, E_6$, the 5-dimensional motion S_1 and the 8-dimensional motion S_2.*

Remark. To get a 1-dimensional F_2 motion we have to choose a curve on a maximal F_2 motion (all parameters are functions of one variable). By this procedure we in general obtain from an s-dimensional F_2 motion a D_s motion, as it depends on s arbitrary functions. If we choose all matrices to be real, we get real F_2 motions in $P_3(\mathbf{R})$. The problem of determination of all real forms of the F_2 motions in Theorem 2 remains open.

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Souhrn

KLASIFIKACE PROJEKTIVNÍCH PROSTOROVÝCH POHYBŮ S POUZE ROVINNÝMI TRAJEKTORIEMI

ADOLF KARGER

V článku je vyřešen problém klasifikace všech projektivních pohybů v komplexním projektivním prostoru, které mají všechny trajektorie rovinné. Je dokázáno, že každý takový pohyb je podvarietou některého z maximálních projektivních pohybů s pouze rovinnými trajektoriemi.

Tyto maximální pohyby jsou určeny speciálními rovinnými podprostory dimensí 2, 3, 5 a 8 v prostoru $GL(4, \mathbf{C})$. Jsou označeny jako R, $E_1, \ldots, E_6$, S_1, S_2 a je pro ně nalezeno explicitní maticové vyjádření.

Резюме

КЛАССИФИКАЦИЯ ПРОЕКТИВНЫХ ДВИЖЕНИЙ С ТОЛЬКО ПЛОСКИМИ ТРАЕКТОРИЯМИ

ADOLF KARGER

Статья занимается классификацией движений в комплексном проективном пространстве, у которых только плоские тракетории. В статье показано, что каждое такое движение есть подмногообразие одного из максимальных движений с плоскими траекториями. Эти максимальные движения обозначены R, $E_1, \ldots, E_6$, S_1, S_2, имеют размерности 2, 3, 5, 8 и найдены их матричные представления.

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