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KORN'S INEQUALITY UNIFORM WITH RESPECT TO A CLASS OF AXISYMMETRIC BODIES

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Summary. The Korn's inequality involves a positive constant, which depends on the domain, in general. We prove that the constants have a positive infimum, if a class of bounded axisymmetric domains and axisymmetric displacement fields are considered.

Keywords: domain optimization, shape optimization, Korn's inequality

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INTRODUCTION

In the shape optimization of elastic bodies we encounter the following question: does a positive constant of the ellipticity condition exist, which is common for the whole class of admissible bodies? The positive answer is crucial in proving the existence of an optimal body. The present paper is devoted to the question in case of axisymmetric problems. We have to work with weighted Sobolev spaces, which stem naturally by transforming the displacement vectors with finite energy into cylindrical coordinate system.

In Section 1 we formulate the Korn's inequalities in the cylindrical coordinate system, restricting moreover the space of displacement functions with finite energy to the subspace of axisymmetrical vectors. Following some ideas of Haslinger, Neittaanmäki and Tiihonen [1], we prove the uniform Korn's inequality for different kinds of boundary conditions in Section 2.

1. FORMULATION OF KORN'S INEQUALITIES IN CYLINDRICAL COORDINATE SYSTEM

We shall consider a bounded body occupying an axisymmetric domain $\Omega \subset R^3$ with Lipschitz boundary (see e.g. [4] – chapter 1). The displacement vector $\boldsymbol{u} = (u_1, u_2, u_3)$ belongs to the space $W(\Omega)$ of functions with finite energy if each component u_i in the Cartesian coordinate system $\mathbf{x} = (x_1, x_2, x_3)$ belongs to the Sobolev space $H^1(\Omega)$ (see [4] – chapt. 6). Let us denote by

$$\varepsilon_{ij}(\mathbf{u}) = \tfrac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$$

the strain tensor components and let us introduce

$$\|\varepsilon(\mathbf{u})\|_\Omega^2 = \int_\Omega \sum_{i,j=1}^{3} \varepsilon_{ij}^2(\mathbf{u})\,\mathrm{d}x\,.$$

It is well-known, that the following inequality

$$(1) \qquad \|\varepsilon(\mathbf{u})\|_\Omega^2 + \|\mathbf{u}\|_{0,\Omega}^2 \geqq c(\Omega)\,\|\mathbf{u}\|_{W(\Omega)}^2$$

holds for all $\mathbf{u} \in W(\Omega) \equiv [H^1(\Omega)]^3$.

Henceforth $\|\cdot\|_{0,\Omega}$ denotes the norm in $[L^2(\Omega)]^3$. For the proof of (1) see e.g. [4] – chapt. 6, where we called it "coerciveness of strains". In the literature it is also called Korn's second inequality.

Assume that the domain Ω is generated by rotation of a two-dimensional domain D around the $z \equiv x_3$ axis (cf. [4] – 7.6.1 and 10.3). Let us pass to cylindrical coordinates r, ϑ, z.

Let Z map each vector function $\mathbf{u} \in W(\Omega)$, defined in Cartesian coordinates, onto the ordered triple $Z\mathbf{u} = \hat{\mathbf{u}} = (u_r, u_\vartheta, u_z)$ of the physical components of the same vector at the corresponding point (r, ϑ, z).

Then the space $W(\Omega)$ is transformed into

$$ZW(D \times [0, 2\pi))\,.$$

Denoting

$$u_r = u\,, \quad u_\vartheta = v\,, \quad u_z = w\,,$$

the inequality (1) can be transformed – via mapping Z – into the following inequality

$$(2) \quad \begin{aligned} &\textstyle\int_0^{2\pi} \mathrm{d}\vartheta \int_D [\varepsilon_{rr}^2(\hat{\mathbf{u}}) + \varepsilon_{\vartheta\vartheta}^2(\hat{\mathbf{u}}) + \varepsilon_{zz}^2(\hat{\mathbf{u}}) + 2\varepsilon_{r\vartheta}^2(\hat{\mathbf{u}}) + 2\varepsilon_{rz}^2(\hat{\mathbf{u}}) + 2\varepsilon_{\vartheta z}^2(\hat{\mathbf{u}})]\, r\,\mathrm{d}r\,\mathrm{d}z\, + \\ &+ \textstyle\int_0^{2\pi} \mathrm{d}\vartheta \int_D (u^2 + v^2 + w^2)\, r\,\mathrm{d}r\,\mathrm{d}z \geqq c(\Omega) \int_0^{2\pi} \mathrm{d}\vartheta \int_D [u^2 + v^2 + w^2 \\ &+ (\partial u/\partial r)^2 + (\partial w/\partial z)^2 + r^{-2}(\partial v/\partial\vartheta + u)^2 + (\partial v/\partial r)^2 + \\ &+ r^{-2}(\partial u/\partial\vartheta - v)^2 + (\partial w/\partial r)^2 + \\ &+ (\partial u/\partial z)^2 + r^{-2}(\partial w/\partial\vartheta)^2 + (\partial v/\partial z)^2]\, r\,\mathrm{d}r\,\mathrm{d}r\,, \end{aligned}$$

where the last integral defines the norm of $\hat{\mathbf{u}}$ in ZW,

$$\varepsilon_{rr}(\hat{\mathbf{u}}) = \partial u/\partial r\,, \quad \varepsilon_{zz}(\hat{\mathbf{u}}) = \partial w/\partial z\,, \quad \varepsilon_{\vartheta\vartheta}(\hat{\mathbf{u}}) = r^{-1}(u + \partial v/\partial\vartheta)\,,$$
$$\varepsilon_{r\vartheta}(\hat{\mathbf{u}}) = 2^{-1}(r^{-1}\,\partial u/\partial\vartheta + \partial v/\partial r - v/r)\,,$$
$$\varepsilon_{rz}(\hat{\mathbf{u}}) = 2^{-1}(\partial w/\partial r + \partial u/\partial z)\,, \quad \varepsilon_{\vartheta z} = 2^{-1}(r^{-1}\,\partial w/\partial\vartheta + \partial v/\partial z)\,.$$

Let $W_0(D)$ be the following subspace of *axisymmetric displacements with finite energy*

$$W_0(D) = \{\hat{\mathbf{u}} \in ZW(D \times [0, 2\pi)) \mid v = 0,\ \partial u/\partial\vartheta = 0,\ \partial w/\partial\vartheta = 0\}\,.$$

For $\hat{\boldsymbol{u}} \in W_0(D)$ we have

$$(3) \qquad (2\pi)^{-1} \|\boldsymbol{u}\|^2_{W(\Omega)} = (2\pi)^{-1} \|\hat{\boldsymbol{u}}\|^2_{ZW} = \\ = \int_D [u^2 + (\partial u/\partial r)^2 + (\partial u/\partial z)^2 + u^2/r^2 + \\ + w^2 + (\partial w/\partial r)^2 + (\partial w/\partial z)^2] \, r \, \mathrm{d}r \, \mathrm{d}z \equiv \|\hat{\boldsymbol{u}}\|^2_{\mathscr{H}(D)} .$$

The *second Korn's inequality* (2) in $W_0(D)$ takes the following form

$$(4) \qquad \int_D [\varepsilon^2_{rr}(\hat{\boldsymbol{u}}) + \varepsilon^2_{\vartheta\vartheta}(\hat{\boldsymbol{u}}) + \varepsilon^2_{zz}(\hat{\boldsymbol{u}}) + 2\varepsilon^2_{rz}(\hat{\boldsymbol{u}}) + u^2 + w^2] \, r \, \mathrm{d}r \, \mathrm{d}z \geqq \\ \geqq c(D) \|\hat{\boldsymbol{u}}\|^2_{\mathscr{H}(D)}, \quad \forall \hat{\boldsymbol{u}} \in W_0(D) .$$

On the basis of the relation (3) the space $W_0(D)$ can be identified with the space

$$\mathscr{H}(D) \equiv (W^{(1)}_{2,r}(D) \cap L_{2,1/r}(D)) \times W^{(1)}_{2,r}(D) ,$$

where $W^{(1)}_{2,r}(D)$ denotes the weighted Sobolev space with the norm

$$\|u\|_{1,r,D} = (\int_D [u^2 + (\partial u/\partial r)^2 + (\partial u/\partial z)^2] \, r \, \mathrm{d}r \, \mathrm{d}z)^{1/2}$$

and $L_{2,1/r}(D)$ is the space of functions with the norm

$$\|u\|_{0,1/r,D} = (\int_D u^2/r \, \mathrm{d}r \, \mathrm{d}z)^{1/2} .$$

Let $L_{2,r}(D)$ be the space of functions with the norm

$$\|u\|_{0,r,D} = (\int_D u^2 r \, \mathrm{d}r \, \mathrm{d}z)^{1/2} .$$

Lemma 1. *The embedding of $\mathscr{H}(D)$ into $[L_{2,r}(D)]^2$ is compact.*

Proof. Let a subset M be bounded in $\mathscr{H}(D)$. Since $\mathscr{H}(D)$ is a Hilbert space, there exists a subsequence $\{\hat{\boldsymbol{u}}_n\} \subset M$ such that

$$\hat{\boldsymbol{u}}_n \longrightarrow \hat{\boldsymbol{u}} \text{ (weakly) in } \mathscr{H}(D) , \quad (\hat{\boldsymbol{u}} = (u, w)) .$$

Then

$$(5) \qquad \hat{\boldsymbol{u}}_n \longrightarrow \hat{\boldsymbol{u}} \text{ (weakly) in } [W^{(1)}_{2,r}(D)]^2 ,$$

since $\mathscr{H}(D)$ is continuously embedded into $[W^{(1)}_{2,r}(D)]^2$. In the paper [3] (– Lemma 4) we have proven that the embedding of $W^{(1)}_{2,r}(D)$ into $L_{2,r}(D)$ is compact, so that from (5) the strong convergence

$$u_n \to u , \quad w_n \to w$$

in $L_{2,r}(D)$ follows. Consequently, the set M is precompact in $[L_{2,r}(D)]^2$. Q.E.D.

We shall consider a specific *class of domains* $D(\alpha)$, where

$$D(\alpha) = \{(r, z) \mid 0 < r < \alpha(z), \ 0 < z < 1\}$$

and the function α belongs to the following set

$$\mathscr{U}_{\mathrm{ad}} = \{\alpha \in C^{(0),1}([0, 1]) \quad \text{(i.e., Lipschitz function)} ,$$

$$\alpha_{\min} \leqq \alpha(z) \leqq \alpha_{\max}\,, \quad |d\alpha/dz| \leqq C_1 \quad \text{a.e.}\}\,,$$

where $\alpha_{\min}$, $\alpha_{\max}$ and C_1 are given real constants.

Lemma 2. *Let there exist a positive constant $\tilde{c}$ such that the inequality (1) holds with the constant $\tilde{c}$ for all domains $\Omega(\alpha) \subset \mathbb{R}^3$, $\alpha \in \mathscr{U}_{ad}$, where $\Omega(\alpha)$ is generated by the rotation of the domain $D(\alpha)$ around z-axis.*

Then the inequality (4) holds for all domains $D(\alpha)$ with the same constant $\tilde{c}$.

Proof is an easy consequence of the above mentioned transformation by means of the mapping Z. To this end – see (3) and the relations (cf. [4] –7.6.2, 7.6.3)

$$(6) \qquad \|\varepsilon(\boldsymbol{u})\|^2_{\Omega(\alpha)} = 2\pi \|\varepsilon(\hat{\boldsymbol{u}})\|^2_{D(\alpha)} =$$
$$= 2\pi \int_{D(\alpha)} [\varepsilon^2_{rr}(\hat{\boldsymbol{u}}) + \varepsilon^2_{\vartheta\vartheta}(\hat{\boldsymbol{u}}) + \varepsilon^2_{zz}(\hat{\boldsymbol{u}}) + 2\varepsilon^2_{rz}(\hat{\boldsymbol{u}})]\, r \,dr\, dz\,,$$
$$\|\boldsymbol{u}\|^2_{0,\Omega(\alpha)} = 2\pi \int_{D(\alpha)} (u^2 + w^2)\, r\, dr\, dz = 2\pi \|\hat{\boldsymbol{u}}\|^2_{0,r,D(\alpha)}\,,$$

holding for all $\hat{\boldsymbol{u}} \in W_0(D)$. Q.E.D.

Proposition 1. *The Korn's second inequality (4) in $\mathscr{H}(D(\alpha))$ holds uniformly with respect to $\alpha \in \mathscr{U}_{ad}$ (i.e., with a constant $\tilde{c}$ independent of α).*

Proof. Let us consider the class of axisymmetric domains $\Omega(\alpha) \subset \mathbb{R}^3$, $\alpha \in \mathscr{U}_{ad}$, as in the proof of Lemma 2. There exists a set of open parallelepipeds $\{K_j\}$, $j = 1, 2, \ldots, I$, covering $\Omega(\alpha)$ for all $\alpha \in \mathscr{U}_{ad}$, independent of α and such that:

(i) any part of the boundary $\partial\Omega(\alpha) \cap K_j$ is described in a system of local Cartesian coordinates parallel with the edges of K_j by a Lipschitz function with a Lipschitz constant $L_j(\alpha)$ (cf. the definition of domains with Lipschitz boundary e.g. in [4] – Def. 1.1.2);

(ii) there exists a constant C_L such that

$$L_j(\alpha) \leqq C_L \quad \forall \alpha \in \mathscr{U}_{ad}\,, \quad j = 1, 2, \ldots, I\,.$$

From the results of Nitsche ([6] – Section 3) we conclude that there exists a constant $\tilde{c}$ such that the second Korn's inequality (1) holds for all $\alpha \in \mathscr{U}_{ad}$ with the constant $\tilde{c} = c(\Omega(\alpha))$. Making use of Lemma 2, we obtain (4) for all domains $D(\alpha)$ uniformly. Q.E.D.

2. THEOREMS ON UNIFORM FIRST KORN'S INEQUALITY

The following inequality

$$(7) \qquad \|\varepsilon(\boldsymbol{u})\|^2_{\Omega} \geqq c_1(\Omega)\, \|\boldsymbol{u}\|^2_{W(\Omega)}$$

is called Korn's first inequality. It does not hold, however, for any $\boldsymbol{u} \in W(\Omega)$. Instead, it is true on same subspaces of $W(\Omega)$, such as the subspace of functions vanishing on a part of the boundary $\partial\Omega$. In the present section, we show several kinds of subspaces, where the inequality (7) holds even uniformly, i.e. with a constant independent of $\Omega(\alpha)$, $\alpha \in \mathcal{U}_{\text{ad}}$. We again restrict ourselves to axisymmetric fields of displacement functions defined on axisymmetric bodies.

Theorem 1. *Let us define*

$$\mathcal{U}^{\varepsilon}_{\text{ad}} = \{\alpha_\varepsilon \mid \alpha_\varepsilon(z) = \alpha(z) - \varepsilon,\ \alpha \in \mathcal{U}_{\text{ad}},\ \varepsilon \in [0, \alpha_{\min}/2]\}\ .$$

Assume that $V_p(D(\alpha))$ and $V^0_p(D(\alpha))$ are subspaces of $\mathcal{H}(D(\alpha))$ satisfying the following two conditions:

$(H1)$ there exists $\varepsilon_0 \in (0, \alpha_{\min}/2)$ such that

$$\{\hat{\boldsymbol{u}} \in V_p(D(\alpha)),\ \alpha \in \mathcal{U}_{\text{ad}},\ \beta \in \mathcal{U}^{\varepsilon}_{\text{ad}},\ 0 < \alpha(z) - \beta(z) < \varepsilon_0\ \forall z \in [0, 1]\} \Rightarrow$$
$$\Rightarrow \hat{\boldsymbol{u}}|_{D(\beta)} \in V^0_p(D(\beta))\ ;$$

$(H2)$ Korn's first inequality holds on $V^0_p(D(\beta))$ for all $\beta \in \mathcal{U}^{\varepsilon}_{\text{ad}}$, i.e., there exists a constant $c_1(\beta)$ such that

$$(8) \qquad \|\varepsilon(\hat{\boldsymbol{u}})\|^2_{D(\beta)} \geqq c_1(\beta)\ \|\hat{\boldsymbol{u}}\|^2_{\mathcal{H}(D(\beta))}$$

holds for all $\hat{\boldsymbol{u}} \in V^0_p(D(\beta))$, where the left-hand side has been defined in (6).

Then Korn's first inequality holds on $V_p(D(\alpha))$ uniformly with respect to α, i.e., there exists a positive constant c, independent of $\alpha \in \mathcal{U}_{\text{ad}}$, such that

$$(9) \qquad \|\varepsilon(\boldsymbol{u})\|^2_{D(\alpha)} \geqq c\|\hat{\boldsymbol{u}}\|^2_{\mathcal{H}(D(\alpha))} \quad \forall \hat{\boldsymbol{u}} \in V_p(D(\alpha))\ .$$

Proof is based on same ideas of the Appendix in the paper [1]. Let (9) be false. Then there exist sequences $\{\alpha_n\}$ and $\{\hat{\boldsymbol{u}}_n\}$, $n = 1, 2, \ldots$, $\hat{\boldsymbol{u}}_n \in V_p(D_n)$, $\alpha_n \in \mathcal{U}_{\text{ad}}$ (where $D(\alpha_n) \equiv D_n$), such that

$$(10) \qquad \|\varepsilon(\hat{\boldsymbol{u}}_n)\|^2_{D_n} < n^{-1}\|\hat{\boldsymbol{u}}_n\|^2_{\mathcal{H}(D_n)}\ .$$

Without any loss of generality we can set

$$(11) \qquad \|\hat{\boldsymbol{u}}_n\|_{\mathcal{H}(D_n)} = 1 \quad \forall n\ .$$

Since the set $\mathcal{U}_{\text{ad}}$ is compact in $C([0, 1])$, we can find a subsequence (and denote it by the same symbol) of $\{\alpha_n\}$, such that

$$\alpha_n \to \alpha \quad \text{in} \quad C([0, 1])\,, \quad \alpha \in \mathcal{U}_{\text{ad}}\ .$$

Then

$$(12) \qquad \|\varepsilon(\hat{\boldsymbol{u}}_n)\|_{D_n} \to 0$$

follows from (10), (11) for $n \to \infty$.

Using Proposition 1, (11) and (12), we conclude that

$$(13) \qquad \|\hat{\boldsymbol{u}}_n\|^2_{0,r,D_n} \geqq \tilde{c}/2$$

for n sufficiently large.

Let us denote $D(\alpha - 1/m)$ by G_m. Since $G_m \subset D_n$ for $n > n_0(m)$, from (11) we obtain

$$\|\hat{u}_n\|_{\mathscr{H}(G_m)} \leqq 1 \quad \forall n > n_0(m).$$

There exists a subsequence $\{\hat{u}_k\}$ of $\{\hat{u}_n\}$ such that

$$\hat{u}_k \longrightarrow \hat{u} \quad \text{(weakly)} \quad \text{in} \quad \mathscr{H}(G_m). \tag{14}$$

The subspace $V_p^0(G_m)$ is weakly closed. Then (14) and $(H1)$ imply that

$$\hat{u} \in V_p^0(G_m) \tag{15}$$

if m is large enough. Since the functional

$$\hat{u} \to \|\varepsilon(\hat{u})\|_{G_m}^2$$

is convex and differentiable, it is weakly lower semicontinuous. Consequently, we have

$$\|\varepsilon(\hat{u})\|_{G_m}^2 \leqq \liminf_{k\to\infty} \|\varepsilon(\hat{u}_k)\|_{G_m}^2 = 0$$

as follows from (12). The assumption $(H2)$ and (15) imply that

$$\hat{u} = 0$$

in G_m, provided m is large enough.

Using (14) and Lemma 1, we obtain

$$\hat{u}_k \to 0 \quad \text{in} \quad [L_{2,r}(G_m)]^2. \tag{16}$$

On the other hand,

$$\|\hat{u}_k\|_{0,r,D_k}^2 = \|\hat{u}_k\|_{0,r,G_m}^2 + \|\hat{u}_k\|_{0,r,D_k - G_m}^2 \tag{17}$$

holds for $k > k_0(m)$.

We can derive the estimate

$$\|\hat{u}_k\|_{0,r,D_k - G_m}^2 \leqq c \max_{z\in[0,1]} |\alpha(z) - 1/m - \alpha_k(z)| \tag{18}$$

with c independent of k, m and α. In fact,

$$\|\hat{u}_k\|_{0,r,D_k - G_m}^2 \leqq \alpha_{\max} \|\hat{u}_k\|_{0,D_k - G_m}^2$$

and for the norm in $[L^2(D_k - G_m)]^2$ the estimate (18) holds (see [1] – Appendix). Consequently, we have

$$\|\hat{u}_k\|_{0,r,D_k - G_m}^2 \leqq \tilde{c}/4 \tag{19}$$

for m and k sufficiently large, $k > k_0(m)$. Combining (19), (17) and (13), we arrive at

$$\|\hat{u}_k\|_{0,r,G_m}^2 \geqq \tilde{c}/4$$

for k, m sufficiently large, $k > k_0(m)$, which contradicts (16). Q.E.D.

Let Γ be a part of the boundary $\partial D - \mathcal{O}$, where $\mathcal{O}$ denotes the z-axis and Γ has a positive length. In $\mathcal{H}(D)$ we can define the trace operator. In fact, since any component u or w of $\hat{\boldsymbol{u}}$ belongs to $W^{(1)}_{2,r}(D)$, we can use the linear continuous mapping

$$\gamma \colon W^{(1)}_{2,r}(D) \to L_{2,r}(\Gamma)$$

(see e.g. [3] – Section 1).

Next we present several examples of boundary value problems, to which Theorem 1 can be applied.

1. Dirichlet conditions on a part of the boundary

Let us define

$$V_p(D(\alpha)) = \{\hat{\boldsymbol{u}} \in \mathcal{H}(D(\alpha)) |\ \gamma u = \gamma w = 0 \ \text{on} \ \Gamma_1(\alpha)\} ,$$

where $\Gamma_1(\alpha) \subset \partial D(\alpha) \dot{-} \mathcal{O} \dot{-} \Gamma(\alpha)$ and $\Gamma(\alpha)$ is the graph of the function α (for $z \in [0, 1]$),

$$\text{meas } \Gamma_1(\alpha) \geqq a > 0 \quad \forall \alpha \in \mathcal{U}_{\text{ad}} ,$$

with $a \in \mathbb{R}$ independent of α;

$$V^0_p(D(\beta)) = \{\hat{\boldsymbol{u}} \in \mathcal{H}(D(\beta)) |\ \gamma u = \gamma w = 0 \ \text{on} \ \Gamma_0(\beta)\} ,$$

where $\Gamma_0(\beta) \subset \partial D(\beta) \dot{-} \mathcal{O}$, meas $\Gamma_0(\beta) > 0$.

It is not difficult to verify $(H1)$. The assumption $(H2)$ follows from an analogue of Lemma 2 and the well-known Korn's (first) inequality for displacement functions vanishing on a part of the boundary of a three-dimensional body (see e.g. [4] – Section 7.4).

2. Traction boundary value problem

Let us define the set of rigid body displacements

$$\mathcal{P} = \{\hat{\boldsymbol{u}} = (u, w) \mid u = 0, \ w = c \in \mathbb{R}\}$$

(see [4] – Section 10.3) and the following functional

$$p(\hat{\boldsymbol{u}}) = \int_0^{\alpha_{\min}/2} \gamma w(r, 0)\, r \,\mathrm{d}r .$$

If we introduce

$$V_p(D(\alpha)) = V^0_p(D(\alpha)) = \{\hat{\boldsymbol{u}} \in \mathcal{H}(D(\alpha)) \mid p(\hat{\boldsymbol{u}}) = 0\} ,$$

then it is readily seen that $(H1)$ is true.

Since

$$\mathcal{P} \cap V^0_p(D(\beta)) = \{0\}$$

holds for all $\beta \in \mathcal{U}^{\varepsilon}_{\text{ad}}$, the Korn's first inequality (8) is true for $\hat{\boldsymbol{u}} \in V^0_p(D(\beta))$. In fact, in Cartesian coordinates the inequality (7) holds in the following subspace of $W(\Omega(\beta))$:

$$V_p^0(\Omega(\beta)) = \{\mathbf{u} \in Z^{-1} W_0(D(\beta)) \mid p^0(\mathbf{u}) = 0\},$$

where

$$p^0(\mathbf{u}) = \int_{S_0} \gamma u_3(x_1, x_2, 0)\, dx_1\, dx_2$$

and $S_0 \subset \partial\Omega(\beta)$ is the disc generated by rotating segment

$$\{(r, z) \mid 0 < r < \alpha_{\min}/2,\ z = 0\}.$$

(For the proof – see e.g. [4] – Section 7.4 or [5] – II.) Using the mapping $\mathbf{Z}$, we are led to the Korn's first inequality in $V_p^0(D(\beta))$.

For both the examples mentioned above, Theorem 1 yields Korn's first inequality (9), uniform with respect to $\alpha \in \mathscr{U}_{ad}$. ∎

If the *displacements vanish on the variable part* $\Gamma(\alpha)$ of the boundary, the following subspace has to be considered

$$V(D(\alpha)) = \{\hat{\mathbf{u}} \in \mathscr{H}(D(\alpha) \mid \gamma\hat{\mathbf{u}} = 0 \text{ on } \Gamma(\alpha)\}.$$

We cannot apply Theorem 1, however, since the condition $(H2)$ is violated. Instead, we are able to verify the uniform Korn's inequality by a simple idea, as follows.

Let us define

$$\mathscr{U}_{ad}^0 = \{\alpha \in C^{(0),1}([0, 1]) \mid \alpha_{\min} \leqq \alpha(z) \leqq \alpha_{\max}\}.$$

Then a positive constant c exists, independent of $\alpha \in \mathscr{U}_{ad}^0$, such that the Korn's inequality (9) holds for all $\hat{\mathbf{u}} \in V(D(\alpha))$ and $\alpha \in \mathscr{U}_{ad}^0$.

In fact, let us extend $\hat{\mathbf{u}} \in V(D(\alpha))$ by zero to the domain $D = (0, \delta) \times (0, 1)$, where δ is any number greater than $\alpha_{\max}$. Denoting the extension by $E\hat{\mathbf{u}}$, we conclude that $E\hat{\mathbf{u}}$ belongs to the space $V(D) = \{\hat{\mathbf{u}} \in \mathscr{H}(D) \mid \gamma\hat{\mathbf{u}} = 0 \text{ on } \Gamma_\delta\}$, where Γ_δ is the graph of the constant function $\alpha = \delta$. It is easy to verify the Korn's inequality

$$\|\varepsilon(E\hat{\mathbf{u}})\|_D^2 \geqq c_\delta \|E\hat{\mathbf{u}}\|_{\mathscr{H}(D)}^2 \quad \forall E\hat{\mathbf{u}} \in V(D)$$

via the relations (3) and (6). Since $E\hat{\mathbf{u}} = 0$ outside $D(\alpha)$, we obtain (9) with $c = c_\delta$ for any $\hat{\mathbf{u}} \in V(D(\alpha))$, $\alpha \in \mathscr{U}_{ad}^0$.

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Souhrn

KORNOVA NEROVNOST STEJNOMĚRNÁ VZHLEDEM KE TŘÍDĚ OSOVĚ SYMETRICKÝCH TĚLES

IVAN HLAVÁČEK

Kornova nerovnost obsahuje kladnou konstantu, která obecně závisí na oblasti — tělese. Dokazuje se, že tyto konstanty mají kladné infimum, uvažujeme-li celou třídu omezených osově symetrických těles a osově symetrická pole posunutí.

Резюме

НЕРАВЕНСТВО КОРНА, РАВНОМЕРНОЕ ПО ОТНОШЕНИЮ К НЕКОТОРОМУ КЛАССУ ОСЕСИММЕТРИЧЕСКИХ ТЕЛ

IVAN HLAVÁČEK

Неравенство Корна содержит положительную постояную, которая в общем случае зависит от области — тела. Доказывается, что эти постояные обладают положительной нижней граню, если рассматривать некоторый класс ограниченных осесимметрических тел и осесимметричес-кие поля перемещений.

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