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STEREOLOGY OF GRAIN BOUNDARY PRECIPITATES

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Summary. Precipitates modelled by rotary symmetrical lens-shaped discs are situated on matrix grain boundaries and the homogeneous specimen is intersected by a plane section. The stereological model presented enables one to express all basic parameters of spatial structure and moments of the corresponding probability distributions of quantitative characteristics of precipitates in terms of planar structure parameters the values of which can be estimated from measurements carried out in the plane section. The derived relationships are transformed into those valid for spherical precipitates.

Keywords: Lens-shaped precipitates, parameter estimations, random tessellation, stereology.

1. INTRODUCTION

The determination of size distribution of precipitates β situated on the $\varrho_{\alpha\alpha}$ matrix grain boundaries is one of the problems intensively studied in connection with precipitate nucleation, growth, dissolution, coarsening etc. — see e.g. Martin and Doherty [7].

The first solution of the problem of size distribution for rotary symmetrical lens-shaped precipitates β lying in a simple grain boundary was presented by Gokhale and Jena [4]. The authors expressed the probability density function (pdf) $g(p)$ of chord length created on the circular bases of precipitates by a random plane section ϱ_s in terms of the pdf $f(y)$ of diameters of these bases located in planar grain boundaries. Further, they proposed a procedure for estimating the contact angle θ.

In comparison with [4], the present paper improves the model assumptions, corrects the results derived there and essentially extends the field of the problems already solved. It introduces the following results:

i) the mutual relationship between the pdf $g(p)$ and the pdfs $f(y)$, $h(x)$ and $v(t)$, where $h(x)$ is the pdf of diameters of two identical spheres the non-empty intersection of which forms a rotary symmetrical lens-shaped precipitate β and $v(t)$ is the pdf of the minor axis of the precipitate β,

ii) the expressions relating the moments of the pdfs $f(y)$, $h(x)$ and $v(t)$ to the moments of the pdf $g(p)$,

iii) stereological estimates of the spatial structure parameters:

$N_{V,\beta}$ and $N_{A(\alpha\alpha),\beta}$, the mean number of precipitates β per unit volume and per unit area of the grain boundary, respectively;

$\bar{V}_\beta$ and $\bar{S}_{\alpha\beta}$, the mean values of the precipitate volume and of the precipitate surface area, respectively, and $\bar{V}_{V,\beta}$, the mean value of the volume fraction of the precipitates β per unit volume;

the contact angle θ of the precipitate β.

These spatial structure parameters are expressed in terms of planar structure parameters that can be estimated from results of measurements gained in the plane ϱ_s e.g. by means of an automatic image analyzer. For our purposes the following planar structure parameters are considered:

the k-th moment γ_k of the pdf $g(p)$ of chord lengths measurable on the traces $c = \varrho_{\alpha\alpha} \cap \varrho_s$ in the plane ϱ_s,

$N_{A(s),\beta}$, the mean number of precipitates β sectioned by the traces c per unit area of the plane ϱ_s,

$N_{L(c),\beta}$, the mean number of precipitates β intersected by unit length of the trace c,

$P_{L,\alpha}$, the mean number of traces c hit by a test line of unit length randomly situated in ϱ_s,

$\bar{A}_{A,\beta}$, the mean value of the area fraction of sections of all precipitates observable in ϱ_s.

Definitions and model assumptions are given in Section 2. The solution of the above formulated problems is the subject of Section 3 for lens-shaped precipitates and of Section 4 for spherical precipitates. Some selected results are discussed in Section 5.

2. DEFINITIONS AND MODEL ASSUMPTIONS

At the beginning we explain some terms that will be used in the formulation of model assumptions.

Homogeneous 3-d *random tessellation*. Let us assume that the three-dimensional euclidean space E_3 is divided into spacefilling and non-overlapping three-dimensional bounded open and connected subsets $G_j \subset E_3$ forming so called grains. Their boundaries ∂G_j are piecewise smooth closed surfaces separating G_j from its exterior. Let $G_* = \{G_j\}$ be the set of all grains in E_3, then $\mathscr{G} \subset G_*$ is called a 3-d tessellation. $\mathscr{T}$ be the class of all tessellations and $\sigma_{\mathscr{T}}$ the σ algebra in $\mathscr{T}$ generated by sets of the form

$$\{\mathscr{G} \in \mathscr{T} : \partial\mathscr{G} \cap K \neq \emptyset\} ,$$

where $\partial\mathscr{G} = \bigcap \partial G_j$ is the union of the boundaries of $G_j \in \mathscr{G}$ and K runs through all compact subsets of E_3. Then the random variable taking the values in $[\mathscr{T}, \sigma_{\mathscr{T}}]$

defines a 3-d random tessellation. If $\mathscr{P}$, the probability measure on $[\mathscr{T}, \sigma_{\mathscr{T}}]$, is invariant under translations in E_3 and invariant with respect to rotations about the origin in E_3, then the 3-d random tessellation is stationary and isotropic, respectively, and such a tessellation is called a homogeneous 3-d random tessellation. For details we refer e.g. to Stoyan's et al. book [13] and Møller's booklet [10].

Types of interphase interfaces. The structure under investigation is a two-phase structure. It consists of space-filling grains G_j formed by the α phase, and of precipitates, formed by the β phase and modelled by rotary symmetrical lens-shaped discs situated on the junction of two grains. This interface is of type $\alpha\alpha$ and for planar grain boundary it will be denoted by $\varrho_{\alpha\alpha}$. On the other hand, on each precipitate β having the surface area $\partial\beta$ and its circular base with the centre of gravity C_i located in $\varrho_{\alpha\alpha}$, two types of interphase interfaces can be observed

– the triple line circle $\alpha\alpha\beta$ containing all points of the intersection $\partial\beta \cap \varrho_{\alpha\alpha}$ and

– two interfaces of type $\alpha\beta$ containing all points of two boundaries of spherical cap shape with the exception of those points belonging to the triple line circle $\alpha\alpha\beta$ bounding the base common to two spherical caps, creating the precipitate β.

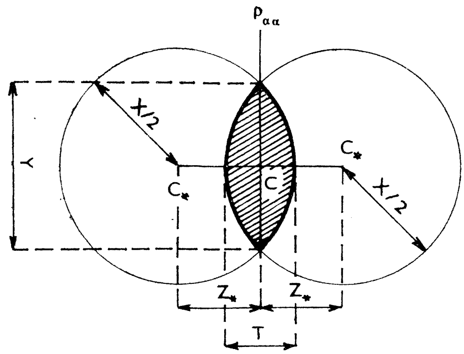

Fig. 1. Rotary symmetrical lens-shaped disc sectioned in a normal plane to the boundary plane $\varrho_{\alpha\alpha}$ the normal plane passes through the centres C_{*1} and C_{*2} of the spheres.

The present stereological model is constructed under the following assumptions:

(a) the grains G_j are convex bodies, namely spacefilling and non-overlapping three-dimensional polyhedra; planar grain boundaries form a spatial homogeneous random tessellation;

(b) the precipitate β is fine and modelled by a rotary symmetrical lens-shaped disc arising as a non-empty intersection of two spheres of the same diameter X (see Fig. 1), having the pdf $h(x)$;

(c) in the planar grain boundaries $\varrho_{\alpha\alpha}$ the centres of gravity C_i, $i = 1, 2, \dots$ of precipitates β form a two-dimensional Poisson field of constant density $N_{A(\alpha\alpha),\beta}$, the area density per unit area of $\varrho_{\alpha\alpha}$ surfaces, low enough in order to remain the $\alpha\alpha\beta$ triple line circles, shared by both spherical caps, disjoint;

(d) the circular bases $\alpha\alpha\beta$ situated in $\varrho_{\alpha\alpha}$ have diameters which are independently and identically distributed with the pdf $f(y)$, $0 < y < \infty$. The random variable (rv) Y takes the value of the major axis of the lens-shaped precipitate β and is independent of the position of the precipitate gravity centre C_i. The minor axis T of the precipitate β has the pdf $v(t)$;

(e) all grain boundary energies are isotropic; the interfacial energy with both matrix grains is constant so that the contact angle θ is fixed for all precipitates β (see Fig. 2);

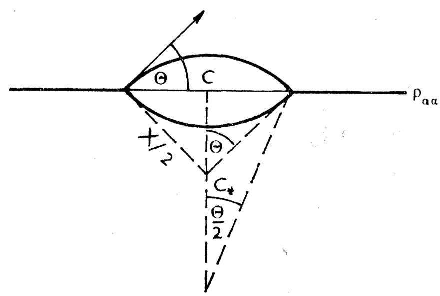

Fig. 2. Lens-shaped disc consisting of two spherical caps both of the same radius $X/2$ and contact angle θ.

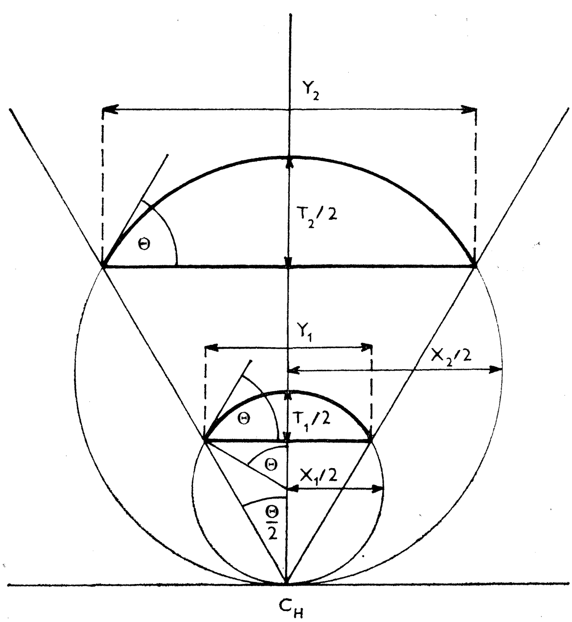

Fig. 3. Similarity of spherical caps with a constant contact angle θ; C_H is the centre of homothety.

(f) the homogeneous specimen is intersected by an arbitrary plane section; from the technical point of view we assume that we are able to prepare it in such a way that the traces $c = \varrho_{\alpha\alpha} \cap \varrho_s$, forming polygons in the plane ϱ_s, are sharply delineated.

The structure possessing the feature specified in points (a) to (f) will be denoted **S**. Let us emphasize that in this structure the precipitate centres of gravity C_i do not form the Poisson point field with a constant volume density $N_{V,\beta}$ per unit volume as it is usual in stereological models concerning precipitates embedded in a matrix (see e.g. models published by Coleman [1], Horálek [5], Saltykov [11], Stoyan et al [13], Underwood [14], Wicksell [15]).

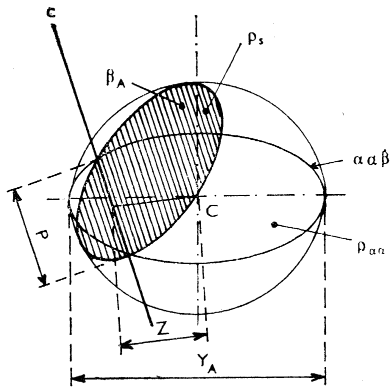

Fig. 4a. Formation of the precipitate section β_A.

As a result of the assumption (e) the precipitates β fulfil the condition of similarity (see Fig. 3) and have a constant ratio of the minor to the major axis

$$(1) \qquad \frac{T}{Y} = k_0 = \tan\frac{\theta}{2} = \text{const.}\,, \quad 0 < k_0 \leqq 1\,; \quad 0 < \theta \leqq \frac{\pi}{2}\,.$$

Therefore, for known k_0 the size distribution of precipitates β can be described only by the pdf $f(y)$. For $k_0 = 1$, the lens-shaped discs convert into spheres.

In the plane ϱ_s we can observe two types of precipitate sections:

- the sections hit by the trace c; these will be called the precipitate sections of type A and denoted by β_A (see Fig. 4a);
- the sections missed by the trace c; these will be called the precipitate sections of type B and denoted by β_B (see Fig. 4b).

Both types of precipitate sections can be easily and uniquely distinguished from one another:

- the section of type A has the shape of a lens in its profile that need not be symmetrical. This section lies always on the trace c;
- the section of type B is always a disc. The sections β_B are located only inside the polygons formed by the grain boundaries intersected by the plane ϱ_s.

As a result of the assumption (a), the planar grain boundaries $\varrho_{\alpha\alpha}$ form a three-dimensional random tessellation invariant under translations in E_3 and invariant with respect to rotations about the origin in E_3 and the plane ϱ_s can be taken as an IUR plane section since the specimen is homogeneous. Further, for all precipitates of the same shape (but irrespective of this shape) embedded in a specimen, the probability of their hitting by an IUR plane section is independent of the spatial position and orientation of these precipitates in the specimen, i.e. the precipitates need not be homogeneous or isotropic in the specimen and the location of centres may even be correlated or correlated with their orientation – Coleman [1].

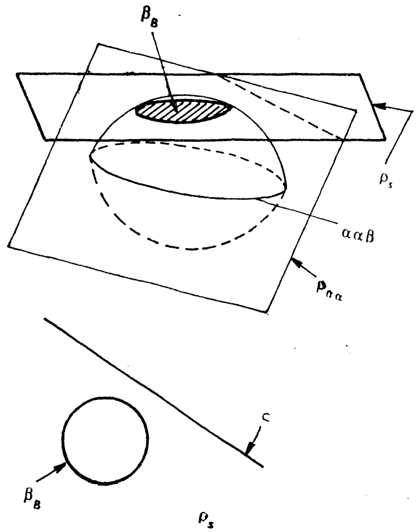

Fig. 4b. Formation of the precipitate section β_B (according to Gokhale and Jena, 1980).

Therefore, the traces $c = \varrho_{\alpha\alpha} \cap \varrho_s$ can be regarded as random segments across the corresponding surface $\varrho_{\alpha\alpha}$ and the intercept length of the section β_A can be viewed as a line section sampling of the circles (C_i, Y_i), $i = 1, 2, \ldots$, contained in $\varrho_{\alpha\alpha}$. In consequence of this fact, in the sequel we will exclude from processing the sections β_B, except for specified situations.

From the above introduced model assumptions it is clear that we ignore special situations arising when the precipitate β is hit by the edge, the intersection of three $\varrho_{\alpha\alpha}$ boundaries, or when two precipitates β, each belonging only to one of two ajdacent $\varrho_{\alpha\alpha}$ boundaries, are located near the edge and interpenetrate. If we wished to take these special situations into account the relationship derived in the following Sections 3 to 5 could be considered only as approximate ones.

3. STEREOLOGICAL RELATIONSHIPS IN STRUCTURE $\mathbf{S}$ FOR $k_0 < 1$

The relationships between the pdfs $g(p)$, $h(x)$, $v(t)$ and $f(y)$ are the subject of Theorem 1, those between the corresponding moments δ_k, γ_k, $\varkappa_k$ and τ_k are formulated in Theorem 2 and, finally, the stereological relationships between spatial and planar structure parameters of $\mathbf{S}$ are covered by Theorems 4 and 5.

Theorem 1. *In the structure* $\mathbf{S}$ *the pdfs* $g(p)$, $h(x)$ *and* $v(t)$ *are related to the pdf* $f(y)$ *in the following way*

$$g(p) = \frac{p}{\delta_1} \int_p^\infty \frac{1}{\sqrt{(y^2 - p^2)}} f(y) \, dy , \quad p > 0 , \tag{2}$$

$$h(x) = (\sin \theta) f(x \sin \theta) , \tag{3}$$

$$v(t) = \frac{1}{\tan(\theta/2)} f\left(\frac{t}{\tan \theta}\right) , \tag{4}$$

where $\delta_1 = \mathsf{E}(Y)$ *is the mean value of the rv* Y.

Proof. Consider the structure $\mathbf{S}$ sectioned by the plane ϱ_s. Let Y_A denote the diameter of the triple line circle $\alpha\alpha\beta$ hit by the trace c, Z the distance of c to the corresponding C_i and, finally, P the chord length created by the trace c on the intersected circle (see Fig. 4a). These three rvs Y_A, Z and P are related by

$$P = Y_A \sqrt{(1 - Q^2)} \tag{5}$$

where the rv

$$Q = 2ZY_A^{-1}$$

is uniformly distributed on the interval $(0, 1)$ and is independent of the rv Y_A. Then, it can be shown – see Coleman [2] and Horálek [5] – that the pdf of the rv P has the form (2).

Due to the similarity of spherical caps with a contact angle θ – see Fig. 3 – the rvs Y and X are related by

$$Y = X \sin \theta \tag{6}$$

and the rvs T and Y by Eq (1) which can be rewritten in the form

$$Y = \frac{T}{\tan(\theta/2)} . \tag{7}$$

Hence the pdfs (3) and (4).

Corollary 1. 1. *If the pdf* $f(y)$ *is unknown and the pdf* $g(p)$ *known, then*

$$f(y) = \frac{-2\delta_1 y}{\pi} \int_y^\infty \frac{1}{\sqrt{(p^2 - y^2)}} \frac{d}{dp} \left(\frac{g(p)}{p}\right) dp . \tag{8}$$

Proof. Regarding the pdf $g(p)$ as known, the equation (2) represents an integral equation for an unknown function $f(y)$. The form of this integral equation coincides with the well-known Abel integral equation, so that the corresponding inversion formula has the form (8) – see e.g. Jakeman and Anderssen [6].

Note. The relationship between $g(p)$ and $f(y)$, analogous to (2) in Theorem 1, has been derived by Gokhale and Jena in their paper [4] as Eq (12). Unfortunately, we have to state that this equation is not correct and that the integral of $g(p)$ over the definition region differs from 1.

Theorem 2. *In the structure* **S** *the moments* $\delta_k, \varkappa_k, \tau_k$ *and* γ_k *are related in the following way for* $k = 0, 1, 2, \ldots$

$$\delta_k = \mathsf{E}(Y^k) = D(k)\frac{\gamma_{k-1}}{\gamma_{-1}}, \tag{9}$$

$$\varkappa_k = \mathsf{E}(X^k) = (\sin\theta)^{-k}\,\delta_k, \tag{10}$$

$$\tau_k = \mathsf{E}(T^k) = \left(\tan\frac{\theta}{2}\right)^k \delta_k, \tag{11}$$

where

$$\gamma_k = \mathsf{E}(P^k)$$

and

$$D(k) = \frac{\Gamma(\frac{1}{2})\,\Gamma[(k+2)/2]}{\Gamma[(k+1)/2]}, \tag{12}$$

$\Gamma(n)$ *being the gamma function.*

Proof. The rvs P, Y_A and Q are related by (5), the rvs Y_A and Q are mutually independent and the pdf $f_A(y)$ of the rv Y_A is

$$f_A(y) = \frac{y}{\delta_1} f(y) \tag{13}$$

(the plane ϱ_s selects proportionally to the diameters Y of the $\alpha\alpha\beta$ circles lying in surfaces $\varrho_{\alpha\alpha}$); then the k-th moment of $g(p)$ satisfies

$$\begin{aligned}\gamma_k &= \mathsf{E}(P^k) = \{\mathsf{E}(Y_A^k)\}\,\{\mathsf{E}[(1-Q^2)^{k/2}\} = \\ &= \frac{1}{2}\,\frac{\Gamma(\frac{1}{2})\,\Gamma[(k+2)/2]}{\Gamma[(k+3)/2]}\,\frac{\delta_{k+1}}{\delta_1} \quad \text{for} \quad k = -1, 0, 1, \ldots,\end{aligned} \tag{14}$$

which could be expected in view of the pdf $g(p)$, given in (2), that has the same form as the Wicksell [15] formula for planar sections of spheres.

Setting $k = -1$ in (14) we have

$$\delta_1 = \pi/(2\gamma_{-1}), \tag{15}$$

so that the k-th moment δ_k of the pdf $f(y)$ can be written as (9).

Recalling (6) and (7) we easily derive (10) and (11).

Theorem 3. *In the structure* **S** *the following stereological relationships hold*

$$N_{A(\alpha\alpha),\beta} = \frac{2}{\pi} N_{L(c),\beta}\gamma_{-1}\,, \tag{16}$$

$$N_{V,\beta} = \frac{4}{\pi} N_{L(c),\beta}P_{L,\alpha}\gamma_{-1} \tag{17a}$$

$$= \frac{8}{\pi^2} N_{A(s),\beta}\gamma_{-1}\,. \tag{17b}$$

Proof. Taking into account the properties of the line section sampling across the planar boundaries $\varrho_{\alpha\alpha}$ containing the triple line circles $\alpha\alpha\beta$ we have for the mean number $N_{A(\alpha\alpha),\beta}$ in accordance with Horálek [5]

$$N_{A(\alpha\alpha),\beta} = \frac{1}{\delta_1} N_{L(c),\beta}\,;$$

after replacing δ_1 by (15) we get (16).

Let $S_{V,\alpha}$ denote the mean value of the surface area of $\varrho_{\alpha\alpha}$ per unit volume. Then

$$N_{V,\beta} = S_{V,\alpha}N_{A(\alpha\alpha),\beta}\,. \tag{18}$$

The properties of the **S** structure under investigation allow us to express $S_{V,\alpha}$ in the form derived e.g. by Saltykov [11], Smith and Guttman [12] and Underwood [14]

$$S_{V,\alpha} = 2P_{L,\alpha}\,. \tag{19}$$

After inserting (19) and (16) into (18) we obtain (17a).

Let $\mathsf{E}(B_A)$ be the mean value of the tessellation length (the length of traces c) per unit area of the plane ϱ_s. Then the mean value $N_{L(c),\beta}$ can be given by

$$N_{L(c),\beta} = \frac{N_{A(s),\beta}}{\mathsf{E}(B_A)} = \frac{2}{\pi}\frac{N_{A(s),\beta}}{P_{L,\alpha}}\,, \tag{20}$$

since

$$\mathsf{E}(B_A) = \frac{\pi}{2} P_{L,\alpha}$$

– see e.g. Underwood [14]. The application of (20) to (17a) proves (17b).

Theorem 4. *In the structure* **S** *the mean value* $\bar{S}_{\alpha\beta}$ *of the precipitate surface area and the mean value* $\bar{V}_\beta$ *of the precipitate volume are given by*

$$\bar{S}_{\alpha\beta} = \frac{2\pi}{1 + \cos\theta}\frac{\gamma_1}{\gamma_{-1}}\,, \tag{21}$$

$$(22) \qquad \bar{V}_\beta = \frac{\pi^2}{64} k_0(3 + k_0^2) \frac{\gamma_2}{\gamma_{-1}}.$$

Proof. Since T is the minor axis of the precipitate β and X is the diameter of both spheres the non-empty intersection of which creates the rotary symmetrical lens-shaped disc the surface area $S_{\alpha\beta}$ of the precipitate β can be expressed by

$$S_{\alpha\beta} = \pi TX .$$

The application of (6) and (7) gives

$$S_{\alpha\beta} = \pi Y^2(1 + \cos\theta)^{-1} .$$

Hence the mean value satisfies

$$(23) \qquad \bar{S}_{\alpha\beta} = E(S_{\alpha\beta}) = \frac{\pi}{1 + \cos\theta} E(Y^2),$$

which can be readily transformed into (21) putting $k = 2$ in (9).

By virtue of (1) the volume V_β of the precipitate β can be written in the form

$$V_\beta = \frac{\pi}{48} k_0(3 + k) Y^3 .$$

After applying the same procedure as when deriving $\bar{S}_{\alpha\beta}$ we prove (22) for $\bar{V}_\beta$. Naturally, in this case we put $k = 3$ in (9).

Theorem 5. *In the structure* **S** *the ratio* $k_0(0 < k_0 < 1)$ *of the minor to the major axis of the precipitate* β *can be expressed in the form*

$$(24) \qquad k_0 = 2 \sinh \psi ,$$

where

$$(25) \qquad \psi = \tfrac{1}{3} \ln \{0{\cdot}5(U + \sqrt{(U^2 + 4)}\}$$

and

$$(26a) \qquad U = \frac{16}{\pi} \frac{\bar{A}_{A,\beta}}{N_{L(c),\beta} P_{L,\alpha} \gamma_2}$$

$$(26b) \qquad = \frac{8\bar{A}_{A,\beta}}{N_{A(s),\beta}\gamma_2},$$

ln *being the natural logarithm.*

Proof. The mean values $\bar{V}_{V,\beta}$, $\bar{V}_\beta$ and $N_{V,\beta}$ are related by

$$\bar{V}_{V,\beta} = \bar{V}_\beta N_{V,\beta} .$$

The model assumptions a), c) and d) enable us to put

$$\bar{V}_{V,\beta} = \bar{A}_{A,\beta} ,$$

where $\bar{A}_{A,\beta}$ the mean value of the area fraction of sections of the precipitates β_A and β_B, can be determined from measurements carried out in the plane section ϱ_s.

Now, by means of (17a) and/or (17b) we find

$$\bar{A}_{A,\beta} = \frac{\pi}{16} N_{L(c),\beta} P_{L,\alpha} \gamma_2 k_0 (3 + k_0^2)$$

$$= \frac{1}{8} N_{A(s),\beta} \gamma_2 k_0 (3 + k_0^2) .$$

These are equations of the third order for the unknown k_0 and they can be written in the form

$$k_0^3 + 3k_0 - U = 0 , \tag{27}$$

since all parameters involved in the dimensionless constant U, given by (26), are parameters of the planar structure in ϱ_s. The discriminant $\mathscr{D} = -[1 + (U^2/4)]$ is negative, therefore the last equation has two complex conjugate roots and one real root ${}_1k_0$ $(0 < {}_1k_0 \leqq 1)$ of the form (24).

Corollary 5.1. *In the structure* **S**

$$\theta = 2 \operatorname{arc\,tan} (e^{\psi} - e^{-\psi}) , \tag{28}$$

ψ *having the form* (25).

Proof. By the well-known relationship $2 \sinh = e^{\psi} - e^{-\psi}$ Eq (28) follows from (1) and (24).

Note. When deriving the expression for the estimate of the contact angle θ, Gokhale and Jena [4] start from an equation similar to (23). For determining the mean value $\bar{S}_{\alpha\beta}$ as a function of θ they propose to use the so called unfolding method developed by Saltykov [11], numerically more complicated than the procedure devised in the present paper.

4. STEREOLOGICAL RELATIONSHIPS IN STRUCTURE S FOR $k_0 = 1$

Let us consider the special case when $k_0 = 1$, i.e. the precipitates β are spheres with diameter X and the pdf $h(x)$ and, moreover, $X = Y = T$ and $\theta = \pi/2$. Under these conditions the basic relationships derived in Section 3 take the following forms:

– the pdfs and the relevant moments satisfy

$$h(x) = f(x) = v(x)$$

and

$$\varkappa_k = \delta_k = \tau_k \quad \text{for} \quad k = 0, 1, 2, \ldots ;$$

– the pdf $g(p)$ of chord length creates on $\alpha\alpha\beta$ circles by the trace c satisfies

$$g(p) = \frac{p}{\varkappa_1} \int_p^\infty \frac{1}{\sqrt{(x^2 - p^2)}} h(x) \, \mathrm{d}x \,, \quad p > 0 \tag{29}$$

with the k-th moment

$$\gamma_k = \frac{\Gamma(\frac{1}{2}) \, \Gamma[(k+2)/2]}{2\Gamma[(k+3)/2]} \frac{\varkappa_{k+1}}{\varkappa_1} \quad \text{for} \quad k = -1, 0, 1, \ldots ; \tag{30}$$

– the expressions for $\bar{S}_{\alpha\beta}$, $\bar{V}_\beta$ and $\bar{V}_{V,\beta}$ read

$$\bar{S}_{\alpha\beta} = 2\pi \frac{\gamma_1}{\gamma_{-1}} \,,$$

$$\bar{V}_\beta = \frac{\pi^2}{16} \frac{\gamma_2}{\gamma_{-1}} \,,$$

$$\bar{V}_{V,\beta} = \frac{\pi}{4} N_{L(c),\beta} P_{L,\alpha} \gamma_2 \,;$$

– the validity of (27) and (25) for the spherical precipitates, i.e. for $\theta = \pi/2$, can be easily verified.

The precipitate sections of type A and of type B observable in the plane ϱ_s are in both case discs. When quantifying such a scene using an automatic image analyzer we can measure:

– either all disc diameters D without any separating sections of type A from those of type B,

– or the chord lengths created by the traces c on sections of type A.

In practice, we usually prefer the method based on diameter measurements for its simplicity and higher precision of results. We will show that for spherical precipitates the replacement of the second method by the first is quite warrantable from the theoretical point of view and does not change the validity of the relationships derived above.

In proving Theorem 2 we have stated that the plane section ϱ_s selects proportionally to the diameters Y of the $\alpha\alpha\beta$ circles located in the grain boundaries $\varrho_{\alpha\alpha}$ and having the pdf $f(y)$. However, due to the arguments introduced at the end of Section 2 concerning the organization of precipitates in the space of the specimen, in the case of spherical precipitates β the plane ϱ_s selects proportionally to the diameters X of these precipitates, too, and further we know that $Y = X, f(x) = h(x)$ and $\delta_k = \varkappa_k$. Therefore, recalling (2) and (14), we have for the pdf $w(d)$ of the rv D, taking the value of the observed disc diameter,

$$w(d) = g(d) \tag{31}$$

and

$$\mu_k = \mathsf{E}(D^k) = \gamma_k \,,$$

i.e.

$$(32) \qquad w(d) = \frac{d}{\varkappa_1} \int_d^\infty \frac{1}{\sqrt{(x^2 - d^2)}} h(x)\,\mathrm{d}x\,, \quad d > 0\,.$$

This implies that for estimating the moments $\varkappa_k$ of the rv X, Eq (9) can be used after replacing δ_k by $\varkappa_k$ and γ_k by μ_k. Further, for $N_{A(\alpha\alpha),\beta}$ and $N_{V,\beta}$ we have according to (16) and (17), respectively,

$$N_{A(\alpha\alpha),\beta} = \frac{2}{\pi} N_{L(c),\beta}\mu_{-1}\,,$$

$$(33a) \qquad N_{V,\beta} = \frac{4}{\pi} N_{L(c),\beta} P_{L,\alpha}\mu_{-1}$$

$$(33b) \qquad = \frac{8}{\pi^2} N_{A(s),\beta}\mu_{-1}\,.$$

5. DISCUSSION OF SOME RESULTS

In comparison with (17a) and (26a) the right hand expressions in (17b) and (26b), respectively, are independent of parameters characterizing the tesselation. It might seem that the model assumption (a) is superfluous. But it is not so. For example, when assuming a specified type of a 3-d tessellation with planar grain boundaries, the so called Poisson-Voronoi tessellation – see e.g. Stoyan et al. [13], Møler [10] – the relevant expressions for $N_{V,\beta}$ and U fully coincide with the analyzed equations, i.e. they do not involve parameters of the PV tessellation. However, when applying the Johnson-Mehl tessellation – see e.g. Gilbert [3], Meijering [8], Miles [9] – having, in general, nonplanar boundaries, the present model could be considered only as an approximation under the condition that the precipitates are very fine so that the areas $\partial G_j \cap \beta$ can be taken as plane surfaces halving the precipitates β and containing the triple line circles $\alpha\alpha\beta$, i.e. the intersections $\partial G_j \cap \beta \cap \varrho_s$ create line segments in the plane section ϱ_s.

In Section 2 we have emphasized the distinction between the structure $\boldsymbol{S}$ investigated in the present paper and the structure $\boldsymbol{S}_P$, in which the centres of gravity C_i of precipitates β form the Poisson point process with a constant volume density $N_{V,\beta}$ per unit volume. Now we will demonstrate some specific features of the $\boldsymbol{S}$ structure in comparison with the $\boldsymbol{S}_P$ structure.

The mean number $N_{V,\beta}$ of precipitates β per unit volume is given

– in the structure $\boldsymbol{S}$ by (33b), i.e.

$$N_{V,\beta} = \frac{8}{\pi^2} N_{A(s),\beta}\mu_{-1} = 0{\cdot}81057 N_{A(s),\beta}\mu_{-1}\,,$$

– in the structure $\boldsymbol{S}_P$ by (see e.g. Coleman [1], Horálek [5], Saltykov [11], Stoyan et al. [13], Underwood [14], Wicksell [15])

$$N_{V,\beta} = \frac{2}{\pi} N_{A,\beta}\mu_{-1} = 0{\cdot}63662\, N_{A,\beta}\mu_{-1}\,,$$

$N_{A,\beta}$ being the area density of precipitate sections per unit area of ϱ_s. Hence

$$N_{A(s),\beta} = \frac{\pi}{4} N_{A,\beta} ,$$

which shows that the mean number per unit area of precipitate sections β_B missed by the traces c is equal to

$$\left(1 - \frac{\pi}{4}\right) N_{A,\beta} .$$

These results stem from the fact that the sampling procedure is carried out in planes occupied by $\alpha\alpha\beta$ circles, and only these bases are subjected to the analysis.

The pdf $g(p)$ of chord length created on circular planar sections of precipitates β by the traces c and the relevant k-th moment γ_k are given

- in the structure $\mathbf{S}$ by (29) and (30), respectively, but
- in the structure $\mathbf{S}_P$ by the following relationships derived by Coleman [2] and Horálek [5]

$$g(p) = \frac{p}{\mu_1} \int_p^\infty \frac{1}{\sqrt{(d^2 - p^2)}} \mathrm{d}d$$

$$= \frac{p}{\mu_1 \varkappa_1} \iint_{\substack{p<d<\infty \\ d<x<\infty}} \frac{d\, h(x)}{\sqrt{[(d^2 - p^2)(x^2 - p^2)]}} \mathrm{d}x \,\mathrm{d}d$$

$$= \frac{2p}{\varkappa_2} \left[1 - \int_p^\infty h(x)\, \mathrm{d}x\right] = \frac{2p}{\varkappa_2} [1 - H(p)] ,$$

$H(x)$ being the distribution function of the rv X, and

$$\gamma_k = \frac{2}{k+2} \frac{\varkappa_{k+2}}{\varkappa_2} \quad \text{for} \quad k > -2 .$$

Note that the equality (30) holds only for the structure $\mathbf{S}$ with $k_0 = 1$.

References

[1] *R. Coleman:* An Introduction to Mathematical Stereology. Memoirs No. 3, Department of Theoretical Statistics, University of Aarhus, Denmark, 1979.

[2] *R. Coleman:* Line section sampling of Wicksell's corpuscles. Acta Stereologica, 6 (1987), 33–36.

[3] *E. N. Gilbert:* Random subdivision of space into crystals. Ann. Math. Statistics, 33 (1962), 958–72.

[4] *A. M. Gokhale, A. K. Jena:* Size distribution of grain-boundary precipitates. Metallography 13 (1980), 307–17.

[5] *V. Horálek:* Statistical Models of Some Testing and Inspection Procedures of Products, Materials and Raw Materials (in Czech). Thesis, Charles University Praha, 1961.

[6] *A. J. Jakeman, R. S. Anderssen:* Abel type integral equation in stereology. I. General discussion. J. Microscopy, 105 (1975), 121—33.
[7] *J. W. Martin, R. D. Doherty:* Stability of Microstructure in Metallic Systems. Cambridge University Press, London, 1975.
[8] *J. L. Meijering:* Interface area, edge length and number of vertices in crystal aggregates with random nucleation. Philips Res. Rep. 8 (1953), 270—90.
[9] *R. E. Miles:* The random division of space. Suppl. Adv. Appl. Prob., 1972, 243—66.
[10] *J. Møller:* Random tessellation in R^d. Memoirs No. 9, Department of Theoretical Statistics, University of Aarhus, Denmark, 1986.
[11] *S. A. Saltykov:* Stereometric Metallography. 2nd. Ed., Metallurgizdat, Moscow, 1958.
[12] *C. S. Smith, L. Guttman:* Measurement of internal boundaries in three-dimensional structure by random sectioning. Trans. AIME 197 (1953), 81—87.
[13] *D. Stoyan, W. S. Kendall, J. Mecke:* Stochastic Geometry and Its Applications. Akademie Verlag, Berlin, 1987.
[14] *E. E. Underwood:* Quantitative Stereology. Addison-Wesley, Reading, Mass., 1970.
[15] *S. D. Wicksell:* The corpuscle problem. A mathematical study of a biometric problem. Biometrika, 17 (1925), 84—99.

Souhrn

STEREOLOGIE PRECIPITÁTŮ UMÍSTĚNÝCH NA ROZHRANÍ ZRN

VRATISLAV HORÁLEK

Precipitáty modelované rotačními symetrickými disky jsou umístěny pouze na rozhraní zrn matrice a homogenní vzorek je protnut rovinou řezu. Předložený stereologický model umožňuje vyjádřit všechny základní parametry prostorové struktury pomocí parametrů rovinné struktury, které lze odhadnout z výsledků měření v rovině řezu. Odvozené vztahy jsou transformovány na případ kulových precipitátů. Výsledky jsou porovnány se vztahy platnými ve struktuře, v níž zakotvené precipitáty jsou v matrici rozmístěny náhodně.

Резюме

СТЕРЕОЛОГИЯ ПРЕЦИПИТАТОВ, НАХОДЯЩИХСЯ НА ГРАНИЦЕ РАЗДЕЛА ЗЕРЕН

VRATISLAV HORÁLEK

Преципитаты, моделируемые осесимметричными дисками, размещаются лишь на границах раздела зерен матрицы; однородный образец пересекается произвольной плоскостью сечения. Предлагаемая стереологическая модель позволяет выразить все основные параметры пространственной структуры с помощью параметров плоскостной структуры, которые могут быть получены по результатам измерений в плоскости сечения. Полученные соотношения трансформированы на случай сферических преципитатов. Результаты сравниваются с соотношениями действующими в структурах, где внедренные преципитаты распределены в матрице произвольным образом.

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