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TIME-DEPENDENT INVARIANT REGIONS FOR PARABOLIC SYSTEMS RELATED TO ONE-DIMENSIONAL NONLINEAR ELASTICITY

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Summary. A parabolic system arising as a viscosity regularization of the quasilinear one-dimensional telegraph equation is considered. The existence of L_∞ — a priori estimates, independent of viscosity, is shown. The results are achieved by means of generalized invariant regions.

Keywords: Invariant region, vanishing viscosity, nonlinear parabolic system.

AMS Classification: 35B45, 35K45.

In this paper we consider the a priori estimates of solutions of parabolic systems $\{S_\varepsilon\}_{\varepsilon>0}$:

$$(S_\varepsilon^1) \qquad u_t - v_x + a_1 u = \varepsilon A_1 u\,,$$

$$(S_\varepsilon^2) \qquad v_t - \sigma(x, t, u)_x + a_2 v = \varepsilon A_2 v + f$$

where the periodicity conditions

$$(P) \qquad u(x + l, t) = u(x, t)\,, \quad v(x + l, t) = v(x, t)$$

are imposed on the couple of unknown functions u, v of $x, t \in \mathbb{R}^1$.

Here $a_1, a_2 > 0$ are strictly positive constants, the symbols A_1, A_2 stand for differential operators representing „artificial viscosity" added to the original system

$$(S^1) \qquad u_t - v_x + a_1 u = 0\,,$$

$$(S^2) \qquad v_t - \sigma(x, t, u)_x + a_2 v = f\,.$$

The problem just outlined arises in the study of the one-dimensional damped wave equation of the form

$$(E) \qquad U_{tt} + dU_t - \sigma(x, t, U_x)_x + aU = f\,,$$

the functions u, v being determined by the relations

$$u = U_x\,, \quad v = U_t + a_1 U \quad \text{where } d = a_1 + a_2\,, \quad a = a_1 \,.\, a_2\,.$$

Taking the recent results of the compensated compactness theory into account (see DiPerna [3], Serre [6], Rascle [4]) we expect some boundary value problems connected with (E) to be solved via the method of vanishing viscosity, that is, the solutions are constructed as limits of solutions of the perturbed systems $\{S_\varepsilon\}_{\varepsilon>0}$, $\varepsilon \searrow 0$.

To carry out such a program, a priori estimates on the functions (u, v), independent of ε, have to be found to induce the convergence of approximate solutions almost everywhere. In view of [3] it suffices to look for L_∞-estimates, which forms the bulk of the present paper. In the forthcoming publication [8] we intend to apply these results to the problem of existence of time-periodic solutions related to (E).

Since a nonautonomous problem is involved (σ does depend on x, t), neither the method of Chueh, Conley, Smoller [1] nor the technique of Dafermos [2] (cf. also Serre [5], Venttsel' [7]) seem to be applicable to our situation in a direct way.

We introduce the concept of time-dependent invariant regions (see Section 1), the existence of which results from the presence of the damping terms $a_1 u$, $a_2 v$ in the equations.

Also, the operators A_1, A_2 are to be chosen properly to obtain positive results. Note in passing that, in the autonomous case, the choice $A_1 = A_2 = \Delta$ is well known to ensure the desired estimates (see [1], [2]).

The main idea we use here is closely related to the paper [1] and seems to be quite simple though some computations turn out to be rather lengthy (see Section 2).

Throughout the whole text, the symbols c or c_i, $i = 1, \dots$ will denote strictly positive real constants.

1. MAIN RESULTS

Consider a Cauchy problem resulting from $\{S_\varepsilon\}_{\varepsilon>0}$ by adding the initial conditions

$$(I) \qquad u(x, 0) = u^0(x), \quad v(x, 0) = v^0(x)$$

where $u^0, v^0 \in C(\mathbb{R}^1)$ satisfy (of course) the condition (P).

Whenever speaking about a solution of the problem $\{S_\varepsilon\}$, (I) we mean a classical one, i.e. a pair (u, v) of continuous functions on the strip

$$Q = \{(x, t) \mid x \in \mathbb{R}^1,\ t \in [0, t_0)\}$$

with all derivatives appearing in the equations continuous on

$$\operatorname{int} Q = \{(x, t) \mid x \in \mathbb{R}^1,\ t \in (0, t_0)\},$$

u, v satisfying (S_ε^1), (S_ε^2) on int Q as well as the conditions (P), (I).

As to the data we deal with, let us assume the following:

(C_1) *The function* $\sigma = \sigma(x, t, u): \mathbb{R}^3 \to \mathbb{R}^1$ *is smooth, satisfies* (P) *with respect to* x, *and the growth restrictions*

$$(1.1) \qquad \sigma_u(x, t, u) \geqq c_1 > 0 \,,$$

$$(1.2) \qquad |\sigma_x|, |\sigma_t|, |\sigma_{xu}|, |\sigma_{tu}|, |\sigma_{xxu}| \leqq c_2$$

hold for all x, t, u.

Moreover,

$$(1.3) \qquad \lim_{u \to \pm\infty} \sigma_u(x, t, u) = +\infty \quad \textit{uniformly in} \quad x, t$$

and

$$(1.4) \qquad \sigma_{uu}(x, t, u)\, u > 0 \quad \textit{whenever} \quad u \neq 0 \,.$$

(C_2) $f = f(x, t): \mathbb{R}^2 \to \mathbb{R}^1$ *is a continuous function satisfying* (P) *and uniformly bounded, i.e.*

$$(1.5) \qquad |f(x, t)| \leqq c_3 \quad \textit{for all} \quad x, t \,.$$

The presence of the variables x, t in σ is the reason why we introduce the notion of a time-dependent invariant region.

Definition 1. *A set* $M \subset \mathbb{R}^4$ *is called an invariant region related to the system* $\{S_\varepsilon\}$ *if any solution* (u, v) *of* $\{S_\varepsilon\}$, (I) *satisfying*

$$(1.6) \qquad [x, 0, u^0(x), v^0(x)] \in M \,, \quad x \in \mathbb{R}^1$$

remains in M *for all* t *belonging to its domain of existence; more specifically,*

$$(1.7) \qquad [x, t, u(x, t), v(x, t)] \in M$$

holds for all $x \in \mathbb{R}^1$, $t \in [0, t_0)$.

Consider now the Riemann invariants

$$r(x, t, u, v) = v + \textstyle\int_0^u \sqrt{(\sigma_u(x, t, z))}\, dz \,,$$

$$s(x, t, u, v) = v - \textstyle\int_0^u \sqrt{(\sigma_u(x, t, z))}\, dz \,.$$

Next, denoting

$$\Psi(x, t, u) = \int_0^u \frac{\sigma_{xu}(x, t, z)}{\sigma_u(x, t, z)}\, dz$$

we set

$$A_1 u = u_{xx} + \Psi(x, t, u)_x \,,$$

$$A_2 v = v_{xx} \,.$$

Finally, being motivated by [1] we define the set

$$M = M(c) = \{(x, t, u, v) \mid -c \leqq r,\ s \leqq c\}\ . \tag{1.8}$$

Our main goal is to establish the following result.

Theorem 1. *Let the conditions* $(C_1), (C_2)$ *be satisfied.*

Then the set M *determined by* (1.8) *forms an invariant region for the system* $\{S_\varepsilon\}$ *provided the number* c *is large enough.*

The sufficient magnitude of c *depends exclusively on the data* σ, f *and is independent of* ε.

2. PROOF OF THEOREM 1

We start with an auxiliary assertion.

Lemma 1. *Suppose* $(x, t, u, v) \notin M(c)$.

Then

$$|u| + |v| \geqq h(c) \tag{2.1}$$

where $\lim_{c \to +\infty} h(c) = +\infty$.

Proof. Let $r > c$, $|r| \geqq |s|$, the other cases being treated in a similar way. Necessarily, we have $u, v \geqq 0$ and, consequently,

$$c < r = |v| + \int_0^{|u|} \sqrt{(\sigma_u(x, t, z))}\, dz \leqq$$

(according to (1.4))

$$\leqq |v| + |u| \sqrt{(\sigma_u(x, t, u))} \leqq$$

(in view of (1.2))

$$\leqq |v| + |u| \sqrt{(c_4 + \sigma_u(0, 0, u))}\ ,$$

which yields the desired result. Q.E.D.

We are about to prove Theorem 1. Consider a solution (u, v) of the Cauchy problem $\{S_\varepsilon\}$, (I) defined for $t \in [0, t_0)$ and satisfying (1.6).

For arbitrary $\tilde{t} \in [0, t_0)$ we set

$$w = \max\left\{ \max_{\substack{t \in [0, \tilde{t}] \\ x \in \mathbf{R}^1}} |r(x, t, u(x, t), v(x, t))|,\ \max_{\substack{t \in [0, \tilde{t}] \\ x \in \mathbf{R}^1}} |s(x, t, u(x, t), v(x, t))| \right\}\ .$$

In view of continuity, the value w must be attained at some point (x_1, t_1), $u^1 = u(x_1, t_1)$, $v^1 = v(x_1, t_1)$.

We need only to show $w \leqq c$.

Assume the contrary, i.e. $w > c$. With (1.6) in mind, we get $t_1 \in (0, \tilde{t}]$ and, consequently, four different cases are to be distinguished, namely

(a) $\quad r(x_1, t_1, u^1, v^1) = w \quad$ with $\quad u^1 \geqq 0, \quad v^1 \geqq 0,$

$(2.2)_a$ $\quad r_x = 0, \quad r_{xx} \leqq 0, \quad r_t \geqq 0 \quad$ in $\quad (x_1, t_1);$

(b) $\quad r(x_1, t_1, u^1, v^1) = -w \quad$ with $\quad u^1 \leqq 0, \quad v^1 \leqq 0,$

$(2.2)_b$ $\quad r_x = 0, \quad r_{xx} \geqq 0, \quad r_t \leqq 0 \quad$ in $\quad (x_1, t_1);$

(c) $\quad s(x_1, t_1, u^1, v^1) = w \quad$ with $\quad u^1 \leqq 0, \quad v^1 \geqq 0,$

$(2.2)_c$ $\quad s_x = 0, \quad s_{xx} \leqq 0, \quad s_t \geqq 0 \quad$ in $\quad (x_1, t_1);$

(d) $\quad s(x_1, t_1, u^1, v^1) = -w \quad$ with $\quad u^1 \geqq 0, \quad v^1 \leqq 0,$

$(2.2)_d$ $\quad s_x = 0, \quad s_{xx} \geqq 0, \quad s_t \leqq 0 \quad$ in $\quad (x_1, t_1).$

Carrying out the necessary computations we derive

$$r_x = v_x + \sqrt{(\sigma_u(x, t, u))}\, u_x + \Phi_1 ,$$
$$s_x = v_x - \sqrt{(\sigma_u(x, t, u))}\, u_x - \Phi_1 ,$$

where

$$\Phi_1 = \frac{1}{2} \int_0^u \frac{\sigma_{xu}(x, t, z)}{\sqrt{(\sigma_u(x, t, z))}}\, \mathrm{d}z ;$$

$$r_{xx} = v_{xx} + \sqrt{(\sigma_u(x, t, u))}\, u_{xx} + \frac{\sigma_{xu}(x, t, u)}{\sqrt{(\sigma_u(x, t, u))}}\, u_x +$$
$$+ \frac{\sigma_{uu}(x, t, u)}{2\sqrt{(\sigma_u(x, t, u))}}\, u_x^2 + \Phi_2 ,$$

$$s_{xx} = v_{xx} - \sqrt{(\sigma_u(x, t, u))}\, u_{xx} - \frac{\sigma_{xu}(x, t, u)}{\sqrt{(\sigma_u(x, t, u))}}\, u_x -$$
$$- \frac{1}{2} \frac{\sigma_{uu}(x, t, u)}{\sqrt{(\sigma_u(x, t, u))}}\, u_x^2 - \Phi_2$$

with

$$\Phi_2 = \frac{1}{2} \int_0^u \frac{\sigma_{xxu}(x, t, z)}{\sqrt{(\sigma_u(x, t, z))}}\, \mathrm{d}z - \frac{1}{4} \int_0^u \frac{\sigma_{xu}^2(x, t, z)}{(\sigma_u(x, t, z))^{3/2}}\, \mathrm{d}z .$$

Next, we have

$$r_t = v_t + \sqrt{(\sigma_u(x, t, u))}\, u_t + \Phi_3 ,$$

where

$$\Phi_3 = \frac{1}{2} \int_0^u \frac{\sigma_{tu}(x, t, z)}{\sqrt{(\sigma_u(x, t, z))}}\, \mathrm{d}z .$$

Taking advantage of the equations (S_ε^1), (S_ε^2) we are led to the conclusion that

$$r_t = \sigma(x, t, u)_x - a_2 v + \varepsilon v_{xx} + f +$$
$$+ \sqrt{(\sigma_u(x, t, u))}\,(v_x - a_1 u + \varepsilon u_{xx} + \varepsilon\Psi(x, t, u)_x) + \Phi_3 ;$$

similarly

$$s_t = \sigma(x, t, u)_x - a_2 v + \varepsilon v_{xx} + f -$$
$$- \sqrt{(\sigma_u(x, t, u))}\,(v_x - a_1 u + \varepsilon u_{xx} + \varepsilon\Psi(x, t, u)_x) - \Phi_3 .$$

Suppose first that (a) holds. In view of the relations just obtained, we have

$$r_t = B_1 + B_2 + B_3 + B_4 + B_5$$

where

$$B_1 = \sqrt{(\sigma_u(x, t, u))}\,(v_x + \sqrt{(\sigma_u(x, t, u))}\, u_x) ,$$

$$B_2 = \varepsilon\left(v_{xx} + \sqrt{(\sigma_u(x, t, u))}\, u_{xx} + \frac{\sigma_{xu}(x, t, u)}{\sqrt{(\sigma_u(x, t, u))}}\, u_x\right),$$

$$B_3 = f + \sigma_x(x, t, u) ,$$

$$B_4 = \Phi_3 + \varepsilon \sqrt{(\sigma_u(x, t, u))}\, \Phi_4$$

with

$$\Phi_4 = \int_0^u \frac{\sigma_{xxu}(x, t, z)}{\sigma_u(x, t, z)}\, dz - \int_0^u \frac{\sigma_{xu}^2(x, t, z)}{\sigma_u^2(x, t, z)}\, dz ,$$

$$B_5 = -a_2 v - a_1 \sqrt{(\sigma_u(x, t, u))}\, u .$$

Seeing that $r_x = 0$ we get the estimate

$$B_1 + B_3 + B_4 \leqq \sqrt{(\sigma_u(x_1, t_1, u^1))}\,(-\Phi_1 + \varepsilon\Phi_4) + \Phi_3 + c_2 + c_3 . \tag{2.3}$$

The relation $r_{xx} \leqq 0$ brings forth

$$B_2 \leqq \varepsilon\left(-\frac{1}{2}\,\frac{\sigma_{uu}(x_1, t_1, u^1)}{\sqrt{(\sigma_u(x_1, t_1, u^1))}}\, u_x^2 - \Phi_2\right).$$

The condition (1.4) together with the inequality $u^1 \geqq 0$ implies

$$B_2 \leqq \varepsilon(-\Phi_2) . \tag{2.4}$$

Our aim is to show $r_t < 0$ for c large enough, which contradicts $(2.2)_a$. With the relations (2.3), (2.4) in mind, we are to cope with the term

$$r_t \leqq c_5(1 + \sqrt{(\sigma_u(x_1, t_1, u^1))}) \sum_{i=1}^{4} |\Phi_i| - a_2|v^1| - a_1 \sqrt{(\sigma_u(x_1, t_1, u^1))}\, |u^1|$$

where Φ_i are of the form

$$\textstyle\int_0^{|u^1|} g(x_1, t_1, z)\, dz , \quad g(x, t, z) \to 0 \quad \text{for} \quad z \to +\infty$$

uniformly in x, t due to (1.2), (1.3).

We conclude that

$$r_t \leqq c_6 - c_7 h(c) \tag{2.5}$$

where h appears in (2.1) and $c_6, c_7 > 0$ depend on the functions σ, f only.

Thus Lemma 1 completes the proof in the case (a).

One easily observes that the case (b) can be treated in a similar way. Note that, since $u^1 \leqq 0$, we have

$$B_2 \geqq -\varepsilon\Phi_2$$

in view of (1.4).

In the conclusion, let us sketch the way how to treat the case (c). The reader will observe that a similar technique applies to (d).

Following the line of arguments from the case (a), we can decompose

$$s_t = D_1 + D_2 + D_3 + D_4 + D_5$$

where

$$D_1 = \sqrt{(\sigma_u(x, t, u))}\,(\sqrt{(\sigma_u(x, t, u))}\, u_x - v_x)\,,$$

$$D_2 = \varepsilon\left(v_{xx} - \sqrt{(\sigma_u(x, t, u))}\, u_{xx} - \frac{\sigma_{xu}(x, t, u)}{\sqrt{(\sigma_u(x, t, u))}}\, u_x\right),$$

$$D_3 = B_3\,, \quad D_4 = -B_4\,,$$

$$D_5 = -a_2 v + a_1 \sqrt{(\sigma_u(x, t, u))}\, u\,.$$

We confine ourselves to the most difficult term D_2. Seeing that $s_{xx} \leqq 0$ we obtain

$$D_2 \leqq \varepsilon\left(\frac{1}{2}\frac{\sigma_{uu}(x_1, t_1, u^1)}{\sqrt{(\sigma_u(x_1, t_1, u^1))}}\, u_x^2 + \Phi_2\right) \leqq$$

(combining (1.4) together with $u^1 \leqq 0$)

$$\leqq \varepsilon\Phi_2\,.$$

Since we have $u^1 \leqq 0$, $v^1 \geqq 0$, the remaining part of the proof is literally the same as in the case (a).

Thus, Theorem 1 has been proved.

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Souhrn

ČASOVĚ ZÁVISLÉ INVARIANTNÍ OBLASTI PRO PARABOLICKÉ SYSTÉMY, TÝKAJÍCÍ SE JEDNODIMENSIONÁLNÍ NELINEÁRNÍ ELASTICITY

EDUARD FEIREISL

Je vyšetřován parabolický systém vznikající při řešení nelineární telegrafní rovnice metodou mizející viskosity. Dokážeme existenci apriorních odhadů v prostoru L_∞ nezávislých na viskositě. Výsledku je dosaženo pomocí metody zobecněných invariantních oblastí.

Резюме

ЗАВИСЯЩИЕ ОТ ВРЕМЕНИ ИНВАРИАНТНЫЕ ОБЛАСТИ ДЛЯ ПАРАБОЛИЧЕСКИХ СИСТЕМ, КАСАЮЩИХСЯ ОДНОМЕРНОЙ НЕЛИНЕЙНОЙ ЗАДАЧИ УПРУГОСТИ

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Изучается параболическая система, возникающая при решении квазилинейного телеграфного уравнения методом искуственной вязкости. Доказано существование независимых от вязкости априорных оценок в пространстве L_∞. Результаты получены при помощи метода обобщённых инвариантных областей.

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