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EQUIVALENT FORMULATIONS OF GENERALIZED VON KÁRMÁN EQUATIONS FOR CIRCULAR VISCOELASTIC PLATES

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Summary. The paper deals with the analysis of generalized von Kármán equations which describe stability of a thin circular viscoelastic clamped plate of constant thickness under a uniform compressive load which is applied along its edge and depends on a real parameter. The meaning of a solution of the mathematical problem is extended and various equivalent reformulations of the problem are considered. The structural pattern of the generalized von Kármán equations is analyzed from the point of view of nonlinear functional analysis.

Keywords: Von Kármán equations, viscoelastic plates.

AMS Classification: 35D99.

1. INTRODUCTION

The analysis of von Kármán equations has its own history. Since their derivation in 1917 they have attracted attention of different groups of specialists. In the first place they have been analyzed from the point of view of mechanics, because their eigenvalues are the critical values at which a plate buckles, and when a load increases above these values it is necessary to investigate its postbuckling behaviour. There exist papers dealing with the approximate numerical analysis of postbuckling behaviour of plates.

Also a great number of mathematicians have dealt with the analysis of mathematical problems connected with von Kármán equations. It would be difficult to give the complete list of authors and papers. However, even now we are stil very far from a fully successful analysis of solutions with "big" norms.

Von Kármán equations were derived for elastic materials. However, many materials exhibit mechanical behaviour different from the elastic ones, i.e., their deformation under the loading which is constant in time increases. They exhibit an instant elastic deformation followed by a viscous flow. They are called viscoelastic materials.

We deal with the mathematical analysis of generalized von Kármán equations for viscoelastic plates. They were derived [1] similarly as the original von Kármán equations. The analysis shows that the behaviour of viscoelastic plates under a pressure along their edges exhibits some qualitatively different features as compared with elastic ones.

We consider a thin circular viscoelastic clamped plate under a uniform compressive load which is applied along its edge in its midplane and depends on a real parameter λ, with zero initial conditions.

In this paper the physical problem to be considered is described. We extended the meaning of a solution of the associated mathematical problem and consider various equivalent reformulations of the problem. We analyze the structural pattern of generalized von Kármán equations from the point of view of nonlinear functional analysis.

We will deal with the detailed analysis of this problem also in the following two papers "Bifurcations of Generalized von Kármán Equations for Circular Viscoelastic Plates" and "Analysis of Postbuckling Solutions of Generalized von Kármán Equations for Circular Viscoelastic Plates".

2. FORMULATION OF THE PROBLEM

We consider generalized von Kármán equations for viscoelastic plates of standard materials [1]

$$K(1 + \alpha D_t)\,\Delta^2 W = (1 + \beta D_t)\,\{\lambda_a[W, F_0] + [W, F]\}\,, \tag{2.1}$$

$$(1 + \beta D_t)\,\Delta^2 F = -\tfrac{1}{2}hE(1 + \alpha D_t)\,[W; W] \tag{2.2}$$

where $W(x, y, t)$ is the transverse displacement of the plate, $F(x, y, t)$ is the stress function, $F_0(x, y, t)$ is the stress function corresponding to the given boundary loading, E is the modulus of elasticity, h is the thickness of the plate, K is the stiffness of the plate, α, β are positive viscous parameters such that $\alpha > \beta$, λ_a is the positive parameter of proportionality of the given boundary loading with respect to F_0, D_t denotes the differentiation with respect to time, Δ^2 is the biharmonic operator and

$$[f, g] = f_{xx}g_{yy} + f_{yy}g_{xx} - 2f_{xy}g_{xy}\,.$$

In what follows we deal with an analysis of a viscoelastic buckling of a circular clamped plate loaded with a uniform compressive load proportional to λ_a and applied in its midplane along its edge. We consider zero initial conditions. Then (2.1) and (2.2) depend only on the polar coordinate r and assume the form

$$K(1 + \alpha D_t)\, r\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}W(r, t)}{\mathrm{d}r}\right)\right) = \tag{2.3}$$
$$= (1 + \beta D_t)\left\{\frac{\mathrm{d}F(r, t)}{\mathrm{d}r}\frac{\mathrm{d}W(r, t)}{\mathrm{d}r} - \lambda_a r\frac{\mathrm{d}W(r, t)}{\mathrm{d}r}\right\},$$

$$(2.4) \qquad (1 + \beta D_t)\, r \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r \frac{\mathrm{d}F(r, t)}{\mathrm{d}r}\right)\right) =$$

$$= -\frac{1}{2} hE(1 + \alpha D_t)\left\{\frac{\mathrm{d}W(r, t)}{\mathrm{d}r}\right\}^2 \qquad r \in (0, R)\,, \quad t \in (0, T\rangle\,, \quad T < \infty$$

with the boundary conditions

$$(2.5) \qquad W(r, t)\big|_{r=R} = 0 \qquad t \in \langle 0, T\rangle$$

$$(2.6) \qquad F(r, t)\big|_{r=R} = 0 \qquad t \in \langle 0, T\rangle$$

$$(2.7) \qquad \frac{\mathrm{d}W(r, t)}{\mathrm{d}r}\bigg|_{r=R} = 0 \qquad t \in \langle 0, T\rangle$$

$$(2.8) \qquad \frac{\mathrm{d}F(r, t)}{\mathrm{d}r}\bigg|_{r=R} = 0 \qquad t \in \langle 0, T\rangle$$

$$(2.9) \qquad \frac{\mathrm{d}W(r, t)}{\mathrm{d}r}\bigg|_{r=0} = 0 \qquad t \in \langle 0, T\rangle$$

$$(2.10) \qquad \frac{\mathrm{d}F(r, t)}{\mathrm{d}r}\bigg|_{r=0} = 0 \qquad t \in \langle 0, T\rangle$$

and the initial conditions

$$(2.11) \qquad W(r, t)\big|_{t=0^-} = 0 \qquad r \in \langle 0, R\rangle$$

$$(2.12) \qquad F(r, t)\big|_{t=0^-} = 0 \qquad r \in \langle 0, R\rangle$$

where R is the radius of the plate.

Using the relations

$$r = xR\,,$$

$$w(x, t) = \frac{1}{x}\left(\frac{E\,h}{2\,K}\right)^{1/2} \frac{\mathrm{d}W(xR, t)}{\mathrm{d}x}\,,$$

$$f(x, t) = \frac{1}{Kx}\frac{\mathrm{d}F(xR, t)}{\mathrm{d}x}\,,$$

$$\lambda = \frac{R^2}{K}\lambda_a$$

the system (2.3)–(2.12) can be reduced to the form

$$(2.13) \qquad (1 + \alpha D_t)\left[x^3 w'(x, t)\right]' = (1 + \beta D_t)\, x^3 w(x, t)\left[f(x, t) - \lambda\right],$$

$$(2.14) \qquad (1 + \beta D_t)\left[x^3 f'(x, t)\right]' = -(1 + \alpha D_t)\, x^3 w^2(x, t)$$

$$x \in (0, 1)\,; \quad t \in (0, T\rangle\,; \quad T < \infty\,,$$

(2.15)	$\lvert w(x,t)\rvert_{x=0}\rvert < \infty$		$t \in \langle 0, T \rangle$
(2.16)	$\lvert f(x,t)\rvert_{x=0}\rvert < \infty$		$t \in \langle 0, T \rangle$
(2.17)	$w(x,t)\rvert_{x=1} = 0$		$t \in \langle 0, T \rangle$
(2.18)	$f(x,t)\rvert_{x=1} = 0$		$t \in \langle 0, T \rangle$
(2.19)	$w(x,t)\rvert_{t=0^-} = 0$		$x \in \langle 0, 1 \rangle$
(2.20)	$f(x,t)\rvert_{t=0^-} = 0$		$x \in \langle 0, 1 \rangle$

where a prime denotes the differentiation with respect to the space variable.

3. VARIOUS DEFINITIONS OF THE SOLUTION

Definition 3.1. *A classical solution of the problem* (2.13)–(2.20) *is a pair of functions* $w(x, t)$, $f(x, t)$ *with the following properties*:

a) $w(x, t), f(x, t) \in C^1((0, T\rangle; C^2(\langle 0, 1\rangle))$ *for arbitrary* T, $0 < T < \infty$;

b) $w(x, t), f(x, t)$ *satisfy* (2.13)–(2.20) *pointwise for some real number* λ.

Using the transformation

$$\frac{1}{\beta}\int_0^t g(\tau)\, K(t-\tau)\,\mathrm{d}\tau = (1+\beta D_t)^{-1}\, g(t)$$

for zero initial conditions where the kernel K has the form

$$K(t-\tau) = \exp\left[-\frac{1}{\beta}(t-\tau)\right]$$

we get from (2.13) and (2.14) the following equations:

$$(3.1)\qquad [x^3 w'(x,t)]' - \frac{1}{\alpha}\left(\frac{\alpha}{\beta}-1\right)\int_0^t [x^3 w'(x,\tau)]'\, K(t-\tau)\,\mathrm{d}\tau = $$
$$= \frac{\beta}{\alpha}\, x^3 w(x,t)\,[f(x,t)-\lambda]\,,$$

$$(3.2)\qquad [x^3 f'(x,t)]' = \frac{1}{\beta}\left(\frac{\alpha}{\beta}-1\right)\int_0^t x^3 w^2(x,\tau)\, K(t-\tau)\,\mathrm{d}\tau - \frac{\alpha}{\beta}\, x^3 w^2(x,t)$$

$$x \in (0,1)\,;\quad t \in \langle 0, T\rangle\,;\quad T < \infty\,.$$

Definition 3.2. *A generalized solution in the time variable of the problem* (2.13)–(2.20) *is a pair of functions* $w(x, t)$, $f(x, t)$ *with the following properties*:

a) $w(x, t), f(x, t) \in L_\infty(0, T; C^2(\langle 0, 1\rangle))$ *for arbitrary* T, $0 < T < \infty$;

b) $w(x, t)$, $f(x, t)$ *satisfy* (3.1), (3.2), (2.15)–(2.18) *pointwise for some real number* λ.

Let $W^{1,2}((0, 1), x^3)$ be the real Sobolev space with the weight x^3 and with the inner product

$$(u, v)_{1,2,x^3} = \int_0^1 x^3\, u(x)\, v(x)\, \mathrm{d}x + \int_0^1 x^3\, u'(x)\, v'(x)\, \mathrm{d}x$$

and the corresponding norm

$$\|u\|_{1,2,x^3} = [(u, u)_{1,2,x^3}]^{1/2}\,. \tag{3.3}$$

We denote

$$M = \{u \in C^\infty(\langle 0, 1\rangle);\ u(1) = 0\}\,.$$

Then we introduce a real Hilbert space H defined as the closure of the set M in the norm (3.3). A more convenient inner product and norm for the space H is [3]

$$\langle u, v\rangle = \int_0^1 x^3\, u'(x)\, v'(x)\, \mathrm{d}x\,, \tag{3.4}$$

$$\|u\|_H = [\langle u, u\rangle]^{1/2}\,. \tag{3.5}$$

In the sequel we equip H with the inner product given by (3.4) and with the norm given by (3.5).

Let φ, ψ be smooth functions in H. Then integrating (3.1) and (3.2) by parts over $(0, 1)$ we obtain

$$\langle w(t), \varphi\rangle = \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right)\int_0^t \langle w(\tau), \varphi\rangle\, K(t - \tau)\, \mathrm{d}\tau + \tag{3.6}$$
$$+ \lambda \frac{\beta}{\alpha}\int_0^1 x^3 w(x, t)\, \varphi(x)\, \mathrm{d}x - \frac{\beta}{\alpha}\int_0^1 x^3 w(x, t) f(x, t)\, \varphi(x)\, \mathrm{d}x\,,$$

$$\langle f(t), \psi\rangle = \frac{\alpha}{\beta}\int_0^1 x^3 w^2(x, t)\, \psi(x)\, \mathrm{d}x - \tag{3.7}$$
$$- \frac{1}{\beta}\left(\frac{\alpha}{\beta} - 1\right)\int_0^t\int_0^1 x^3 w^2(x, \tau)\, K(t - \tau)\, \psi(x)\, \mathrm{d}x\, \mathrm{d}\tau$$

for a.e. $t \in \langle 0, T\rangle$.

Definition 3.3. *A generalized solution of the problem* (2.13)–(2.20) *is a pair of functions* $w(x, t), f(x, t)$ *with the following properties*:

a) $w(x, t)$, $f(x, t) \in L_\infty(0, T; H)$ *for arbitrary* T, $0 < T < \infty$;

b) $w(x, t), f(x, t)$ *satisfy* (3.6) *and* (3.7) *for all test functions* $\varphi, \psi \in H$, *some real number* λ *and a.e.* $t \in \langle 0, T\rangle$.

Theorem 3.1. *Any generalized solution in the time variable of the problem* (2.13)–(2.20) *is a generalized solution. Conversely, any generalized solution of the problem* (2.13)–(2.20) *is a generalized solution in the time variable.*

In the proof of this theorem we use the following results [5], [2].

Assertion 3.1. *A solution of the equation*

$$v(x, t) - \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right)\int_0^t v(x, \tau)\, K(t - \tau)\, d\tau = u(x, t)$$

where $u(x, t) \in L_\infty(0, T; B)$, *B is a Banach space, belongs to the space* $L_\infty(0, T; B)$ *and has the form*

$$v(x, t) = u(x, t) + \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right)\int_0^t \exp\left[-\frac{1}{\alpha}(t - \tau)\right] f(x, \tau)\, d\tau\,.$$

Assertion 3.2. *Let* $u \in H$, *then*

$$\int_0^1 x|u(x)|^2\, dx \leqq \int_0^1 x^3|u'(x)|^2\, dx\,.$$

Proof of Theorem 3.1. The first assertion is obvious. We prove the second. Let $w(x, t), f(x, t) \in L_\infty(0, T; H)$ be the generalized solution of the problem (2.13)–(2.20). From (3.6) and (3.7) it follows for a.e. $t \in \langle 0, T\rangle$ that

$$\int_0^1 \Big\{x^3 w'(x, t) - \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right)\int_0^t x^3 w'(x, \tau)\, K(t - \tau)\, d\tau + \tag{3.8}$$
$$+ \frac{\beta}{\alpha}\lambda\int_0^x s^3 w(s, t)\, ds - \frac{\beta}{\alpha}\int_0^x s^3 w(s, t) f(s, t)\, ds\Big\}\, \varphi'(x)\, dx = 0\,,$$

$$\int_0^1 \Big\{x^3 f'(x, t) + \frac{\alpha}{\beta}\int_0^x s^3 w^2(s, t)\, ds - \tag{3.9}$$
$$- \frac{1}{\beta}\left(\frac{\alpha}{\beta} - 1\right)\int_0^x\int_0^t s^3 w^2(s, \tau)\, K(t - \tau)\, d\tau\, ds\Big\}\, \psi'(x)\, dx = 0\,.$$

We denote

$$p_1(x, t) = x^3 w'(x, t) - \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right)\int_0^t x^3 w'(x, \tau)\, K(t - \tau)\, d\tau +$$
$$+ \frac{\beta}{\alpha}\lambda\int_0^x s^3 w(s, t)\, ds - \frac{\beta}{\alpha}\int_0^x s^3 w(s, t) f(s, t)\, ds\,,$$

$$p_2(x, t) = x^3 f'(x, t) + \frac{\alpha}{\beta}\int_0^x s^3 w^2(s,t)\, ds -$$
$$- \frac{1}{\beta}\left(\frac{\alpha}{\beta} - 1\right)\int_0^x\int_0^t s^3 w^2(s, \tau)\, K(t - \tau)\, d\tau\, ds$$

and let

$$\varphi(x, t) = \int_0^x \{p_1(s, t) - c_1(t)\}\, ds\,,$$
$$\psi(x, t) = \int_0^x \{p_2(s, t) - c_2(t)\}\, ds$$

where

$$c_1(t) = \int_0^1 p_1(s, t)\,\mathrm{d}s\,,$$
$$c_2(t) = \int_0^1 p_2(s, t)\,\mathrm{d}s\,.$$

Then (3.8) and (3.9) yield

$$x^3 w'(x, t) - \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right)\int_0^t x^3 w'(x, \tau)\,K(t - \tau)\,\mathrm{d}\tau =$$
$$= -\frac{\beta}{\alpha}\lambda\int_0^x s^3 w(s, t)\,\mathrm{d}s + \frac{\beta}{\alpha}\int_0^x s^3 w(s, t) f(s, t)\,\mathrm{d}s + c_1(t)\,,$$
$$x^3 f'(x, t) = -\frac{\alpha}{\beta}\int_0^x s^3 w^2(s, t)\,\mathrm{d}s +$$
$$+ \frac{1}{\beta}\left(\frac{\alpha}{\beta} - 1\right)\int_0^x\int_0^t s^3 w^2(s, \tau)\,K(t - \tau)\,\mathrm{d}\tau\,\mathrm{d}s + c_2(t)$$

for a.e. $x \in \langle 0, 1\rangle$ and a.e. $t \in \langle 0, T\rangle$. Then

$$h_1(x, t) = x^3 h_2'(x, t) =$$
$$= x^3\left\{w'(x, t) - \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right)\int_0^t w'(x, \tau)\,K(t - \tau)\,\mathrm{d}\tau\right\},$$
$$g(x, t) = x^3 f'(x, t)$$

are continuous functions in the space variable x on $\langle 0, 1\rangle$ for a.e. $t \in \langle 0, T\rangle$, and according to Assertion 3.1

$$h(x, t) = x^3 w'(x, t)$$

is also a continuous function in the space variable x on $\langle 0, 1\rangle$ for a.e. $t \in \langle 0, T\rangle$. We assert $c_i(t) = 0$; $i = 1, 2$ for a.e. $t \in \langle 0, T\rangle$. If this were not true, we would have

$$h_1(0, t_1) = \lim_{x\to 0^+} h_1(x, t_1) = \lim_{x\to 0^+} x^3 h_2'(x, t_1) = c_1(t_1) \neq 0 \tag{3.10}$$

for $t_1 \in M_{t_1}$ where $M_{t_1} \subset \langle 0, T\rangle$ and $\operatorname{mes} M_{t_1} > 0$. Without loss of generality we may suppose $c_1(t_1) > 0$ for $t_1 \in M_{t_1}$. In virtue of (3.10) we have for $t_1 \in M_{t_1}$

$$\forall\varepsilon(t_1) > 0\ \{c_1(t_1) > \varepsilon(t_1) > 0\}\ \exists\delta(t_1) < 1 \mid 0 < x < \delta(t_1) \Rightarrow$$
$$\Rightarrow 0 < \frac{c_1(t_1) - \varepsilon(t_1)}{x^3} < h_2'(x, t) < \frac{c_1(t_1) + \varepsilon(t_1)}{x^3}\,.$$

As $h_2(x, t) \in L_\infty(0, T; H)$ we conclude that for $t_1 \in M_{t_1}$

$$\infty > \|h_2(t)\|_H^2 = \int_0^1 x^3\{h_2'(x, t_1)\}^2\,\mathrm{d}x \geqq \int_0^{\delta(t_1)} x^3\{h_2'(x, t_1)\}^2\,\mathrm{d}x >$$
$$> \{c_1(t_1) - \varepsilon(t_1)\}\int_0^{\delta(t_1)} \frac{1}{x^3}\,\mathrm{d}x$$

which is a contradiction. The same holds for $c_2(t)$. We have

$$(3.11)\qquad x^3 w'(x, t) - \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right)\int_0^t x^3 w'(x, \tau)\, K(t - \tau)\, d\tau =$$

$$= -\frac{\beta}{\alpha}\lambda \int_0^x s^3 w(s, t)\, ds + \frac{\beta}{\alpha}\int_0^t s^3 w(s, t) f(s, t)\, ds\,,$$

$$(3.12)\qquad x^3 f'(x, t) = -\frac{\alpha}{\beta}\int_0^x s^3 w^2(s, t)\, ds +$$

$$+ \frac{1}{\beta}\left(\frac{\alpha}{\beta} - 1\right)\int_0^x \int_0^t s^3 w^2(s, \tau)\, K(t - \tau)\, d\tau\, ds\,.$$

The functions $h(x, t)$ and $g(x, t)$ are continuous in the space variable on $\langle 0, 1\rangle$ for a.e. $t \in \langle 0, T\rangle$, hence $w'(x, t)$ and $f'(x, t)$ are continuous in the space variable on $(0, 1\rangle$ for a.e. $t \in \langle 0, T\rangle$, and

$$(3.13)\qquad w(x, t) = \int_1^x w'(s, t)\, ds\,,$$

$$(3.14)\qquad f(x, t) = \int_1^x f'(s, t)\, ds$$

are continuous functions in the space variable on $(0, 1\rangle$ for a.e. $t \in \langle 0, T\rangle$. Now we show that for a.e. $t \in \langle 0, T\rangle$ there exist

$$\lim_{x \to 0^+} w'(x, t) \quad \text{and} \quad \lim_{x \to 0^+} f'(x, t)$$

and so the functions $w'(x, t)$ and $f'(x, t)$ are continuous in the space variable on $\langle 0, 1\rangle$ for a.e. $t \in \langle 0, T\rangle$.

From (3.11)–(3.14) we get

$$(3.15)\qquad w(x, t) - \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right)\int_0^t w(x, \tau)\, K(t, - \tau)\, d\tau =$$

$$= \lambda \frac{\beta}{\alpha}\int_x^1 \frac{1}{s^3}\int_0^s p^3 w(p, t)\, dp\, ds - \frac{\beta}{\alpha}\int_x^1 \frac{1}{s^3}\int_0^s p^3 w(p, t) f(p, t)\, dp\, ds\,,$$

$$(3.16)\qquad f(x, t) = \frac{\alpha}{\beta}\int_x^1 \frac{1}{s^3}\int_0^s p^3 w^2(p, t)\, dp\, ds -$$

$$- \frac{1}{\beta}\left(\frac{\alpha}{\beta} - 1\right)\int_x^1 \frac{1}{s^3}\int_0^s p^3 \int_0^t w^2(p, \tau)\, K(t - \tau)\, d\tau\, dp\, ds$$

for $x \in (0, 1\rangle$ and a.e. $t \in \langle 0, T\rangle$. From (3.16) using Assertion 3.2 we get for $x \in (0, 1\rangle$ and a.e. $t \in \langle 0, T\rangle$

$$(3.17)\qquad 0 \leqq |f(x, t)| \leqq -\ln x \left\{\frac{\alpha}{\beta} + \left(\frac{\alpha}{\beta} - 1\right)\left[1 - \exp\left(-\frac{1}{\beta}t\right)\right]\right\}\,.$$

$$\cdot \|w\|^2_{L_\infty(0,T;H)} = -k_1 \ln x < \infty\,.$$

With help of Assertion 3.1 (3.15) can be written in the form

$$(3.18)\qquad w(x, t) = \lambda \frac{\beta}{\alpha} \int_x^1 \frac{1}{s^3} \int_0^s p^3 w(p, t)\,dp\,ds - \frac{\beta}{\alpha} \int_x^1 \frac{1}{s^3} \int_0^s p^3 w(p, t)\,.$$

$$. f(p, t)\,dp\,ds + \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right) \int_0^t \left\{\lambda \frac{\beta}{\alpha} \int_x^1 \frac{1}{s^3} \int_0^s p^3 w(p, \tau)\,dp\,ds - \right.$$

$$\left. - \frac{\beta}{\alpha} \int_x^1 \frac{1}{s^3} \int_0^s p^3 w(p, \tau) f(p, \tau)\,dp\,ds\right\} \exp\left\{-\frac{1}{\alpha}(t - \tau)\right\} d\tau\,.$$

From (3.18) using the Cauchy-Schwarz inequality and Assertion 3.2 we get for $x \in (0, 1\rangle$ and for a.e. $t \in \langle 0, T\rangle$

$$(3.19)\qquad 0 \leqq |w(x, t)| \leqq -\frac{\beta}{\alpha} \ln x \left\{1 + \left(\frac{\alpha}{\beta} - 1\right)\left[1 - \exp\left(-\frac{1}{\alpha} t\right)\right]\right\} .$$

$$. \left\{\frac{\lambda}{2} + \|f\|_{L_\infty(0,T;H)}\right\} \|w\|_{L_\infty(0,T;H)} = k_2 \ln x < \infty\,.$$

With help of Assertion 3.1 (3.11) can be written in the form

$$(3.20)\qquad w'(x, t) = -\frac{\beta}{\alpha} \lambda \frac{1}{x^3} \int_0^x s^3 w(s, t)\,ds + \frac{\beta}{\alpha} \frac{1}{x^3} \int_0^x s^3 w(s, t) f(s, t)\,ds +$$

$$+ \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right) \int_0^t \left\{-\frac{\beta}{\alpha} \lambda \frac{1}{x^3} \int_0^x s^3 w(s, \tau)\,ds + \right.$$

$$\left. + \frac{\beta}{\alpha} \frac{1}{x^3} \int_0^x s^3 w(s, \tau) f(s, \tau)\right\} \exp\left\{-\frac{1}{\alpha}(t - \tau)\right\} d\tau\,.$$

From (3.20) using (3.17) and (3.19) and the l'Hospital rule we have for $x \in (0, 1\rangle$ and a.e. $t \in \langle 0, T\rangle$

$$\lim_{x \to 0^+} |w'(x, t)| \leqq \lim_{x \to 0^+} \left\{-\lambda k_2 \frac{1}{x^3} \int_0^x s^3 \ln s\,ds + \right.$$

$$\left. + k_1 k_2 \frac{1}{x^3} \int_0^x s^3 \ln^2 s\,ds\right\} = 0\,.$$

Analogously from (3.12) using (3.19) and the l'Hospital rule we get for $x \in (0, 1\rangle$ and a.e. $t \in \langle 0, T\rangle$

$$\lim_{x \to 0^+} |f'(x, t)| \leqq k_2^2 \left(\frac{\alpha}{\beta}\right)^2 \lim_{x \to 0^+} \frac{1}{x^3} \int_0^x s^3 \ln^2 s\,ds = 0\,.$$

Now we show that for a.e. $t \in \langle 0, T\rangle$ there exist

$$\lim_{x \to 0^+} w''(x, t) \quad \text{and} \quad \lim_{x \to 0^+} f''(x, t)\,.$$

Differentiating (3.20) and (3.12) with respect to the space variable on $(0, 1\rangle$ for a.e. $t \in \langle 0, T\rangle$ we have

$$(3.21) \qquad w''(x, t) = -\frac{3}{x^4}\left\{-\frac{\beta}{\alpha}\lambda\int_0^x s^3 w(s, t)\,\mathrm{d}s + \frac{\beta}{\alpha}\int_0^x s^3 w(s, t) f(s, t)\,\mathrm{d}s + \right.$$

$$+ \frac{\beta}{\alpha^2}\left(\frac{\alpha}{\beta} - 1\right)\int_0^t\left\{-\lambda\int_0^x s^3 w(s, \tau)\,\mathrm{d}s + \int_0^x s^3 w(s, \tau) f(s, \tau)\,\mathrm{d}s\right\}.$$

$$\left.. \exp\left[-\frac{1}{\alpha}(t - \tau)\right]\mathrm{d}\tau\right\} - \frac{\beta}{\alpha}\lambda w(x, t) + \frac{\beta}{\alpha} w(x, t) f(x, t) +$$

$$+ \frac{\beta}{\alpha^2}\left(\frac{\alpha}{\beta} - 1\right)\int_0^t \{-\lambda w(x, \tau) + w(x, \tau) f(x, \tau)\}\exp\left[-\frac{1}{\alpha}(t - \tau)\right]\mathrm{d}\tau\,,$$

$$(3.22) \qquad f''(x, t) = -\frac{3}{x^4}\left\{-\frac{\alpha}{\beta}\int_0^x s^3 w^2(s, t)\,\mathrm{d}s + \right.$$

$$\left. + \frac{1}{\beta}\left(\frac{\alpha}{\beta} - 1\right)\int_0^t\int_0^x s^3 w^2(s, \tau)\,K(t - \tau)\,\mathrm{d}s\,\mathrm{d}\tau\right\} -$$

$$- \frac{\alpha}{\beta} w^2(x, t) + \frac{1}{\beta}\left(\frac{\alpha}{\beta} - 1\right)\int_0^t w^2(x, \tau)\,K(t - \tau)\,\mathrm{d}\tau\,.$$

Let

$$M = \operatorname*{ess\,sup}_{t\in\langle 0,T\rangle}\ \max_{x\in\langle 0,1\rangle}\ \{w(x, t), f(x, t)\}\,.$$

From (3.21) and (3.22) we get for $x \in \langle 0, 1\rangle$ and a.e. $t \in \langle 0, T\rangle$

$$|w''(x, t)| \leqq \frac{7}{4} M(\lambda + M)\,,$$

$$|f''(x, t)| \leqq \frac{7}{4} M^2\left(2\frac{\alpha}{\beta} - 1\right),$$

hence

$$\lim_{x\to 0^+} |w''(x, t)| < \infty\,, \quad \lim_{x\to 0^+} |f''(x, t)| < \infty\,.$$

Now differentiating (3.11) and (3.12) with respect to the space variable we obtain the desired result.

Using (3.15) and (3.16) we have

Definition 3.4. *A w-generalized solution of the problem* (2.13)–(2.20) *is a pair of functions* $w(x, t), f(x, t)$ *with the following properties*:

a) $w(x, t), f(x, t) \in L_\infty(0, T;\ L_\infty((0, 1)))$ *for arbitrary* T, $0 < T < \infty$;

b) $w(x, t), f(x, t)$ *satisfy* (3.15) *and* (3.16) *for some real number* λ.

Theorem 3.2. *Any generalized solution of the problem* (2.13)–(2.20) *is a w-generalized solution. Conversely, any w-generalized solution of the problem* (2.13)–(2.20) *is a generalized solution.*

The proof of this theorem follows from the proof of Theorem 3.1.

4. OPERATOR FORMULATION OF THE PROBLEM

Theorem 4.1. *Any generalized solution of the problem* (2.13)–(2.20) *is a solution of the pair of operator equations of the form*

$$(4.1)\qquad w(t) = \lambda \frac{\beta}{\alpha} L w(t) - C[w(t)] + \\ + \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right) \int_0^t \{w(\tau) + G[w(t), w^2(\tau)]\}\, K(t-\tau)\, \mathrm{d}\tau\,,$$

$$(4.2)\qquad f(t) = \frac{\alpha}{\beta} B[w(t), w(t)] - \frac{1}{\beta}\left(\frac{\alpha}{\beta} - 1\right) \int_0^t B[w(\tau), w(\tau)]\, K(t-\tau)\, \mathrm{d}\tau$$

defined on the space $L_\infty(0, T; H)$. *Conversely, any solution of the pair of operator equations* (4.1) *and* (4.2) *is a generalized solution of the problem* (2.13)–(2.20). *Here*

$$\langle Lu(t), \varphi \rangle = \textstyle\int_0^1 x^3 u(x,t)\, \varphi(x)\, \mathrm{d}x\,,$$
$$\langle B[u(t), v(t)], \varphi \rangle = \textstyle\int_0^1 x^3 u(x,t)\, v(x,t)\, \varphi(x)\, \mathrm{d}x\,,$$
$$C[u(t)] = B[u(t), B[u(t), u(t)]]\,,$$
$$G[u(t), u^2(\tau)] = B[u(t), B[u(\tau), u(\tau)]]$$

for a.e. $t \in \langle 0, T\rangle$ *and for* $\varphi \in H$, $u, v \in L_\infty(0, T; H)$. L *is a linear bounded self-adjoint compact operator mapping* H *into itself for a.e.* $t \in \langle 0, T\rangle$, B *is a bilinear bounded symmetric compact operator defined on* $H \times H$ *with the range in* H *for a.e.* $t \in \langle 0, T\rangle$, C *is a bounded compact operator mapping* H *into itself for a.e.* $t \in \langle 0, T\rangle$,

Proof of this theorem is analogous to the stationary case (see for example [3]).

5. INTEGRO-OPERATOR FORMULATION OF THE PROBLEM

Using the notation

$$(5.1)\qquad L_1 u(x,t) = \int_x^1 \frac{1}{s^3} \int_0^s p^3 u(p,t)\, \mathrm{d}p\, \mathrm{d}s\,,$$

$$(5.2)\qquad B_1[u(x,t), v(x,t)] = \int_x^1 \frac{1}{s^3} \int_0^s p^3 u(p,t)\, v(p,t)\, \mathrm{d}p\, \mathrm{d}s\,,$$

(5.3) $$C_1[u(x,t)] = B_1[u(x,t), B_1[u(x,t), u(x,t)]] = = \int_x^1 \frac{1}{s^3} \int_0^s p^3 u(p,t) \int_p^1 \frac{1}{y^3} \int_0^y z^3 u^2(z,t)\,dz\,dy\,dp\,ds\,,$$

(5.4) $$G_1[u(x,t), u^2(x,\tau)] = B_1[u(x,t), B_1[u(x,\tau), u(x,\tau)]] = = \int_x^1 \frac{1}{s^3} \int_0^s p^3 u(p,t) \int_p^1 \frac{1}{y^3} \int_0^y z^3 u^2(z,\tau)\,dz\,dy\,dp\,ds$$

we can rewrite (3.11) and (3.12) from Definition 3.4 of the w-generalized solution to the following so called integro-operator formulation of the problem (2.13)–(2.20):

(5.5) $$w(x,t) = \lambda \frac{\beta}{\alpha} L_1 w(x,t) - C_1[w(x,t)] + + \frac{1}{\alpha}\left(\frac{\alpha}{\beta} - 1\right) \int_0^t \{w(x,\tau) + G_1[w(x,t), w^2(x,\tau)]\}\, K(t-\tau)\,d\tau\,,$$

(5.6) $$f(x,t) = \frac{\alpha}{\beta} B_1[w(x,t), w(x,t)] - - \frac{1}{\beta}\left(\frac{\alpha}{\beta} - 1\right) \int_0^t B_1[w(x,\tau), w(x,\tau)]\, K(t-\tau)\,d\tau$$

defined according to Theorems 3.1 and 3.2 in the space $L_\infty(0, T; C^2\langle 0, 1\rangle))$. Equations (5.5) and (5.6) are uncoupled in the sense that w can be determined independently of f. Thus it is sufficient to consider only (5.5) if we wish to determine w.

Summarizing Theorems 3.1, 3.2 and 4.1 we have:

Corollary 5.1. *Solutions of the problem* (2.13)–(2.20) *generalized in the time variable, generalized solutions, w-generalized solutions and solutions of the operator and integro-operator formulations are equivalent.*

Lemma 5.1. *The operators* L_1, C_1 *and the integral Volterra operators*

$$\int_0^t w(x,\tau)\,K(t-\tau)\,d\tau \quad \text{and} \quad \int_0^t G_1[w(x,t), w^2(x,\tau)]\,K(t-\tau)\,d\tau$$

are monotone on the positive cone of the space $L_\infty(0, T; C^2(\langle 0, 1\rangle))$.

Here we use the concept of the monotone operator in the following sense:

Definition 5.1. *An operator* S *mapping* $L_\infty(0, T; C^2(\langle 0, 1\rangle))$ *into itself is monotone if*

$$u(x,t) \geqq v(x,t)$$

for $x \in \langle 0, 1\rangle$ *and a.e.* $t \in \langle 0, T\rangle$ *implies*

$$S[u] \geqq S[v]$$

for $x \in \langle 0, 1\rangle$ *and a.e.* $t \in \langle 0, T\rangle$.

The proof of Lemma 5.1 follows from the definitions of the operators.
In the proofs of the next lemmas we use the following assertion [4].

Assertion 5.1. *A sequence of functions $f_n(x) \in C(\langle a, b\rangle)$ is compact if the functions $f_n(x)$ are differentiable and*

$$|f_n(x)| \leqq N ; \quad \left|\frac{\mathrm{d}f_n(x)}{\mathrm{d}x}\right| \leqq N' .$$

Lemma 5.2. *For arbitrary $t_1 \in \langle 0, T\rangle$, L_1 is a linear bounded compact operator mapping the space $C^2(\langle 0, 1\rangle)$ into itself.*

Proof. Linearity and boundedness follow from (5.1). Compactness we show with the help of Assertion 5.1. Let

$$P(t_1) = \{u(x, t_1) \in C^2(\langle 0, 1\rangle) \mid \|u(t_1)\|_{C^2(\langle 0,1\rangle)} \leqq N(t_1)\} ,$$

then using (5.1) we have

$$\begin{aligned}
|L_1 u(x, t_1)| &\leqq \tfrac{1}{8} N(t_1) , \\
|\{L_1 u(x, t_1)\}'| &\leqq \tfrac{1}{4} N(t_1) , \\
|\{L_1 u(x, t_1)\}''| &\leqq \tfrac{7}{4} N(t_1) , \\
|\{L_1 u(x, t_1)\}'''| &\leqq N(t_1) + N_n(t_1)
\end{aligned}$$

for all $u(x, t_1)$ $P(t_1)$, where

$$N_n(t_1) > \left|\frac{3}{x} u(x, t_1) - \frac{12}{x^5} \int_0^x p^3 u(p, t_1)\, \mathrm{d}p\right|$$

because the l'Hospital rule implies

$$\lim_{x \to 0^+} \left|\frac{3}{x} u(x, t_1) - \frac{12}{x^5} \int_0^x p^3 u(p, t_1)\, \mathrm{d}p\right| = \lim_{x \to 0^+} \frac{3}{5} u'(x, t_1) .$$

So, according to Assertion 5.1 the operator L_1 is compact.

Lemma 5.3. *For arbitrary $t_1 \in \langle 0, T\rangle$ the operator B_1 mapping the space $C^2(\langle 0, 1\rangle) \times C^2(\langle 0, 1\rangle)$ into $C^2(\langle 0, 1\rangle)$ has the following properties:*

a) *it is symmetric operator, i.e. for every $u(x, t_1)$, $v(x, t_1) \in C^2(\langle 0, 1\rangle)$*

$$B_1[u(x, t_1), v(x, t_1)] = B_1[v(x, t_1), u(x, t_1)] ;$$

b) *for every $u(x, t_1)$, $v(x, t_1) \in C^2(\langle 0, 1\rangle)$*

$$\|B_1[u(t_1), v(t_1)]\|_{C^2(\langle 0,1\rangle)} \leqq \tfrac{1}{8} \|u(t_1)\|_{C^2(\langle 0,1\rangle)} \|v(t_1)\|_{C^2(\langle 0,1\rangle)} ;$$

c) *for every* $u(x, t_1), v_1(x, t_1), v_2(x, t_1) \in C^2(\langle 0, 1 \rangle)$

$$\|B_1[u(t_1), v_1(t_1)] - B_1[u(t_1), v_2(t_1)]\|_{C^2(\langle 0,1\rangle)} \leqq$$
$$\leqq \tfrac{1}{8}\|u(t_1)\|_{C^2(\langle 0,1\rangle)} \|v_1(t_1) - v_2(t_1)\|_{C^2(\langle 0,1\rangle)} ;$$

d) *it is a compact operator*;

e) *the equality*

$$B_1[u(x, t_1), u(x, t_1)] = 0 , \quad u(x, t_1) \in C^2(\langle 0, 1 \rangle)$$

holds if and only if $u(x, t_1) = 0$ *for every* $x \in \langle 0, 1 \rangle$.

Proof. a), b), c) follow from (5.2), d) is analogous to the proof of compatness of the operator L_1.

e) If $u(x, t_1) = 0$ for $x \in \langle 0, 1 \rangle$ then (5.2) yields that $B_1[u(x, t_1), u(x, t_1)] = 0$ for $x \in \langle 0, 1 \rangle$. Let now $B_1[u(x, t_1), u(x, t_1)] = 0$ for $x \in \langle 0, 1 \rangle$. Since $u(x, t_1) \in C^2(\langle 0, 1 \rangle)$ hence also $u(x, t_1) \in H$ and for every $\varphi(x) \in H$ we have

$$\langle B_1[u(t_1), u(t_1)], \varphi \rangle = \int_0^1 x^3 u^2(x, t_1) \, \varphi(x) \, dx .$$

If we put $\varphi(x) = 1 - x \in C^2(\langle 0, 1 \rangle) \subset H$ we get

$$x^3(1 - x) \, u^2(x, t_1) = 0$$

for a.e. $x \in \langle 0, 1 \rangle$ and so $u(x, t_1) = 0$ for a.e. $x \in \langle 0, 1 \rangle$. Since $u(x, t_1) \in C^2(\langle 0, 1 \rangle)$ hence $u(x, t_1) = 0$ for $x \in \langle 0, 1 \rangle$.

Lemma 5.4. *For arbitrary* $t_1 \in \langle 0, T \rangle$ *the operator* C_1 *mapping* $C^2(\langle 0, 1 \rangle)$ *into itself has the following properties*:

a) *for every* $u(x, t_1) \in C^2(\langle 0, 1 \rangle)$

$$\|C_1[u(t_1)]\|_{C^2(\langle 0,1\rangle)} \leqq \tfrac{1}{64} \|u(t_1)\|^3_{C^2(\langle 0,1\rangle)} ;$$

b) *for every* $u(x, t_1), v(x, t_1) \in C^2(\langle 0, 1 \rangle)$

$$\|C_1[u(t_1)] - C_1[v(t_1)]\|_{C^2(\langle 0,1\rangle)} \leqq \tfrac{3}{64} \max \{\|u(t_1)\|^2_{C^2(\langle 0,1\rangle)},$$
$$\|v(t_1)\|^2_{C^2(\langle 0,1\rangle)}\} \cdot \|u(t_1) - v(t_1)\|_{C^2(\langle 0,1\rangle)} ;$$

c) *it is a compact operator.*

Proof. a), b) follow from (5.3) and c) is analogous to the proof of compactness of the operator L_1.

6. CONCLUSION

We have derived different formulations of the problem of buckling and post-buckling behaviour of circular viscoelastic plates. We will use the results of this paper in forthcoming papers.

The operator formulation of the problem will be used in the paper "Bifurcations of Generalized von Kármán Equations for Circular Viscoelastic Plates" which will deal with an analysis of relations between the critical points of the linearized problem and the bifurcation points.

The integro-operator formulation of the problem and the relations between different concepts of solutions of the problem will be used in the paper "Analysis of Postbuckling Solutions of Generalized von Kármán Equations for Circular Viscoaleastic Plates". In this paper we shall derive results concerning the number and properties of solutions of the problem in the neighbourhood of the first critical point λ_1.

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Súhrn

EKVIVALENTNÉ FORMULÁCIE ZOVŠEOBECNENÝCH VON KÁRMÁNOVYCH ROVNÍC PRE KRUHOVÉ VÄZKOPRUŽNÉ DOSKY

IGOR BRILLA

Článok sa zaoberá analýzou zovšeobecnených von Kármánovych rovníc popisujúcich stabilitu tenkej kruhovej väzkopružnej dosky na okraji upevnenej a radiálne symetricky zaťaženej. V článku sú zavedené rôzne pojmy riešenia uvažovaného matematického problému a sú ukázané vzťahy medzi týmito riešeniami.

Резюме

ЭКВИВАЛЕНТНЫЕ ФОРМУЛИРОВКИ ОБОБЩЕННЫХ УРАВНЕНИЙ ФОН КАРМАНА ДЛЯ КРУГЛЫХ ВЯЗКОУПРУГИХ ПЛАСТИНОК

IGOR BRILLA

Рассматриваются обобщенные уравнения фон Кармана для осесимметричного изгиба тонкой круглой жестко защемленной вязкоупругой пластинки постоянной толщины, подвергающейся по своему контуру действию равномерных сжимающих сил, интенсивность которых пропорциональна вещественному параметру. Расширается понятие решения математической проблемы. Рассматриваются эквивалентные формулировки проблемы с точки зрения нелинейного функционального анализа.

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