Applications of Mathematics

Jiří Neustupa; Antonín Novotný

Global weak solvability to the regularized viscous compressible heat conductive flow

Applications of Mathematics, Vol. 36 (1991), No. 6, 417–431

Persistent URL: `http://dml.cz/dmlcz/104479`

Terms of use:

© Institute of Mathematics AS CR, 1991

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* `http://dml.cz`

36 (1991) APPLICATIONS OF MATHEMATICS No. 6, 417–431

GLOBAL WEAK SOLVABILITY TO THE REGULARIZED VISCOUS COMPRESSIBLE HEAT CONDUCTIVE FLOW

JIŘÍ NEUSTUPA, ANTONÍN NOVOTNÝ

(Received December 7, 1989)

Summary. The concept of regularization to the complete system of Navier-Stokes equations for viscous compressible heat conductive fluid is developed. The existence of weak solutions for the initial boundary value problem for the modified equations is proved. Some energy and entropy estimates independent of the parameter of regularization are derived.

Keywords: Compressible heat conductive fluid, global existence, initial boundary value problem, energy inequality.

AMS Classification: 35Q, 76N

I. INTRODUCTION

The paper deals with the complete regularized system of Navier-Stokes type for the viscous compressible fluid. It is closely related to [11], [12]. We use the same type of regularization as R. Rautman in [14] in the incompressible case. We discuss
the following initial boundary value problem for the density ϱ, velocity $u = (u_1, \ldots, u$
the following initial boundary value problem for the density ϱ, velocity $u = (u_1, \ldots, u_N)$ and temperature θ:

(1.1) $$\varrho_{,t} + (\varrho \tilde{u}_j)_{,j} = 0 \quad \text{in} \quad Q_{T,h}\,,$$

(1.2) $$(\varrho u_i)_{,t} + (\varrho u_i \tilde{u}_j)_{,j} - \mu u_{i,jj} - (\mu/3)\, u_{j,ij} = -\tilde{p}_{,i} \quad \text{in} \quad Q_T\,, \quad u = 0 \quad \text{in} \quad Q_{T,h} - Q_T\,,$$

(1.3) $$c_V(\varrho\theta)_{,t} + c_V(\varrho \tilde{u}_i \theta)_{,i} - \lambda \theta_{,ii} = -p\tilde{u}_{j,j} + \Psi(u) \quad \text{in} \quad Q_{T,h}\,,$$

(1.4) $$p = R\varrho\theta\,,$$

(1.5) $$\Psi(u) = \tau_{ij}(u)\, u_{i,j}$$
$$\left(\tau_{ij}(u) = 2\mu e_{ij} - \tfrac{2}{3}\mu e_{kk}\delta_{ij}\,, \quad e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})\right),$$

(1.6) $$\varrho(0) = \varrho_0 \quad \text{in} \quad \Omega_h\,, \quad u(0) = u_0 = (u_{01}, \ldots, u_{0N}) \quad \text{in} \quad \Omega\,,$$
$$\theta(0) = \theta_0 \quad \text{in} \quad \Omega_h\,,$$

(1.7) $u = 0$ on $(0, T) \times \partial\Omega$, $\frac{\partial\theta}{\partial\nu} = 0$ on $I \times \partial\Omega_h$ (ν is the outer normal to $\partial\Omega_h$).

The coefficients μ (viscosity), c_V (specific heat at constant volume), R (universal gas constant), λ (heat conductivity) are positive constants. $\Omega \subset R^N$ ($N = 2, 3$) is a bounded domain with a smooth boundary. For $h > 0$, one defines $\Omega_h = \{x \in R^N, \operatorname{dist}(x, \Omega) < h\}$. We assume that h can be chosen so small that the boundary $\partial\Omega_h$ is also smooth. Let $T > 0$, $I = (0, T)$; for $t \in I$, let $Q_t = (0, t) \times \Omega$ and $Q_{t,h} = (0, t) \times \Omega_h$. For $g \in L^1(\Omega_h)$ we define $\tilde{g}(x) = \int_{\Omega_h} \omega_h(x - y)\, g(y)\, dy$, where $\omega_h(x) = K(h) \exp(-|x|^2/(h^2 - |x|^2))$ for $|x| < h$, $\omega_h(x) = 0$ for $|x| \geqq h$ and $K(h)$ is such that $\int_{R^N} \omega_h(x)\, dx = 1$. If we write $\tilde{}$ over a function g which is originally defined in Ω only, we assume automatically that $g = 0$ outside Ω. The tilde above a function depending on $(t, x) \in R^{N+1}$ means the regularization only in the space variable x. One easily verifies the following results (see e.g. [11])

(1.8) $\int_{\Omega_h} \tilde{f} g\, dx = \int_{\Omega_h} f \tilde{g}\, dx$ for every $f, g \in L^1(\Omega_h)$,

(1.9) $\max_{x \in \Omega_h} |D^k \tilde{f}(x)| \leqq c_1 \|f\|_{L^1(\Omega_h)}$

(D^k ($k = 0,1, \ldots$) represents any differentiation of the k-th order with respect to the space variables, $c_1 > 0$ depends on h, D^k).

The purpose of our paper is to prove the global in time existence of weak solutions to the problem (1.1)–(1.7). In spite of great efforts, no general global results concerning the systems of Navier-Stokes equations for compressible fluid in more space dimensions have been obtained up to now. Global theorems were proved only in the case of "sufficiently small" initial conditions and external forces (see e.g. [6], [7], [15], [17]). An interesting attempt without these assumptions is Padula's paper [13] concerned with the two dimensional isothermal viscous compressible flow. However, her approach and results are not quite correct. For another approach to the global existence problems see e.g. [9], [10].

We use the current notation $L^p(\Omega)$, $P^p(\Omega, R^N)$, $L^p(Q_T)$, $L^p(Q_T, R^N)$, $W^{k,2}(\Omega)$, $W^{k,2}(\Omega, R^N)$, $L^q(I, W^{k,2}(\Omega))$, $L^2(I, W^{k,2}(\Omega, R^N))$, $\mathscr{C}^s(\bar{\Omega})$, $\mathscr{C}^s(\bar{\Omega}), R^N)$, $\mathscr{C}^s(\bar{Q}_t)$, $\mathscr{C}^s(\bar{Q}_T, R^N)$, $\mathscr{C}^s(\bar{I}, \mathscr{C}^r(\bar{\Omega}))$ for functional spaces (see e.g. [3]). A norm in a Banach space X is denoted by $\| \ \|_X$.

Let us recall the very useful Lions lemma (see e.g. [5], [16]).

Lemma 1.1. *Let* $1 < p_i < +\infty$, *let* B_i ($i = 0, 1$) *be reflexive Banach spaces and* B *Banach space such that* $B_0 \subset\subset B \subset B_1$ ($\subset\subset$ *denotes compact imbedding*). *Then the Banach space*

$$X = \{f; f \in L^{p_0}((0, T), B_0), \quad f_{,t} \in L^{p_1}((0, T), B_1)\}$$

satisfies

$$X \subset\subset L^{p_0}((0, T), B).$$

We define

$$(2.1)\qquad \mathcal{T}_0(\delta) \equiv \{(\varrho_0, u_0, \theta_0);\ \varrho_0 \in W^{2,2}(\Omega_h),\quad \varrho_0 \geqq \delta > 0 \text{ in } \bar{\Omega}_h,\ u_0 \in W_0^{1,2}(\Omega, R^N),\quad \Theta_0 \in L^2(\Omega_h), \Theta_0 > 0 \text{ a.e. in } \Omega_h\},$$

$$(2.2)\qquad \mathcal{S}_{\delta_1}(t) \equiv \{(\varrho, u, \theta); \varrho \in L^\infty((0,t), W^{2,2}(\Omega_h)), \varrho_{,t} \in L^2((0,t), W^{1,2}(\Omega_h)), \varrho \geqq \delta_1 > 0 \text{ a.e. in } Q_{t,h}, u \in L^2((0,t), W^{2,2}(\Omega, R^N) \cap W_0^{1,2}(\Omega, R^N)), u_{,t} \in L^2(Q_t, R^N), \theta \in L^2((0,t), W^{1,2}(\Omega_h) \cap L^\infty((0,t), L^2(\Omega_h)), \theta \geqq 0 \text{ a.e. in } Q_{t,h}\}.$$

Definition 2.1. *Let $\delta > 0$, $T > 0$, $(\varrho_0, u_0, \theta_0) \in \mathcal{T}_0(\delta)$. By the weak solution to the problem (1.1)–(1.7) we call the triplet $(\varrho, u, \theta) \in \mathcal{S}_{\delta_1}(T)$ such that*

i) *equations* (1.1) ((1.2)) *are fulfilled a.e. in $Q_{T,h}$ (in Q_T, respectively),*

ii) (1.4), (1.5) *hold,*

iv) $\rho(0) = \varrho_0, u(0) = u_0$,

iii) *the equation* (1.3) *is fulfilled in the weak sense, i.e.*

$$(2.3)\qquad -c_V \int_{Q_{T,h}} \varrho\theta\eta_{,t}\,dx\,dt - \int_{\Omega_h} \varrho_0\theta_0\,\eta(0)\,dx + \int_{Q_{T,h}} (\lambda\theta_{,i} - c_V\varrho\tilde{u}_i\theta)\,\eta_{,i}\,dx\,dt = \int_{Q_{T,h}} (\Psi - p\tilde{u}_{j,j})\,\eta\,dx\,dt$$

for every $\eta \in L^2(I, W^{1,2}(\Omega_h))$, $\eta_{,t} \in L^2(Q_{T,h})$, $\eta(T) = 0$.

Remark 2.1 Sometimes it is convenient to replace (2.3) by

$$(2.4)\qquad -c_V \int_{Q_{t',h}} \varrho\theta\eta_{,t}\,dx\,dt - c_V\int_{\Omega_h} \varrho_0\theta_0\,\eta(0)\,dx + \int_{\Omega_h}\varrho\theta\eta\,dx\big|_{t=t'} - \int_{Q_{t',h}} (\Psi - p\tilde{u}_{j,j})\,\eta\,dx\,dt + \int_{Q_{t',h}} (\lambda\theta_{,i} - c_V\varrho\tilde{u}_i\theta)\,\eta_{,i}\,dx\,dt = 0$$

for every $\eta \in L^2(I, W^{1,2}(\Omega_h))$, $\eta_{,t} \in L^2(Q_{T,h})$ and for a.e. $t' \in I$.

Our aim is to prove the following main theorem.

Theorem 2.1. *Let $\delta, T > 0$, $(\varrho_0, u_0, \theta_0) \in \mathcal{T}_0(\delta)$. Then the initial boundary value problem (1.1)–(1.7) has at least one weak solution $(\varrho, u, \theta) \in \mathcal{S}_{\delta_1}(T)$ for some $\delta_1 > 0$.*

III. CONSTRUCTION OF APPROXIMATIONS

Remark 3.1. We shall use a modified Galerkin method with a special basis. Let the operator $\mathcal{A}: W^{2,2}(\Omega, R^N) \cap W_0^{1,2}(\Omega, R^N) \to L^2(\Omega, R^N)$ be given by $(\mathcal{A}v)_i = v_{i,jj} + \frac{1}{3}v_{j,ij}$ $(v = (v_1, \ldots, v_N))$. One constructs an orthogonal basis $\{w^k\}_{k=1}^{+\infty}$ $(w^k = (w_1^k, \ldots, w_N^k))$ in $L^2(\Omega, R^N)$ by solving the eigenvalue problem

$$(3.1)\qquad \mathcal{A}w^k = \lambda_k w^k\ (k = 1,2,\ldots; 0 < \lambda_1 \leqq \lambda_2 \leqq \ldots).$$

Solutions of (3.1) exist due to the compactness of $\mathscr{A}^{-1}$ and the selfadjointness of $\mathscr{A}$ in $L^2(\Omega, R^N)$ (see e.g. [5]). Due to the regularity of elliptic systems $w^k \in \in \mathscr{C}^\infty(\bar{\Omega}, R^N)$, see [1]. Let us denote by P_n $(n = 1, 2, \ldots)$ the orthogonal projection of $L^2(\Omega, R^N)$ onto the n-dimensional subspace spanned by $\{w^1, \ldots, w^n\}$. Due to (3.1), we have

$$\mathscr{A} P_n w = P_n \mathscr{A} w \tag{3.2}$$

for every $w \in W^{2,2}(\Omega, R^N) \cap W_0^{1,2}(\Omega, R^N)$.

Remark 3.2. We look for a sequence of approximative solutions $\{(\varrho^n, u^n, \theta^n)\}_{n=1}^{+\infty}$, $\varrho^n \in \mathscr{C}^1(\bar{Q}_{T,h}) \cap \mathscr{C}^0([0, T], \mathscr{C}^2(\bar{\Omega}_h))$, $u^n = \sum_{k=1}^n \gamma_k w^k$, where $\Gamma = (\gamma_1, \ldots, \gamma_n) \in \in \mathscr{C}^1([0, T], R^n)$ and $\{w^k\}_{k=1}^{+\infty}$ is the basis described in Remark 3.1, $\theta^n \in \mathscr{C}^1(\bar{Q}_{T,h}) \cap \cap \mathscr{C}^0([0, T], \mathscr{C}^2(\bar{\Omega}_h))$, which satisfy the following system:

$$\varrho^n_{,t} + (\varrho^n \tilde{u}^n_j)_{,j} = 0 \quad \text{in} \quad Q_{T,h}, \tag{3.3}$$

$$\int_\Omega (\varrho^n u^n_i)_{,t}\, w^k_i \,\mathrm{d}x + ((u^n, w)) = \int_\Omega \varrho^n u^n_i \tilde{u}^n_j w^k_{i,j} \,\mathrm{d}x + \int_\Omega \tilde{p}(\varrho^n, \theta^n)\, w^k_{j,j} \,\mathrm{d}x \quad \text{in} \quad I, \tag{3.4}$$

where $((u, w)) = \mu \int_\Omega u_{i,j} w_{i,j} \,\mathrm{d}x + \frac{1}{3}\mu \int_\Omega u_{i,i} w_{j,j} \,\mathrm{d}x$,

$$c_V(\varrho^n \theta^n)_{,t} + c_V(\varrho^n \tilde{u}^n_j \theta^n)_{,j} - \lambda \theta^n_{,jj} = -R \varrho^n \theta^n \tilde{u}^n_{j,j} + \Psi(u^n) \quad \text{in} \quad Q_{T,h}, \tag{3.5}$$

$$\varrho^n(0) = \varrho^n_0, \quad \text{where} \quad \varrho^n_0 \in \mathscr{C}^\infty(\bar{\Omega}_h), \quad \varrho^n_0 \geqq \delta \quad \text{in} \quad \bar{\Omega}_h, \quad \varrho^n_0 \to \varrho_0 \tag{3.6}$$
$$\text{strongly in} \quad W^{2,2}(\Omega_h); \quad \gamma_k(0) = \int_\Omega u_{0i} w^k_i \,\mathrm{d}x; \quad \theta^n(0) = \theta^n_0,$$
$$\text{where} \quad \theta^n_0 \in \mathscr{C}^2(\bar{\Omega}_h), \quad \frac{\partial}{\partial \nu}\theta^n_0 = 0 \quad \text{on} \quad \partial\Omega_h, \quad \theta^n_0 \to \theta_0$$
$$\text{strongly in} \quad L^2(\Omega_h).$$

Lemma 3.1. *The system* (3.3)–(3.6) *has at least one solution* $\varrho^n \in \mathscr{C}^1(\bar{Q}_{T,h}) \cap \cap \mathscr{C}^0(\bar{I}, \mathscr{C}^2(\bar{\Omega}_h))$, $\varrho^n > 0$ *in* $\bar{Q}_{T,h}$, $\theta^n \in \mathscr{C}^1(\bar{Q}_{T,h}) \cap \mathscr{C}^0(\bar{I}, \mathscr{C}^2(\bar{\Omega}_h))$, $\theta \geqq 0$ *in* $\bar{Q}_{T,h}$, $\Gamma \in \in \mathscr{C}^1(\bar{I}, R^n)$.

Proof. Let $v = \sum_{k=1}^n \bar{\gamma}_k w^k$, $\bar{\Gamma} = (\bar{\gamma}_1, \ldots, \bar{\gamma}_n) \in \mathscr{C}^0(\bar{I}, R^n)$, $\bar{\Gamma}(0) = (\gamma_1(0), \ldots, \gamma_n(0))$. We are looking for ϱ^n which satisfies the initial condition $\varrho^n(0) = \varrho^n_0$ and (3.3) with $u^n = v$. For $\varrho^n > 0$, (3.3) is equivalent to

$$(1/\varrho^n)\, \varrho^n_{,t} + (1/\varrho^n)\, \varrho^n_{,i} \tilde{v}_i = -\tilde{v}_{i,i}. \tag{3.7}$$

Let $\dot{x}_i(t) = \tilde{v}_i(t, x^n(t))$, $x^n(0) = y \in \bar{\Omega}_h(1, 2, 3,)$. For every $t \in \bar{I}$, $y \to x^n(t)$ is a diffeomorphism of $\bar{\Omega}_h$ onto $\bar{\Omega}_h$. From (3.7) one gets

$$\varrho^n(t, x) = \varrho^n_0(y) \exp\left(-\int_0^t \tilde{v}_{i,i}(\tau, x^n(\tau))\,\mathrm{d}\tau\right), \quad \text{where} \quad x = x^n(t), \tag{3.8}$$
$$y = x^n(0).$$

Thus for every $\bar{\Gamma} \in \mathscr{C}^0(\bar{I}, R^n)$, $\|\bar{\Gamma}\|_{\mathscr{C}^0(I,R^n)} \leqq \sigma_1$, $\sigma_1 > 0$ we have

(3.9) $$\max_{\bar{Q}_{T,h}} \varrho^n(t, x) \geqq \sigma_2 \; (\sigma_2 > 0)$$

According to Misohata [8], $\varrho^n \in \mathscr{C}^1(\bar{Q}_{T,h}) \cap \mathscr{C}^0(\bar{I}, \mathscr{C}^2(\bar{\Omega}_h))$ and the following estimates hold:

(3.10) $$\|\varrho^n\|_{\mathscr{C}^1(\bar{Q}_{T,h}) \cap \mathscr{C}^0(I, \mathscr{C}^2(\bar{\Omega}_h))} \leqq \sigma_3 (\sigma_3 > 0)$$

provided $\|\bar{\Gamma}\|_{\mathscr{C}^0(I,R^n)} \leqq \sigma_1$,

(3.11) $$\|\varrho_1^n - \varrho_2^n\|_{\mathscr{C}^1(\bar{Q}_{T,h})} \leqq \sigma_3 \|\Gamma^1 - \Gamma^2\|_{\mathscr{C}^0(I,R^n)}$$

provided $\|\bar{\Gamma}^l\|_{\mathscr{C}^0(I,R^n)} \leqq \sigma_1$ $(l = 1, 2)$ (ϱ_1^n and ϱ_2^n are the solutions to (3.3) with the initial condition ϱ_0^n and with $u^n = v^1 = \sum_{k=1}^{n} \bar{\gamma}_k^1 w^k$ and $v^2 = \sum_{k=1}^{n} \bar{\gamma}_k^2 w^k$, respectively.)

For such ϱ^n and $u^n = v$ one solves the parabolic equation (3.5) with the initial condition $\theta^n(0) = \theta_0^n$ and boundary condition $\partial/\partial v(\theta^n) = 0$ on $I \times \partial\Omega_h$. Because of the smoothness of its coefficients (see the definition of v and (3.9), (3.10) above) one obtains that $\theta^n \in \mathscr{C}^1(\bar{Q}_{T,h}) \cap \mathscr{C}^0(\bar{I}, \mathscr{C}^2(\bar{\Omega}_h))$. The following estimates hold:

(3.12) $$\|\theta^n\|_{\mathscr{C}^1(\bar{Q}_{T,h}) \cap \mathscr{C}^0(I, \mathscr{C}^2(\bar{\Omega}_h))} \leqq \sigma_4 \quad (\sigma_4 > 0)$$

provided $\|\bar{\Gamma}\|_{\mathscr{C}^0(I,R^n)} \leqq \sigma_1$ and

(3.13) $$\|\theta_1^n - \theta_2^n\|_{\mathscr{C}^1(\bar{Q}_{T,h})} \leqq \sigma_5 \|\bar{\Gamma}^1 - \bar{\Gamma}^2\|_{\mathscr{C}^0(I,R^n)}, \quad \sigma_5 > 0$$

provided $\|\bar{\Gamma}^l\|_{\mathscr{C}^0(I,R^n)} \leqq \sigma_1 (1 = 1, 2)$. ($\theta_1^n$ and θ_2^n are the solutions to (3.5) corresponding to $\varrho^n = \varrho_1^n$ or ϱ_2^n and $u^n = v^1$ or v^2, respectively.) For details see [2] [4].

For ϱ^n, θ^n computed above we solve the problem

(3.14) $$\int_\Omega (\varrho^n u_i^n)_{,t} \, w_i^k \, dx + ((u^n, w^k)) = \int_\Omega \varrho^n u_i^n \tilde{v}_j w_{i,j}^k \, dx + \int_\Omega \tilde{p}(\varrho^n, \theta^n) \, w_{j,j}^k \, dx$$

with the initial condition $\gamma_k(0) = \int_\Omega u_{0i} w_i^k \, dx$, $k = 1, \ldots, n$.

(i) We multiply $(3.14)_k$ by γ_k, add the obtained equalities and integrate over $(0, t)$, $t \in (0, T]$.

(ii) We multiply (3.3) (where $u^n = v$) by $\frac{1}{2}|u^n|^2$ and integrate over $Q_{T,h}$. After adding (i), (ii) we get

(3.15) $$\frac{1}{2} \int_\Omega \varrho^n |u^n|^2 \, dx\big|_t + \int_0^t ((u^n, u^n)) \, d\tau = \frac{1}{2} \int_\Omega \varrho_0^n |u_0^n|^2 \, dx + \\ + \int_0^t \int_\Omega p(\varrho^n, \theta^n) \, \tilde{u}_{j,j}^n \, dx \, d\tau \, .$$

From (3.15), (3.9), (3.10), (3.12) we get

(3.16) $$\|\Gamma\|_{\mathscr{C}^0(I,R^n)} \leqq \sigma_6 \; (\sigma_6 > 0)$$

provided $\|\bar{\Gamma}\|_{\mathscr{C}^0(\bar{I},R^n)} \leqq \sigma_1$.

From (3.14), (3.9), (3.10), (3.12), (3.16) we obtain

(3.17) $$\|\Gamma\|_{\mathscr{C}^1(I,R^n)} \leqq \sigma_7 \quad (\sigma_7 > 0)$$

provided $\|\bar{\Gamma}\|_{\mathscr{C}^0(I,R^n)} \leqq \sigma_1$.
Since $\det\left(\int_\Omega \varrho^n w_i^k w_i^l \, dx\right) > 0$, one gest from (3.14) the system of ordinary differential equations

(3.18) $$\dot{\Gamma} = \mathscr{F}(\Gamma, \bar{\Gamma}, \Gamma^I, \Gamma^{II}, \Gamma^{III}), \; \Gamma(0) = \Gamma_0 = (\gamma_1(0), \ldots, \gamma_n(0)),$$

where $\Gamma^I = (\gamma_k^I)_{k=1}^n$, $\gamma_k^I = \int_\Omega \tilde{p}(\varrho^n, \sigma^n) \, w_{j,j}^k \, dx$, $\Gamma^{II} = (\gamma_{rk}^{II})_{r,k=1}^n$, $\gamma_{rk}^{II} = \int_\Omega \varrho^n w_i^r w_i^k \, dx$, $\Gamma^{III} = (\gamma_{rak}^{III})_{r,a,k=1}^n$, $\gamma_{rak}^{III} = \int_\Omega \varrho^n \tilde{w}_j^r w_{i,j}^a w_i^k \, dx$. The explicit form of $\mathscr{F}$ can be found from the following system of linear equations for $\dot{\Gamma}$:

(3.19) $$\gamma_{rk}^{II} \dot{\gamma}_r = \gamma_{rak}^{III} \bar{\gamma}_r \gamma_a - ((w^r, w^k)) \, \gamma_r + \gamma_k^I \quad (k = 1, \ldots, n) .$$

The equation (3.18) defines the operator $\mathscr{G}: \mathscr{C}^0(\bar{I}, R^n) \to \mathscr{C}^0(\bar{I}, R^n)$ such that $\Gamma = \mathscr{G}(\bar{\Gamma})$. For every t, t_1 such that $0 \leqq t_1 < t \leqq T$ we have:

$$\Gamma(t) - \Gamma(t_1) = \int_{t_1}^t \mathscr{F} \, d\tau .$$

Using the estimates (3.9), (3.10), (3.11), (3.12), (3.13), (3.17) and taking into account the form of $\mathscr{F}$ (see (3.19)), we derive that for every $q \in (0, +\infty)$ there exists $\bar{t} \in (0, T]$ such that

(3.20) $$\|\Gamma - \Gamma(0)\|_{\mathscr{C}^0([0,\bar{t}],R^n)} \leqq q ,$$
$$\|\Gamma^1 - \Gamma^2\|_{\mathscr{C}^0([0,\bar{t}],R^n)} \leqq \sigma_9 \|\bar{\Gamma}^1 - \bar{\Gamma}^2\|_{\mathscr{C}^0([0,\bar{t}],R^n)}$$

($\sigma_9 > 0$) provided the norms $\|\bar{\Gamma} - \Gamma(0)\|_{\mathscr{C}^0(I,R^n)}$ and $\|\bar{\Gamma}^l - \Gamma(0)\|_{\mathscr{C}^0(I,R^n)}$ ($l = 1, 2$) are less or equal to q. Γ^l corresponds to the solution of (3.18) with $\bar{\Gamma} = \bar{\Gamma}^l$.

The operator $\mathscr{G}: \mathscr{C}^0([0, \bar{t}], R^n) \to \mathscr{G}^0([0, \bar{t}], R^n)$ maps the sphere $\mathscr{K}(q) = \{\Gamma \in \mathscr{C}^0([0, \bar{t}], R^n), \|\Gamma - \Gamma(0)\|_{\mathscr{C}^0([0,\bar{t}],R^n)} \leqq q\}$ into itself. It is compact and continuous (see (3.17), (3.20)). One obtains a fixed point of $\mathscr{G}$ using the Schauder theorem. From (3.18) we get that $(d/dt) \, |\Gamma^1 - \Gamma^2| \leqq \sigma_8(q) \, |\Gamma^1 - \Gamma^2|$, hence the fixed point Γ is unique in $\mathscr{K}(q)$. Let

$\mathscr{N} = \{\bar{t}$, the mapping $\bar{\Gamma}|_{[0,\bar{t}]} \to \Gamma_{[0,\bar{t}]}$ has at least one fixed point in at least one sphere $\mathscr{K}(q)$, $q \in (0, +\infty)\}$

and put $\bar{\alpha} = \sup \mathscr{N}$. It is clear that $\bar{\alpha} > 0$. If $\bar{\alpha} < T$ we get a contradiction with (3.15). Hence $\bar{\alpha} = T$.

Let us prove that $\theta^n \geqq 0$. Put $\hat{\theta}^n = \theta^n \exp(-\Lambda t)$, $\Lambda > 0$. Due to (3.5), $\hat{\theta}^n$ satisfies the equation

(3.21) $$c_V \varrho^n \hat{\theta}^n_{,t} + \Lambda c_V \varrho^n \hat{\theta}^n + c_V \varrho^n \tilde{u}_j^n \hat{\theta}^n_{,j} = -R \varrho^n \hat{\theta}^n \tilde{u}^n_{j,j} + \exp(-\Lambda t) \, \Psi(u^n) + \\ + \lambda \hat{\theta}^n_{,jj} .$$

We suppose that $\min_{\bar{Q}_{T,h}} \hat{\theta}^n = \hat{\theta}^n(t_0, x_0) = 0$. Thus for (t_0, x_0) we have

(3.22) $$\hat{\theta}^n_{,j}(t_0, x_0) = 0 , \quad \hat{\theta}^n_{,jj}(t_0, x_0) \geqq 0 , \quad \hat{\theta}^n_{,t}(t_0, x_0) \leqq 0 .$$

From (3.21), (3.22) we get the inequality

$$(3.23)\qquad \varrho^n\theta^n(c_V\Lambda + R\tilde{u}^n_{j,j})(t_0, x_0) \geqq 0\,.$$

If Λ is sufficiently great then $(c_V\Lambda + R\tilde{u}^n_{j,j})(t_0, x_0) > 0$. Thus $\theta^n(t_0, x_0) \geqq 0$.

IV. ESTIMATES OF APPROXIMATIONS

(i) We multiply $(3.4)_k$ by γ_k $(k = 1, \ldots, n)$, add these equations and integrate over $(0, t)$, $t \in (0, T]$.

(ii) We multiply (3.3) by $\frac{1}{2}|u^n|^2$ and integrate over $Q_{t,h}$.

(iii) We integrate (3.5) over $Q_{t,h}$. We add equalities (i), (ii), (iii). After some computation, we derive the estimate

$$(4.1)\qquad \tfrac{1}{2}\int_\Omega \varrho^n|u^n|^2\,\mathrm{d}x\big|_t + c_V\int_{\Omega_h}\varrho^n\theta^n\,\mathrm{d}x\big|_t \leqq \tfrac{1}{2}\int_\Omega \varrho^n_0|u^n_0|^2\,\mathrm{d}x + c_V\int_{\Omega_h}\varrho^n_0\theta^n_0\,\mathrm{d}x \leqq$$

$$\leqq K_1\quad \Big(K_1 > 0\,,\quad u^n_0 = \sum_{k=1}^{n}\gamma_k(0)\,u^k\Big)\,.$$

Adding the results of (i) and (ii), we obtain

$$(4.2)\qquad \int_0^t ((u^n, u^n))\,\mathrm{d}\tau \leqq \int_0^t\int_{\Omega_h}\tilde{p}(\varrho^n, \theta^n)\,u^n_{i,i}\,\mathrm{d}x\,\mathrm{d}\tau + \tfrac{1}{2}\int_\Omega \varrho^n_0|u^n_0|^2\,\mathrm{d}x \leqq$$

$$\leqq (2\mu)^{-1}\,\|\tilde{p}(\varrho^n, \theta^n)\|^2_{L^2(Q_{T,h})} + \tfrac{1}{2}\mu\,\|u^n_{i,i}\|^2_{L^2(Q_T)} + \tfrac{1}{2}\int_\Omega \varrho^n_0|u^n_0|^2\,\mathrm{d}x\,.$$

Using (1.9) we get

$$(4.3)\qquad \|u^n\|_{L^2(I,W^{1,2}(\Omega,R^N))} \leqq K_2\quad (K_2 > 0)\,.$$

(iv) We apply to (3.3) the differentiation D^j $(j = 0, 1, 2)$ and multiply it by $D^j\varrho^n$.
(v) We add all equalities obtained in (iv) and integrate over $Q_{t,h}$. Using the Green formula, (1.9) and the Gronwall lemma, we can derive

$$(4.4)\qquad \|\varrho^n\|_{L^\infty(I,W^{2,2}(\Omega_h))} \leqq K_3\quad (K_3 > 0)\,.$$

From (3.3), (1.9), (4.4) we obtain

$$(4.5)\qquad \|\varrho^n_{,t}\|_{L^2(I,W^{1,2}(\Omega_h))} \leqq K_4\quad (K_4 > 0)\,.$$

Using the Sobolev imbedding theorem, we arrive at

$$(4.6)\qquad \|\varrho^n\|_{L^\infty(Q_{T,h})} \leqq K_5\quad (K_5 > 0)\,.$$

Due to (3.8), (1.9), (4.3) there exists $\delta_1 > 0$ such that

$$(4.7)\qquad \varrho^n > \delta_1\quad \text{a.e. in}\quad Q_{T,h}\,.$$

From (4.1), (4.7) one derives

$$(4.8)\qquad \|u^n\|_{L^\infty(I,L^2(\Omega,R^N))} \leqq K_6\quad (K_6 > 0)\,.$$

This estimate together with (3.3), (4.4), (1.9) gives

(4.9) $$\|\varrho^n_{,t}\|_{L^\infty(I,W^{1,2}(\Omega_h))} \leqq K_7 \quad (K_7 > 0)\,.$$

(vi) We multiply $(3.4)_k$ by $\dot{\gamma}_k$ and integrate over $(0, t)$; $k = 1, \ldots, n$.

(vii) We add the equations $(\text{vi})_k$.

After some computation, we get

(4.10) $$\int_0^t \int_\Omega \varrho^n |u^n_{,t}|^2 \, dx \, d\tau + ((u^n(t), u^n(t))) = \\ = -\int_0^t \int_\Omega (\varrho^n \tilde{u}^n_j u^n_{i,j} u^n_{i,t} + \tilde{p}_{,i}(\varrho^n, \theta^n) \, u^n_{i,t}) \, dx \, d\tau + ((u^n_0, u^n_0))\,.$$

Using the Hölder inequality and (1.9), we can verify that the following estimates hold (indexes are omitted):

(4.11) $$\int_0^t \int_\Omega \varrho^n \tilde{u}^n D u^n u^n_{,t} \, dx \, d\tau \leqq K_8 \|\varrho^n\|_{L^\infty(Q_{t,h})} \times \\ \times \|u^n\|_{L^\infty((0,t),L^2(\Omega,R^N))} \|Du^n\|_{L^2(Q_t,R^N)} \|u^n_{,t}\|_{L^2(Q_t,R^N)} \quad (K_8 > 0)\,,$$

(4.12) $$\int_0^t \int_\Omega D\tilde{p}(\varrho^n, \theta^n) \, u^n_{,t} \, dx \, d\tau \leqq K_8 \|\varrho^n \theta^n\|_{L^\infty((0,t),L^1(\Omega_h))} \|u^n_{,t}\|_{L^2(Q_t,R^N)}\,.$$

Applying the estimates (4.11), (4.12) to the r.h.s. of (4.10), using the Young inequality and the estimates (4.1), (4.3), (4.6), (4.8), we obtain

(4.13) $$\|u^n_{,t}\|_{L^2(Q_T,R^N)} \leqq K_9\,, \quad K_9 > 0\,,$$

(4.14) $$\|u^n\|_{L^\infty(I,W^{1,2}(\Omega,R^N))} \leqq K_9\,.$$

We multiply $(3.4)_k$ by $\lambda_k \gamma_k$ and integrate over $(0, t)$. We add the obtained equalities $(k = 1, \ldots, n)$. We get

(4.15) $$\mu(\|\mathscr{A}u^n\|_{L^2(Q_t)})^2 = \int_0^t \int_\Omega \tilde{p}_{,i}(\varrho^n, \theta^n) \, (\mathscr{A}u^n)_i \, dx \, d\tau + \\ + \int_0^t \int_\Omega (\varrho^n u^n_{i,t} + \varrho^n \tilde{u}^n_j u^n_{i,j}) \, (\mathscr{A}u^n)_i \, dx \, d\tau\,.$$

The Hölder inequality and (1.9) give the estimates of all terms on the r.h.s. of (4.15) (indexes are omitted):

(4.16) $$\int_0^t \int_\Omega D\tilde{p}(\varrho^n, \theta^n) \, \mathscr{A}u^n \, dx \, d\tau \leqq K_{10} \|\varrho^n \theta^n\|_{L^\infty((0,t),L^1(\Omega_h))} \times \\ \times \|\mathscr{A}u^n\|_{L^2(Q_t,R^N)})\,, \quad K_{10} > 0\,,$$

(4.17) $$\int_0^t \int_\Omega \varrho^n \tilde{u}^n D u^n \mathscr{A}u^n \, dx \, d\tau \leqq K_{10} \|\varrho^n\|_{L^\infty(Q_t)} \times \\ \times \|u^n\|_{L^\infty(I,L^2(\Omega,R^N))} \|Du^n\|_{L^2(Q_t,R^N)} \|\mathscr{A}u^n\|_{L^2(Q_t,R^N)}\,,$$

(4.18) $$\int_0^t \int_\Omega \varrho^n u^n_{,t} \mathscr{A}u^n \, dx \, d\tau \leqq K_{10} \|\varrho^n\|_{L^\infty(Q_t)} \|u^n_{,t}\|_{L^2(Q_t,R^N)} \|\mathscr{A}u^n\|_{L^2(Q_t,R^N)}\,.$$

Applying the estimates (4.16)–(4.18) to the r.h.s. of (4.15), using the Young inequality and (4.3), (4.6), (4.8), (4.13), one gets

(4.19) $$\mu(\|\mathscr{A}u^n\|_{L^2(Q_T,R^N)})^2 \leqq K_{11} + \tfrac{1}{2}\mu(\|\mathscr{A}u^n\|_{L^2(Q_T,R^N)})^2 \quad (K_{11} > 0)\,.$$

Thus

(4.20) $$(\|\mathscr{A}u^n\|_{L^2(Q_T,R^N)})^2 \leqq (2/\mu) \, K_{11}\,.$$

Using the operator P_n (see Remark 3.1), we can rewrite (3.4) into the form

$$(4.21) \qquad \mu \mathscr{A} u^n = P_n(\varrho^n u^n_{,t}) + P_n((\varrho^n \tilde{u}^n_j u^n)_{,j}) + P_n(\nabla \tilde{p}(\varrho^n, \theta^n)) .$$

The right hand side of (4.21) is bounded in $L^2(Q_T, R^N)$. Thus, according to classical Agmon, Douglis, Nirenberg results for elliptic systems (see [1]),

$$(4.22) \qquad \|u^n\|_{L^2(I, W^{2,2}(\Omega, R^N))} \leqq K_{12} , \quad K_{12} > 0 .$$

Let us suppose that $\theta_0 \in W^{1,2}(\Omega_h)$ and the sequence θ^n_0 (see (3.6)) satisfies $\theta^n_0 \to \theta_0$ strongly in $W^{1,2}(\Omega_h)$. We multiply (3.5) by θ^n and integrate over $Q_{t,h}$. After some computation we get

$$(4.23) \qquad \tfrac{1}{2} c_V \int_{\Omega_h} \varrho^n (\theta^n)^2 \, dx\big|_t + \lambda \int_0^t \int_{\Omega_h} \theta^n_{,i} \theta^n_{,i} \, dx \, d\tau =$$
$$= - \int_0^t \int_{\Omega_h} R \varrho^n \tilde{u}^n_{j,j} (\theta^n)^2 \, dx \, d\tau + \int_0^t \int_{\Omega_h} \Psi(u^n) \, \theta^n \, dx \, d\tau +$$
$$+ \tfrac{1}{2} c_V \int_{\Omega_h} \varrho^n_0 (\theta^n_0)^2 \, dx .$$

The following estimates hold (indexes are omitted):

$$(4.24) \qquad \int_0^t \int_{\Omega_h} \varrho^n D \tilde{u}^n (\theta^n)^2 \, dx \, d\tau \leqq K_{13} \|\varrho^n\|_{L^\infty(Q_{t,h})} \times$$
$$\times \|u^n\|_{L^\infty((0,t), L^2(\Omega, R^N))} \|\theta^n\|^2_{L^2(Q_{t,h})} ,$$

$$(4.25) \qquad \int_0^t \int_{\Omega_h} \Psi(u^n) \, \theta^n \, dx \, d\tau \leqq K_{13} \|u^n\|_{L^\infty((0,t), W^{1,2}(\Omega))} \times$$
$$\times \|u^n\|_{L^2((0,t), W^{2,2}(\Omega, R^N))} \|\theta^n\|_{L^2((0,t), W^{1,2}(\Omega_h))} .$$

If we use the estimates (4.24), (4.25) on the r.h.s. of (4.23), take into account (4.6), (4.9), (4.7), (4.8), (4.14), (4.22) and apply the Young inequality, we obtain

$$(4.26) \qquad \int_{\Omega_h} (\theta^n)^2 \, dx\big|_t + \int_0^t \int_{\Omega_h} \theta^n_{,i} \theta^n_{,i} \, dx \, d\tau \leqq K_{14} (1 + \int_0^t \int_{\Omega_h} (\theta^n)^2 \, dx \, d\tau) ,$$
$$K_{14} > 0 .$$

Using the Gronwall inequality, we can derive

$$(4.27) \qquad \|\theta^n\|_{L^\infty(I, L^2(\Omega_h))} \leqq K_{15} , \quad K_{15} > 0 ,$$

$$(4.28) \qquad \|\theta^n\|_{L^2(I, W^{1,2}(\Omega_h))} \leqq K_{15} .$$

Due to (4.4) and (4.29) we have

$$(4.29) \qquad \|\varrho^n \theta^n\|_{L^2(I, W^{1,2}(\Omega_h))} \leqq K_{16} , \quad K_{16} > 0 .$$

One verifies that the expressions $(\varrho^n u^n_i \theta^n)_{,i}$, $\varrho^n \tilde{u}^n_{j,j} \theta^n$, $\Psi(u^n)$, $\theta^n_{,jj}$ are bounded in $L^2(I, W^{-1,2}(\Omega_h))$. We can see from (3.5) that

$$(4.30) \qquad \|(\varrho^n \theta^n)_{,t}\|_{L^2(I, W^{-1,2}(\Omega_h))} \leqq K_{17} , \quad K_{17} > 0 .$$

Lemma 5.1. *Let $\{(\varrho^n, u^n, \theta^n)\}_{n=1}^{+\infty}$ be a sequence of solutions to the problem (3.3)–(3.6). Then there exists (ϱ, u, θ) and a subsequence (denoted $\{(\varrho^n, u^n, \theta^n)\}_{n=1}^{+\infty}$ again) such that*

i) $\varrho^n \to \varrho$ *strongly in* $L^p(I, W^{1,2}(\Omega_h))$, $1 < p < +\infty$, $\varrho \geqq \delta_1 > 0$ *a.e. in* $Q_{T,h}$;

ii) $u^n \to u$ *strongly in* $L^2(I, W^{1,2}(\Omega, R^N))$;

iii) $\theta^n \to \theta$, $\theta^n_{,i} \to \theta_{,i}$ *weakly in* $L^2(Q_{T,h})$, $\varrho^n\theta^n \to \varrho\theta$ *strongly in* $L^2(Q_{T,h})$, $\theta \geqq 0$ *a.e. in* $Q_{T,h}$;

iv) $\varrho^n_{,t} \to \varrho_{,t}$ *weakly in* $L^2(Q_{T,h})$;

v) $\varrho^n u^n_{,t} \to \varrho u_{,t}$ *weakly in* $L^2(Q_T, R^N)$;

vi) $\varrho^n \tilde{u}^n \to \varrho \tilde{u}$ *strongly in* $L^2(Q_{T,h}, R^N)$;

vii) $\int_0^T \int_\Omega \varrho^n \tilde{u}^n_j u^n_{i,j} w_i \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \varrho \tilde{u}_j u_{i,j} w_i \,\mathrm{d}x\,\mathrm{d}t$ *for every* $w \in \mathscr{C}_0^\infty(Q_T, R^N)$;

viii) $\int_0^T \int_{\Omega_h} \varrho^n \theta^n \tilde{u}^n_i \eta_{,i} \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_{\Omega_h} \varrho\theta\tilde{u}_i \eta_{,i} \,\mathrm{d}x\,\mathrm{d}t$ *for every* $\eta \in \mathscr{C}^\infty(\bar{Q}_{T,h})$;

ix) $\int_0^T \int_{\Omega_h} \Phi(u^n)\, \eta \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_{\Omega_h} \Psi(u)\, \eta \,\mathrm{d}x\,\mathrm{d}t$ *for every* $\eta \in \mathscr{C}^\infty(\bar{Q}_{T,h})$.

Proof. i) In order to show the strong convergence, we use the estimates (4.4), (4.9) and apply the Lions lemma (see Lemma 1.1) with $B_0 = W^{2,2}(\Omega_h)$, $B = W^{1,2}(\Omega_h)$, $B_1 = W^{1,2}(\Omega_h)$, $1 < p_l < +\infty$ $(l = 0, 1)$. Due to (4.7) we have $\varrho \geqq \delta_1$ a.e. in $Q_{T,h}$ for some $\delta_1 > 0$.

ii) We apply the Lions lemma with $B_0 = W^{2,2}(\Omega, R^N)$, $B = W^{1,2}(\Omega, R^N)$, $B_1 = L^2(\Omega, R^N)$, $p_0 = p_1 = 2$.

iii) Weak convergence follows directly from (4.28). According to the Lions lemma $\varrho^n\theta^n \to a$ strongly in $L^2(Q_{T,h})$ (see (4.29), (4.30)). We can verify that $\varrho^n\theta^n \to \varrho\theta$, in the sense of distributions, hence $a = \varrho\theta$, $\theta \geqq 0$ a.e. in $Q_{T,h}$ because of $\theta^n \geqq 0$ in $Q_{T,h}$.

iv) follows directly from (4.9).

v) Due to (4.13), $u^n_{,t} \to u_{,t}$ weakly in $L^2(Q_T, R^N)$. This fact and i) give the desired result.

vi) Using the Hölder inequality and (1.8), we get

$$\|\varrho^n \tilde{u}^n - \varrho\tilde{u}\|^2_{L^2(Q_{T,h})} \leqq \|\varrho^n - \varrho\|^2_{L^2(Q_{T,h})} \|\check{u}^n\|^2_{L^\infty(Q_{T,h}, R^N)} + \\ + \|\tilde{b}^n\|_{L^2(Q_{T,h}, R^N)} \|u^n - u\|_{L^2(Q_T, R^N)}, \quad \text{where} \quad b^n = \varrho^2(u^n + u).$$

The r.h.s. tends to zero due to i), ii).

vii) follows directly from ii), vi).

viii) We use (1.8) and the Green formula. We obtain

$$\int_0^t \int_{\Omega_h} (p(\varrho^n, \theta^n)\, \tilde{u}_i^n - p(\varrho, \theta)\, \tilde{u}_i)\, \eta_{,i}\, dx\, d\tau =$$
$$= \int_0^t \int_{\Omega_h} R(\varrho^n\theta^n - \varrho\theta)\, \tilde{u}_i^n \eta_{,i}\, dx\, d\tau + \int_0^t \int_{\Omega_h} \tilde{p}(\varrho, \theta)\,(u_i^n - u_i)\, \eta_{,i}\, dx\, d\tau\,.$$

The r.h.s. tends to zero due to ii), iii).

ix) holds due to ii), because Du^n and $Du = 0$ a.e. in $Q_{T,h} - Q_T$.

Now we can turn our attention to the proof of the main theorem. Applying the results of Lemma 5.1 and considering that $n \to +\infty$ in (3.3)–(3.6), we can show that

(5.1) $$\int_0^T \int_{\Omega_h} (\varrho_{,t}\zeta - \varrho\tilde{u}_j\zeta_{,j})\, dx\, dt = 0 \quad \text{for every} \quad \zeta \in \mathscr{C}_0^\infty(Q_{T,h})\,,$$

(5.2) $$\int_0^T \int_{\Omega_h} (\varrho u_{i,t} + \varrho\tilde{u}_j u_{i,j} + \tilde{p}_{,i}(\varrho, \theta))\, w_i\, dx\, dt +$$
$$+ \int_0^T ((u, w))\, dx\, dt = 0 \quad \text{for every} \quad w \in \mathscr{C}_0^\infty(Q_T, R^N)\,,$$

(5.3) $$- c_V \int_{Q_{T,h}} \varrho\theta\eta_{,t}\, dx\, dt - c_V \int_\Omega \varrho_0\theta_0\, \eta(0)\, dx +$$
$$+ \int_{Q_{T,h}} (\lambda\theta_{,i} - c_V\varrho\tilde{u}_i\theta)\eta_{,i}\, dx\, dt = \int_{Q_{T,h}} (\Psi - p\tilde{u}_{j,j})\, \eta\, dx\, dt$$
$$\text{for every} \quad \eta \in \mathscr{C}^\infty(\bar{Q}_{T,h}) \quad \text{such that} \quad \eta(T) = 0\,.$$

Due to the density argument, one derives from (5.1), (5.2) that (1.1), (1.2) are satisfied a.e. in $Q_{T,h}$ and (2.3) holds. It is clear that $\int_{\Omega_h} \varrho^n\theta^n\eta\, dx|_t \to \int_{\Omega_h} \varrho\theta\eta\, dx|_t$ for a.e. $t \in I$ and for every $\eta \in L^2(I, W^{1,2}(\Omega_h))$, $\eta_{,t} \in L^2(Q_{T,h})$, hence (2.4) holds, too.

Remark 5.1 One can easily extend our results to the case of a nonzero external force $F \in L^\infty(Q_T, R^N)$ and nonzero heat sources $q \in L^\infty(Q_{T,h})$.

Remark 5.2 If $\varrho_0 \in W^{k,2}(\Omega_h)$ ($k \geqq 2$) then one obtains $\varrho \in L^\infty(I, W^{k,2}(\Omega_h))$, $\varrho_{,t} \in L^\infty(I, W^{k-1,2}(\Omega_h))$. This can be proved by the same method as the one used in the derivation of (4.4), (4.5), (4.9).

VI. ENERGY AND ENTROPY ESTIMATES

We derive some estimates independent of the parameter of regularization in this section. We start with the following lemma.

Lemma 6.1 *Let* $\{(\varrho^n, u^n, \theta^n)\}_{n=1}^{+\infty}$ *be a sequence of solutions to* (3.3)–(3.6). *Let* (2.1) *be satisfied and let* $\theta_0 \in \mathscr{C}^2(\bar{Q}_h)$, $\theta_0 > 0$ *in* $\bar{\Omega}_h$; *put* $\theta_0^n = \theta_0$. *Then*

i) $\theta^n > 0$ *in* $\bar{Q}_{T,h}$,

ii) *there exists a subsequence (denoted* $\{(\varrho^n, u^n, \theta^n)\}_{n=1}^{+\infty}$ *again) such that* ii)$_1$ $\quad 1/\theta^n \to 1/\theta$ *strongly in* $L^p(Q_{T,h})$, $1 < p < +\infty$;

ii)$_2$ $\quad \int_{\Omega_h} \varrho^n |\ln \theta^n| \, dx \to \int_{\Omega_h} \varrho |\ln \theta| \, dx$ *for a.e.* $t \in I$;

ii)$_3$ $\quad \int_0^t \int_{\Omega_h} (\Psi(u^n)/\theta^n) \, dx \, d\tau \to \int_0^t \int_{\Omega_h} (\Psi(u)/\theta) \, dx \, d\tau$, $t \in (0, T]$;

ii)$_4$ $\quad (\ln \theta^n)_{,i} \to (\ln \theta)_{,i}$ *weakly in* $L^4(Q_{T,h})$.

Proof. i) Due to the smoothness of Θ^n we can find a minimal number σ, $0 < \sigma \leqq T$ such that $\min_{\bar{\Omega}_h} \theta^n(\sigma, x) = 0$. Hence $\theta^n(t, x) > 0$ in $Q_{\sigma,h}$. Let us put $\vartheta = \ln \theta^n$. From (3.5) we derive

$$(6.1) \qquad c_V \varrho^n \vartheta_{,t} + c_V \varrho^n \tilde{u}_j^n \vartheta_{,j} - \lambda \vartheta_{,ii} = \lambda (1/\theta^n)^2 \, \theta^n_{,i} \theta^n_{,i} + \Psi(u^n)/\theta^n - R \varrho^n \tilde{u}^n_{j,j} .$$

For $t < \sigma$ let us put $\vartheta = \vartheta_1 + \vartheta_2$, where

$$\vartheta_1(0) = \ln \theta_0^n , \quad \frac{\partial \vartheta_1}{\partial \nu} = 0 \quad \text{on} \quad (0, \sigma) \times \partial\Omega_h ,$$

$$c_V \varrho^n \vartheta_{1,t} + c_V \varrho^n \tilde{u}_j^n \vartheta_{1,j} - \lambda \vartheta_{1,ii} = \lambda (1/\theta^n)^2 \, \theta^n_{,i} \theta^n_{,i} + \Psi(u^n)/\theta^n$$

and

$$\vartheta_2(0) = 0 , \quad \frac{\partial \vartheta_2}{\partial \nu} = 0 \quad \text{on} \quad (0, \sigma) \times \partial\Omega_h ,$$

$$c_V \varrho^n \vartheta_{2,t} + c_V \varrho^n \tilde{u}_j^n \vartheta_{2,j} - \lambda \vartheta_{2,ii} = -R \varrho^n \tilde{u}^n_{j,j} .$$

According to the maximum principle $\vartheta_1 \geqq \ln \theta_0^n$ and ϑ_2 can be continuously extended to $\bar{Q}_{\sigma,h}$. This is a contradiction.

ii)$_1$ We multiply (3.5) by $1/(k(\theta^n)^{k+1})$, $k > 0$ and integrate over $Q_{T,h}$. We get

$$(6.2) \qquad c_V \int_{\Omega_h} (\varrho^n/(\theta^n)^k) \, dx|_{t=T} + \frac{1}{k} \int_0^T \int_{\Omega_h} (\Psi(u^n)/(\theta^n)^{k+1}) \, dx \, dt +$$

$$+ \frac{k+1}{k} \int_0^T \int_{\Omega_h} (\theta^n_{,i} \theta^n_{,i}/(\theta^n)^{k+2}) \, dx \, dt =$$

$$= \frac{1}{k} \int_0^T \int_{\Omega_h} (R \varrho^n \tilde{u}^n_{i,i}/(\theta^n)^k) \, dx \, dt + c_V \int_{\Omega_h} (\varrho_0^n/(\theta_0^n)^k) \, dx .$$

From (6.2) we obtain

$$\|(\theta^n)^{-\sqrt{k}}\|_{L^\infty(I, L^2(\Omega_h))} \leqq K_{19} , \quad K_{19} > 0 .$$

In particular $1/\theta^n$ is bounded in $L^p(Q_{T,h})$ for $p > 1$. Due to Lemma 5.1, i) and iii) $\theta^n \to \theta$ a.e. in $Q_{T,h}$, hence $1/\theta^n \to 1/\theta$ strongly in $L^p(Q_{T,h})$.

ii)$_2$ $\ln \theta^n(t) \to \ln \theta(t)$ and $\varrho^n(t) \to \varrho(t)$ a.e. in Ω_h for a.e. $t \in I$. $\varrho^n(t) \, |\ln \theta^n(t)|$ is bounded in $L^2(\Omega_h)$ for a.e. $t \in I$. Thus $\varrho^n(t) \, |\ln \theta^n(t)| \to \varrho(t) \, |\ln \theta(t)|$ weakly in $L^2(\Omega_h)$ for a.e. $t \in I$.

ii)$_3$ holds due to ii)$_1$ and part ii) of Lemma 5.1.

ii)$_4$ is consequence of ii)$_1$ and iii) in Lemma 5.1.

We prove the following theorem

Theorem 6.1 *Let $(\varrho_h, u_h, \theta_h)$ be a weak solution to the system (1.1)–(1.7) corresponding to the parameter $h > 0$. Let (2.1) be satisfied and let $\theta_0 \in \mathscr{C}^2(\bar{Q}_h)$, $\theta_0 > 0$ in $\bar{Q}_h$. Then it holds*

(6.3) $$\int_{\Omega_h} \varrho_h \, dx|_t = \int_{\Omega_h} \varrho_0 \, dx ,$$

(6.4) $$\tfrac{1}{2} \int_\Omega \varrho_h |u_h|^2 \, dx|_t + c_V \int_{\Omega_h} \varrho_h \theta_h \, dx|_t \leqq \tfrac{1}{2} \int_\Omega \varrho_0 |u_0|^2 \, dx + c_V \int_{\Omega_h} \varrho_0 \theta_0 \, dx ,$$

(6.5) $$R \int_{\Omega_h} \varrho_h |\ln \varrho_h| \, dx|_t + \int_0^t \int_{\Omega_h} (\Psi(u_h)/\theta_h) \, dx \, d\tau + \lambda \int_0^t \int_{\Omega_h} (\ln \theta_h)_{,i} (\ln \theta_h)_{,i} \, dx \, d\tau \leqq \tfrac{1}{2} \int_\Omega \varrho_0 |u_0|^2 \, dx + 2 \frac{R}{e} m(\Omega_h) - R \int_{\Omega_h} \varrho_0 \ln \varrho_0 \, dx - c_V \int_{\Omega_h} \varrho_0 \ln \theta_0 \, dx + c_V \int_{\Omega h} \varrho_0 \theta_0 \, dx ,$$

(6.6) $$c_V \int_{\Omega h} \varrho_h |\ln \theta_h| \, dx|_t \leqq \frac{R}{e} m(\Omega_h) + R \int_{\Omega_h} \varrho_0 \ln \varrho_0 \, dx + 2c_V \int_{\Omega_h} \varrho_0 \theta_0 \, dx + \int_\Omega \varrho_0 |u_0|^2 \, dx - c_V \int_{\Omega_h} \varrho_0 \ln \theta_0 \, dx \, dt$$ *for a.e.* $t \in I$.

Proof. The equalities (6.3), (6.4) follow directly from (3.3), (4.1) by passing to the limit $n \to +\infty$. Due to i) in Lemma 6.1, we can multiply (3.5) by $1/\theta^n$. After integrating over $Q_{t,h}$, we get

(6.7) $$c_V \int_{\Omega_h} \varrho^n \ln \theta^n \, dx|_t = \int_0^t \int_{\Omega_h} (\Psi(u^n)/\theta^n) \, dx \, d\tau + \lambda \int_0^t \int_{\Omega_h} (\ln \theta^n)_{,i} (\ln \theta^n)_{,i} \, dx \, d\tau - \int_0^t \int_{\Omega_h} R \varrho^n \tilde{u}^n_{i,i} \, dx \, d\tau + c_V \int_{\Omega_h} \varrho^n_0 \ln \theta^n_0 \, dx .$$

From (3.3) one gets

(6.8) $$- \int_0^t \int_{\Omega_h} R \varrho^n \tilde{u}^n_{i,i} \, dx \, d\tau = \int_{\Omega_h} R \varrho^n \ln \varrho^n \, dx|_t - \int_{\Omega_h} R \varrho^n_0 \ln \varrho^n_0 \, dx|_t .$$

Putting (6.8) into (6.7), we get

(6.9) $$c_V \int_{\Omega_h} \varrho^n \ln \theta^n \, dx|_t = \int_{\Omega_h} R \varrho^n \ln \varrho^n \, dx|_t - \int_{\Omega_h} R \varrho^n_0 \ln \varrho^n_0 \, dx + \int_0^t \int_{\Omega_h} (\Psi(u^n)/\theta^n) \, dx \, d\tau + \lambda \int_0^t \int_{\Omega_h} (\ln \theta^n)_{,i} (\ln \theta^n)_{,i} \, dx \, d\tau + c_V \int_{\Omega_h} \varrho^n_0 \ln \theta^n_0 \, dx .$$

We know that $\ln \theta < \theta$, $\varrho \ln \varrho = \varrho(\ln \varrho)^+ - \varrho(\ln \varrho)^-$, $\ln \theta = (\ln \theta)^+ - (\ln \theta)^-$ (a^+, a^- are the positive and negative parts of a, respectively) and $R\varrho(\ln \varrho)^- \leqq R/e$.

Using these facts and (4.1) we arrive at

(6.10) $$R \int_{\Omega_h} \varrho^n |\ln \varrho^n| \, dx|_t + \int_0^t \int_{\Omega_h} (\Psi(u^n)/\theta^n) \, dx \, d\tau + \\ + \lambda \int_0^t \int_{\Omega_h} (\ln \theta^n)_{,i} (\ln \theta^n)_{,i} \, dx \, d\tau \leqq \tfrac{1}{2} \int_\Omega \varrho_0^n |u_0^n|^2 \, dx + 2 (R/e) \, m(\Omega_h) + \\ + R \int_{\Omega_h} \varrho_0^n \ln \varrho_0^n \, dx - c_V \int_{\Omega_h} \varrho_0^n \ln \theta_0^n \, dx + c_V \int_{\Omega_h} \varrho_0^n \theta_0^n \, dx ,$$

(6.11) $$c_V \int_{\Omega h} \varrho^n |\ln \theta^n| \, dx|_t \leqq (R/e) \, m(\Omega_h) + R \int_{\Omega_h} \varrho_0^n \ln \varrho_0^n \, dx + \\ + 2c_V \int_{\Omega_h} \varrho_0^n \theta_0^n \, dx + \int_\Omega \varrho_0^n |u_0^n|^2 \, dx - c_V \int_{\Omega_h} \varrho_0^n \ln \theta_0^n \, dx \, dt .$$

Due to (6.10), $(\ln \theta^n)_{,i} \to a$ weakly in $L^2(Q_{T,h})$ (at least for a chosen subsequence). According to Lemma 6.1 ii$_4$), $a = (\ln \theta)_{,i}$, hence due to the Fatou lemma

(6.12) $$\liminf_{n \to +\infty} \|(\ln \theta^n)_{,i}\|_{L^2(Q_{T,h})} \geqq \|(\ln \theta)_{,i}\|_{L^2(Q_{T,h})} .$$

We use Lemma 5.1, Lemma 6.1 and (6.12). We can pass to the limit in (6.10), (6.11) and we get (6.5), (6.6). Thus the theorem is proved.

References

[1] *S. Agmon, A. Douglis, L. Nirenberg:* Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II. Comm. Pure Appl. Math. 17 (1964), 35—92.

[2] *A. Friedman:* Partial differential equations of parabolic type. Prentice-Hall, INC (1964).

[3] *A. Kufner, O. John, S. Fučík:* Function spaces. Praha, Academia (1977).

[4] *O. A. Ladzhenskaya, V. A. Solonnikov, N. N. Uralceva:* Linear and quasilinear equations of parabolic type (Russian). Moskva, Nauka (1967).

[5] *J. L. Lions:* Quelques méthodes des résolution des problemes aux limites nonlinéaires. Dunod, Paris (1969).

[6] *A. Matsumura, T. Nishida:* Initial boundary value problems for the equation of motion of compressible viscous and heat conductive fluids. Comm. Math. Phys. 89 (1983), 445—464.

[7] *A. Matsumura, T. Nishida:* The initial value problem for the equations of motion of viscous and heat conductive gasses. J. Math. Kyoto Univ. 20 (1980), 67—104.

[8] *S. Mizohata:* Theory of partial differential equations (Russian). Moskva, Mir (1977).

[9] *J. Nečas, A. Novotný, M. Šilhavý:* Global solution to the compressible isothermal multipolar fluid, to appear J. Math. Anal. Appl. (1991).

[10] *J. Nečas, M. Šilhavý:* Multipolar viscous fluids, to appear Quart. Appl. Math.

[11] *J. Neustupa:* The global weak solvability of a regularized system of the Navier-Stokes equations for compressible fluid. Apl. Mat. 33 (1988), 389—409.

[12] *J. Neustupa, A. Novotný:* Uniqueness to the regularized viscous compressible heat conductive flow, to appear.

[13] *M. Padula:* Existence of global solutions for 2-dimensional viscous compressible flow. J. Funct. Anal. 69 (1986), 1—20.

[14] *R. Rautman:* The uniqueness and regularity of the solutions of Navier-Stokes problems. Lecture Notes in Math. Vol. 561, Springer-Verlag (1976).

[15] *A. Tani:* On the first initial boundary value problem of compressible viscous fluid, Publ. RIMS Kyoto Univ. 13 (1977), 193—253.

[16] *R. Temam:* Navier-Stokes equations. Amsterdam—New York—Oxford (1979).

[17] *A. Valli:* An existence theorem for compressible viscous fluids, Ann. Mat. Pura Appl. 130 (1982), 197—213.

Souhrn

SLABÁ ŘEŠITELNOST REGULARIZOVANÉHO SYSTÉMU ROVNIC PRO POHYB VAZKÉ STLAČITELNÉ TEKUTINY

JIŘÍ NEUSTUPA, ANTONÍN NOVOTNÝ

V práci je navržena fyzikálně přijatelná regularizace úplného systému Navier-Stokesových rovnic pro vazkou stlačitelnou tepelně vodivou tekutinu. Je dokázána věta o existenci slabých řešení počáteční a okrajové úlohy pro regularizovaný systém. Jsou odvozeny energetické odhady nezávislé na parametru regularizace.

Authors' addresses: Doc. RNDr. *Jiří Neustupa*, CSc., Department of Applied Mathematics, Faculty of Engineering Czech Technical University, Technická 4, 166 07 Praha 6, Czechoslovakia; RNDr. *Antonín Novotný*, CSc., Institute of the Physics of the Atmosphere Cz. Acad. Sci., Boční II, 1401, 141 31 Praha 4, Czechoslovakia.