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SOLVABILITY OF A FORCED AUTONOMOUS DUFFING'S EQUATION WITH PERIODIC BOUNDARY CONDITIONS IN THE PRESENCE OF DAMPING

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Summary. Let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous function, $e\colon [0,1] \to \mathbb{R}$ a function in $L^2[0,1]$ and let $c \in \mathbb{R}$, $c \neq 0$ be given. It is proved that Duffing's equation $u'' + cu' + g(u) = e(x)$, $0 < x < 1$, $u(0) = u(1)$, $u'(0) = u'(1)$ in the presence of the damping term has at least one solution provided there exists an $R > 0$ such that $g(u)u \geqslant 0$ for $|u| \geqslant R$ and $\int_0^1 e(x)\,\mathrm{d}x = 0$. It is further proved that if g is strictly increasing on $\mathbb{R}$ with $\lim\limits_{u\to-\infty} g(u) = -\infty$, $\lim\limits_{u\to\infty} g(u) = \infty$ and is Lipschitz continuous with Lipschitz constant $\alpha < 4\pi^2 + c^2$, then Duffing's equation given above has exactly one solution for every $e \in L^2[0,1]$.

Keywords: Duffing's equation, damping

AMS classification: 34B15, 34C25, 47H15

1. Introduction

Let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous function, $e\colon [0,1] \to \mathbb{R}$ a function in $L^2[0,1]$ and let $c \in \mathbb{R}$, $c \neq 0$ be given. This paper is devoted to the existence of a solution of the forced autonomous Duffing's equation

$$(1.1)\qquad \begin{gathered} u'' + cu' + g(u) = e, \qquad 0 < x < 1, \\ u(0) = u(1), \qquad u'(0) = u'(1). \end{gathered}$$

We call the equation in (1.1) "autonomous" since the nonlinear function g is independent of x. When g is a function of both the variables x and u, i.e. $g\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is a function satisfying Caratheodory's conditions, the non-autonomous Duffing's equation

$$
(1.2) \qquad \begin{gathered} u'' + cu' + g(x,u) = e, \qquad 0 < x < 1, \\ u(0) = u(1), \qquad u'(0) = u'(1) \end{gathered}
$$

has been extensively studied earlier (see e.g. [1], [2], [3], [4], [8], among others). It was shown, for example, by Gupta in [1] that if there exists a $\varrho > 0$ such that $g(x,u)u \leqslant 0$ for a.e. $x \in [0,1]$ and all $u \in \mathbf{R}$ with $|u| \geqslant R$ then (1.2) has at least one solution provided $\int_0^1 e(x)\,dx = 0$. In the case when there exists a $\varrho > 0$ such that $g(x,u)u \geqslant 0$ for a.e. $x \in [0,1]$ and $|u| \geqslant \varrho$, it was shown in [3] that (1.2) has at least one solution provided $\int_0^1 e(x)\,dx = 0$ and $\lim \sup\limits_{|u|\to\infty} \frac{g(x,u)}{u}$ is strictly less than $4\pi^2 + c^2$. Now when $c \neq 0$, then $4\pi^2 + c^2 > 4\pi^2$, which is the second eigenvalue of the linear eigenvalue problem

$$
(1.3) \qquad \begin{gathered} -u'' = \lambda u, \\ u(0) = u(1), \qquad u'(0) = u'(1). \end{gathered}
$$

It was remarked in [3] that $\lambda = 0$ is the only eigenvalue of the linear eigenvalue problem when $c \neq 0$,

$$
(1.4) \qquad \begin{gathered} u'' + cu' = \lambda u, \\ u(0) = u(1), \qquad u'(0) = u'(1) \end{gathered}
$$

to explain that the nonlinearity in $g(x,u)$ can resonate beyond the second eigenvalue $4\pi^2$ of the linear eigenvalue problem (1.3). Indeed, the author feels that when $c \neq 0$ and $g(x,u)u \geqslant 0$ for a.e. $x \in [0,1]$ and $|u| \geqslant \varrho$, then the boundary value problem (1.2) should have at least one solution when $\int_0^1 e(x)\,dx = 0$. But this is not known at this time. The purpose of this paper is to prove this conjecture in the case of the autonomous boundary value problem (1.1) when $c \neq 0$. The autonomous problem (1.1) was studied, when $c \neq 0$, by Nieto and Rao in [8] in the case when $g\colon \mathbf{R} \to \mathbf{R}$ is increasing and $\lim\limits_{u\to\pm\infty} g(u) = g(\pm\infty)$ exists. But this case was already covered in [1] because then g is bounded on $\mathbf{R}$ and accordingly, $\lim\limits_{|u|\to\infty} \frac{g(u)}{u} = 0 < 4\pi^2$.

Our methods involve using Mawhin's version of the Leray-Schauder continuation theorem and Wirtinger type inequalities to get the needed estimates. We also present some uniqueness results for the boundary value problem (1.1).

2. Main results

Let X, Y denote the Banach spaces $X = C[0,1]$ and $Y = L^1[0,1]$ with their usual norms. Let Y_2 be the subspace of Y spanned by the constant function 1 on $[0,1]$, i.e.,

$$Y_2 = \{u \in Y \mid u(x) \equiv c \text{ for a.e. } x \in [0,1],\ c \in \mathbf{R}\},$$

and let Y_1 be the subspace of Y such that $Y = Y_1 \oplus Y_2$. We note that for $u \in Y$ we can write

$$u(x) = \Big(u(x) - \int_0^1 u(x)\,\mathrm{d}x\Big) + \Big(\int_0^1 u(x)\,\mathrm{d}x\Big) \tag{2.1}$$

for $x \in [0,1]$. We define the canonical projection operators $P\colon Y \to Y_1$, $Q\colon Y \to Y_2$ by

$$\begin{aligned} P(u)(x) &= u(x) - \int_0^1 u(x)\,\mathrm{d}x, \\ Q(u) &= \int_0^1 u(x)\,\mathrm{d}x \end{aligned} \tag{2.2}$$

for $u \in Y$. Clearly, $Q = I - P$, where I denotes the identity mapping on Y, and the projections P and Q are continuous. Now let $X_2 = X \cap Y_2$. Clearly X_2 is a closed subspace of X. Let X_1 be the closed subspace of X such that $X = X_1 \oplus X_2$. We note that $P(X) \subset X_1$, $Q(X) \subset X_2$ and the projections $P|X\colon X \to X_1$, $Q|X\colon X \to X_2$ are continuous. In the following, X, Y, P, Q will refer to the Banach spaces and projections as defined and we will not distinguish between P, $P|X$ (resp. Q, $Q|X$) and rely on the context for proper meaning.

Also for $u \in X$, $v \in Y$, let $(u,v) = \int_0^1 u(x)v(x)\,\mathrm{d}x$ denote the duality pairing between X and Y. We note that for $u \in X$, $v \in Y$, such that $u = Pu + Qu$, $v = Pv + Qv$ we have

$$(u,v) = (Pu, Pv) + (Qu, Qv). \tag{2.3}$$

Let $c \in \mathbf{R}, c \neq 0$ be given. Define a linear operator $L\colon D(L) \subset X \to Y$ by setting

$$D(L) = \{u \in X \mid u'(x) \in AC[0,1],\ u(0) = u(1),\ u'(0) = u'(1)\}, \tag{2.4}$$

and for $u \in D(L)$,

$$Lu = u'' + cu'. \tag{2.5}$$

(Here $AC[0,1]$ denotes the space of real-valued absolutely continuous functions on $[0,1]$.) It is easy to see that L is a linear Fredholm mapping with $\ker L = X_2, \operatorname{Im} L = Y_1$. Further, the mapping $K: Y_1 \to X_1$, defined for $u \in Y_1$ by

$$(Ku)(x) = v(x) - \int_0^1 v(x)\,\mathrm{d}x, \tag{2.6}$$

where

$$v(x) = \int_0^x \int_0^\xi \mathrm{e}^{c(t-\xi)} u(t)\,dt\,d\xi - \frac{\mathrm{e}^{-cx} - 1}{c(\mathrm{e}^c - 1)} \int_0^1 \mathrm{e}^{ct} u(t)\,dt, \tag{2.7}$$

(note that we have assumed $c \neq 0$), satisfies the following conditions:

$$\begin{aligned} &\text{(i) for } u \in Y \text{, we have } KP(u) \in D(L),\ LKP(u) = P(u),\\ &\text{(ii) } (KP(u), P(u)) \geqslant -\tfrac{1}{(4\pi^2+c^2)} \|P(u)\|^2_{L^2[0,1]}. \end{aligned} \tag{2.8}$$

Indeed, note that for $v = KP(u) \in D(L)$,

$$(KP(u), P(u)) = (v, Lv) = -\int_0^1 v'^2 \geqslant -\frac{1}{4\pi^2 + c^2} \|Lv\|^2_{L^2[0,1]}$$

and so $(KP(u), P(u)) \geqslant -\frac{1}{4\pi^2+c^2} \|P(u)\|^2_{L^2[0,1]}$ since

$$\begin{aligned} \|Lv\|^2_{L^2[0,1]} &= \int_0^1 (v'' + cv')^2\,\mathrm{d}x = \int_0^1 \left[(v'')^2 + 2cv'v'' + c^2(v')^2\right]\,\mathrm{d}x \\ &= \int_0^1 \left[(v'')^2 + c^2(v')^2\right]\,\mathrm{d}x \geqslant (4\pi^2 + c^2) \int_0^1 v'^2\,\mathrm{d}x. \end{aligned}$$

Let now $g: \mathbf{R} \to \mathbf{R}$ be a given continuous function. Let $N: X \to X \subset Y$ be the non-linear mapping defined by

$$(Nu)(x) = g(u(x)), \qquad x \in [0,1]$$

for $u \in X$. It is then easy to see, using Arzèla-Ascoli theorem, that $KPN: X \to X_1$ is continuous and compact.

Theorem 1. *Let $g: \mathbf{R} \to \mathbf{R}$ be a given continuous function. Let c, a, A, r, R with $a \leqslant A$, $r < 0 < R$, $c \neq 0$ be such that*

$$g(u) \geqslant A \text{ for } u \geqslant R, \quad \text{and} \quad g(u) \leqslant a \text{ for } u \leqslant r. \tag{2.9}$$

Then, for every given function $e(x) \in L^2[0,1]$ with $a \leqslant \int_0^1 e(x)\,dx \leqslant A$, Duffing's equation

$$(2.10)\qquad \begin{aligned} u'' + cu' + g(u) &= e, \qquad 0 < x < 1, \\ u(0) = u(1), &\quad u'(0) = u'(1) \end{aligned}$$

has at least one solution.

Proof. Define functions $g_1 : \mathbf{R} \to \mathbf{R}$ and $e_1 : [0,1] \to \mathbf{R}$ by setting

$$g_1(u) = g(u) - \frac{A+a}{2},$$
$$e_1(x) = e(x) - \frac{A+a}{2}.$$

Then $g_1 : \mathbf{R} \to \mathbf{R}$ is a continuous function and $e_1 : [0,1] \to \mathbf{R}$ is such that $e_1(x) \in L^2[0,1]$. Furthermore,

$$g_1(u) \geqslant \frac{1}{2}(A-a) \geqslant 0 \quad \text{for } u \geqslant R,$$

$$g_1(u) \leqslant \frac{1}{2}(a-A) \leqslant 0 \quad \text{for } u \leqslant R,$$

and

$$\frac{1}{2}(a-A) \leqslant \int_0^1 e_1(x)\,dx \leqslant \frac{1}{2}(A-a).$$

Duffing's equation (2.10) is equivalent to the equation

$$(2.11)\qquad \begin{aligned} u'' + cu' + g_1(u) &= e_1, \quad 0 < x < 1, \\ u(0) = u(1), &\quad u'(0) = u'(1). \end{aligned}$$

Now, for $X = C[0,1]$ and $Y = L^1[0,1]$ we consider the Niemytski operator $N : X \to Y$ defined for $u \in X$ by

$$(Nu)(x) = g_1(u(x)), \qquad x \in [0,1],$$

and the linear operator $L : D(L) \subset X \to Y$ defined in (2.4), (2.5).

The equation (2.11) is equivalent to the operator equation

$$(2.12)\qquad Lu + Nu = e_1$$

in X. To solve (2.12) it suffices to solve the system of equations

$$
\begin{aligned}
Pu + KPNu &= KPe_1, \\
QNu &= Qe_1
\end{aligned}
\tag{2.13}
$$

in X. Indeed, if $u \in X$ solves (2.13), then $u \in D(L)$ and

$$
\begin{aligned}
LPu + LKPNu = Lu + PNu = LKPe_1 &= Pe_1, \\
QNu &= Qe_1,
\end{aligned}
$$

which gives, on adding, that $Lu + Nu = e_1$.

Now, (2.13) is clearly equivalent to the single equation

$$
Pu + QNu + KPNu = KPe_1 + Qe_1, \tag{2.14}
$$

which has the form of a compact perturbation of the Fredholm operator P of index zero. We can, therefore, apply the version given in [6, Theorem 1, Corollary 1] or [7, Theorem IV.4] or [5] of the Leray-Schauder continuation theorem, which ensures the existence of a solution for (2.14) if the set of all possible solutions of the family of equations

$$
Pu + (1-\lambda)Qu + \lambda QNu + \lambda KPNu = \lambda KPe_1 + \lambda Qe_1, \tag{2.15}
$$

$\lambda \in]0,1[$, is *a priori* bounded independently of λ. Now (2.15) is equivalent to the system of equations

$$
\begin{aligned}
Pu + \lambda KPNu &= \lambda KPe_1, \\
(1-\lambda)Qu + \lambda QNu &= \lambda Qe_1.
\end{aligned}
\tag{2.16}
$$

Let $u_\lambda \in X$ be a solution of (2.16) for some $\lambda \in]0,1[$, then $u_\lambda \in D(L)$ and

$$
\begin{aligned}
Pu_\lambda + \lambda KPNu_\lambda &= \lambda KPe_1, \\
(1-\lambda)Qu_\lambda + \lambda QNu_\lambda &= \lambda Qe_1.
\end{aligned}
\tag{2.17}
$$

It follows that

$$
Lu_\lambda + (1-\lambda)Qu_\lambda + Nu_\lambda = \lambda e_1,
$$

i.e.

$$
\begin{aligned}
u_\lambda'' + cu_\lambda' + (1-\lambda)\int_0^1 u_\lambda(x)\,\mathrm{d}x + g_1(u_\lambda) &= \lambda e_1, \\
u_\lambda(0) = u_\lambda(1), \quad u_\lambda'(0) &= u_\lambda'(1).
\end{aligned}
\tag{2.18}
$$

Multiplying the equation in (2.17) by u'_λ and integrating over $[0,1]$ we obtain that

$$c\int_0^1 u'^2_\lambda = \lambda \int_0^1 e_1(x)u'_\lambda(x)\,dx,$$

which implies, using the Cauchy-Schwarz inequality, that

$$|c|\|u'_\lambda\|_{L^2[0,1]} \leqslant \|e_1\|_{L^2[0,1]}. \tag{2.19}$$

Now, we claim that there exists a $\xi \in [0,1]$ such that $r \leqslant u_\lambda(\xi) \leqslant R$. Indeed, suppose that $u_\lambda(x) \geqslant R$ for every $x \in [0,1]$, then we get from the second equation in (2.17) and our assumptions on g_1 and e_1 that

$$(1-\lambda)R + \lambda \cdot \frac{1}{2}(A-a) \leqslant (1-\lambda)Qu_\lambda + \lambda QNu_\lambda = \lambda Qe_1 \leqslant \lambda \cdot \frac{1}{2}(A-a),$$

so that $(1-\lambda)R \leqslant 0$, which is a contradiction since $\lambda \in]0,1[$ and $R > 0$. Similarly, $u_\lambda \leqslant r$ for $x \in [0,1]$ leads to a contradiction. This proves the claim.

Next it follows that for every $x \in [0,1]$

$$\begin{aligned}|u_\lambda(x)| &\leqslant \max(-r, R) + \int_0^1 |u'_\lambda(x)|\,dx \\ &\leqslant \max(-r, R) + \|u'_\lambda\|_{L^2[0,1]} \\ &\leqslant \max(-r, R) + \frac{1}{|c|}\|e_1\|_{L^2[0,1]} \equiv C.\end{aligned}$$

Hence

$$\|u_\lambda\|_X \leqslant C,$$

where C is a constant independent of $\lambda \in]0,1[$.

This completes the proof of the theorem. □

Corollary 1. *Let $g: \mathbf{R} \to \mathbf{R}$ be a continuous function and let $c \in \mathbf{R}$, $c \neq 0$ be given. Suppose there exists an $R > 0$ such that $g(u)u \geqslant 0$ for $u \in \mathbf{R}$, $|u| \geqslant R$.*

Then for every $e(x) \in L^2[0,1]$ with $\int_0^1 e(x)\,dx = 0$, Duffing's equation (2.9) has at least one solution.

Proof. The proof follows immediately from Theorem 1 with $a = A = 0$ and $r = -R$. □

Theorem 2. *Let $g\colon \mathbf{R} \to \mathbf{R}$ be a strictly increasing function with $\lim_{u\to-\infty} g(u) = -\infty$, $\lim_{u\to\infty} g(u) = \infty$ and let $c \in \mathbf{R}$, $c \neq 0$. Suppose that g is a Lipschitz continuous function with a Lipschitz constant α, i.e.*

$$|g(u) - g(v)| \leqslant \alpha |u - v| \tag{2.20}$$

for $u, v \in \mathbf{R}$, with

$$\alpha < 4\pi^2 + c^2 \tag{2.21}$$

for all $u \in \mathbf{R}$.

Then for every $e \in L^2[0,1]$, the boundary value problem

$$\begin{aligned} u'' + cu' + g(u) &= e(x), \quad 0 < x < 1 \\ u(0) = u(1), &\quad u'(0) = u'(1), \end{aligned} \tag{2.22}$$

has exactly one solution u in $X = C[0,1]$.

P r o o f. Under our assumptions, it is easy to see that there exist a, A, r, R with $a \leqslant A$, $r < 0 < R$ such that

$$\begin{aligned} g(u) &\leqslant A \text{ for } u \geqslant R, \\ g(u) &\leqslant a \text{ for } u \leqslant r, \end{aligned}$$

and

$$a \leqslant \int_0^1 e(x)\,\mathrm{d}x \leqslant A.$$

Accordingly, Theorem 1 implies that (2.22) has at least one solution u in X.

Let, now, $u_1, u_2 \in X$ be two different solutions for (2.22). Then

$$u_1'' - u_2'' + c(u_1' - u_2') + g(u_1) - g(u_2) = 0, \quad 0 < x < 1. \tag{2.23}$$

It follows that

$$\begin{aligned} 0 &= -\int_0^1 (u_1' - u_2')^2\,\mathrm{d}x + \int_0^1 (g(u_1) - g(u_2))(u_1 - u_2)\,\mathrm{d}x \\ &= -\int_0^1 (u_1' - u_2')^2\,\mathrm{d}x + \int_0^1 |g(u_1) - g(u_2)|\,|u_1 - u_2|\,\mathrm{d}x \\ &\geqslant -\frac{1}{4\pi^2 + c^2}\|Lu_1 - Lu_2\|^2_{L^2[0,1]} + \frac{1}{\alpha}\int_0^1 |g(u_1) - g(u_2)|^2\,\mathrm{d}x \\ &= \Big(\frac{1}{\alpha} - \frac{1}{4\pi^2 + c^2}\Big)\int_0^1 |g(u_1) - g(u_2)|^2\,\mathrm{d}x, \end{aligned}$$

in view of (2.23). Using (2.21), we get that

$$g(u_1(x)) = g(u_2(x))$$

for a.e. $x \in [0,1]$, which implies $u_1(x) = u_2(x)$ for a.e. $x \in [0,1]$, since g is strictly increasing on $\mathbf{R}$. Hence $u_1(x) = u_2(x)$ for every $x \in [0,1]$ since u_1, u_2 are continuous in $[0,1]$.

This completes the proof of the theorem. □

Remark 1. Theorem 2 seems to imply that Duffing's equation (2.22) in the presence of the non-zero damping term cu' has a unique solution as long as the non-linearity $g(u)$ does not resonate against too many eigenvalues of the linear eigenvalue problem

$$\begin{gathered} -u'' = \lambda u, \quad 0 < x < 1, \\ u(0) = u(1), \quad u'(0) = u'(1). \end{gathered}$$

Also, it indicates that while the presence of even a small amount of damping gives existence, the presence of large enough damping ensures uniqueness.

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