Archivum Mathematicum

Václav Havel

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Archivum Mathematicum, Vol. 3 (1967), No. 2, 65--68

Persistent URL: http://dml.cz/dmlcz/104631

FREE EXTENSIONS OF COUPLED SYSTEMS

VÁCLAV HAVEL (BRNO)

Received May 21, 1965

A coupled system consists of ,,points" and ,,lines" such that each point (line) can be understood as a set of certain pairs of lines (points). An important particular case of a coupled system is of course an ,,incidence structure".[1]) In the present Note we deduce some results on free extensions of coupled systems parallelly to any known properties of free extensions of incidence structures.[2])

A *coupled system* is defined as a quadruple (S_1, f_1, S_2, f_2) where S_1, S_2 are nonempty sets and f_i is a mapping of a certain set $\mathrm{Dom}\, f_i \subset \subset \{X \subset S_i \mid \mathrm{card}\, X = 2\}$ into S_j; $(i, j) = (1, 2), (2, 1)$. If $\mathrm{Dom}\, f_i = \{X \subset S_i \mid \mathrm{card}\, X = 2\}$ for $i = 1, 2$, we get a *complete* coupled system If S_1, S_2 are finite sets we get a *finite* coupled system.

Let $\mathfrak{C} = (S_i, f_i; i = 1, 2)$, $\mathfrak{C}' = (S'_i, f'_i; i = 1, 2)$ be coupled systems such that $S_i \subset S'_i$, $\mathrm{Dom}\, f_i \subset \mathrm{Dom}\, f'_i$, $f_i\{a, b\} = c \Rightarrow f'_i\{a, b\} = c$ for $i = 1, 2$. Then we say that $\mathfrak{C}$ is a *coupled subsystem* of $\mathfrak{C}'$ and write $\mathfrak{C} \prec \mathfrak{C}'$.

A family $\mathfrak{S} = (\mathfrak{C}^\gamma)_{\gamma \in \Gamma}$ of coupled systems $\mathfrak{C}^\gamma = (S^\gamma_i, f^\gamma_i; i = 1, 2)$ is said to be *compatible* if $f^\alpha_i\{a, b\} = c$, $f^\beta_i\{a, b\} = d \Rightarrow c = d$ (for $\alpha, \beta \in \Gamma$ and $i = 1, 2$).

If $\mathfrak{S}$ is such a compatible family then there exists the coupled system $\bigcup_{\gamma\in\Gamma} \mathfrak{C}^\gamma = (\bigcup_{\gamma\in\Gamma} S^\gamma_i, \bigcup_{\gamma\in\Gamma} f^\gamma_i; i = 1, 2)$ such that, for $i = 1, 2$, $\mathrm{Dom}\, (\bigcup_{\gamma\in\Gamma} f^\gamma_i) = \bigcup_{\gamma\in\Gamma} \mathrm{Dom}\, f^\gamma_i$ and $(\bigcup_{\gamma\in\Gamma} f^\gamma_i)\{a, b\} = c \Leftrightarrow \exists \gamma \in \Gamma : f^\gamma_i\{a, b\} = c$. A compatible family $\mathfrak{S}$ is said to be *intersecting* if $\bigcap_{\gamma\in\Gamma} \mathrm{Dom}\, f^\gamma_i \neq \emptyset$ for $i = 1, 2$. If $\mathfrak{S}$ is an intersecting family then there is the coupled system $\bigcap_{\gamma\in\Gamma} \mathfrak{C}^\gamma = (\bigcap_{\gamma\in\Gamma} S^\gamma_i, \bigcap_{\gamma\in\Gamma} f^\gamma_i; i = 1, 2)$ such that, for $i = 1, 2$, $\mathrm{Dom}\, (\bigcap_{\gamma\in\Gamma} f^\gamma_i) = \bigcap_{\gamma\in\Gamma} \mathrm{Dom}\, f^\gamma_i$ and $\bigcap_{\gamma\in\Gamma} f^\gamma_i\{a, b\} = c \Leftrightarrow \forall \gamma \in \Gamma : f^\gamma_i\{a, b\} = c$.

Let $\mathfrak{C} = (S_i, f_i; i = 1, 2)$ be a coupled subsystem of a complete system $\overline{\mathfrak{C}} = (\overline{S}_i, \overline{f}_i; i = 1, 2)$. If $\mathfrak{S}$ is now the family of all complete coupled systems $\mathfrak{C}'$ satisfying $\mathfrak{C} \prec \mathfrak{C}' \prec \overline{\mathfrak{C}}$ then $\mathfrak{S}$ is intersecting so that there exists the coupled system $\bigcap_{\mathfrak{C}'\in\mathfrak{S}} \mathfrak{C}'$. It will be called *generated* by $\mathfrak{C}$

[1]) See G. Pickert: Projektive Ebenen, Berlin—Göttingen—Heidelberg 1955; p. 2.
[2]) Ibid., pp. 12—26.

with respect to $\overline{\mathfrak{C}}$. If, in particular $\bigcap_{\mathfrak{C}' \in \mathfrak{S}} \mathfrak{C}' = \overline{\mathfrak{C}}$, then $\overline{\mathfrak{C}}$ is said to be *generated* by $\mathfrak{C}$.

Let $\mathfrak{C} = (S_i, f_i; i = 1, 2)$, $\mathfrak{C}' = (S'_i, f'_i; i = 1, 2)$ be coupled systems. Any *mapping of* $\mathfrak{C}$ onto $\mathfrak{C}'$ is defined as a pair $\sigma = (\sigma_1, \sigma_2)$ where σ_i is a mapping of S_i onto S'_i for $i = 1, 2$. Such a mapping σ is called an *epimorphism* between $\mathfrak{C}$ and $\mathfrak{C}'$ if for every $\{x, y\} \in \operatorname{Dom} f_i$ with $\sigma_i x \neq \sigma_i y$ it follows $\{\sigma_i x, \sigma_i y\} \in \operatorname{Dom} f'_i$ and $\sigma_j f_i\{x, y\} = f'_i\{\sigma_i x, \sigma_i y\}$ for $(i, j) = (1, 2), (2, 1)$ and if $\{\{\sigma_i x, \sigma_i y\} \mid \{x, y\} \in \operatorname{Dom} f_i, \sigma_i x \neq \sigma_i y\} = \operatorname{Dom} f'_i$ for $i = 1, 2$.

If, moreover, there is a coupled subsystem $\mathfrak{C}'' \langle \mathfrak{C}, \mathfrak{C}'$ such that the restriction of σ with respect to $\mathfrak{C}''$ is the identity mapping upon $\mathfrak{C}''$ then we say that σ is an epimorphism *over* $\mathfrak{C}''$. By an *isomorphism* we shall mean a bijective epimorphism.

Let $\mathfrak{C} = (S_i, f_i; i = 1, 2)$ be a coupled subsystem of a complete coupled system $\mathfrak{C}' = (S'_i, f'_i; i = 1, 2)$. Construct a sequence $(\overline{\mathfrak{C}}^n)_{n=0}^{\infty}$ of coupled subsystems $\overline{\mathfrak{C}}^n = (\overline{S}_i^n, \overline{f}_i^n; i = 1, 2)$ in $\mathfrak{C}'$ (this sequence will be denoted as the *extension chain over* $\mathfrak{C}$ *in* $\mathfrak{C}'$) as follows: Set $\overline{\mathfrak{C}}_0 = \mathfrak{C}$. If $\overline{\mathfrak{C}}^n$ is already formed, determine $\overline{\mathfrak{C}}^{n+1}$ in such a way that $\operatorname{Dom} \overline{f}_j^{n+1} = \{X \subset \overline{S}_j^n \mid \operatorname{card} X = 2\}$ and $\overline{S}_i^{n+1} = \overline{S}_i^n \cup \overline{T}_i^n$ where $\overline{T}_i^n = \{f'_j\{a, b\} \mid \{a, b\} \in \operatorname{Dom} \overline{f}_j^{n+1} \setminus \operatorname{Dom} \overline{f}_j^n\}$; $(i, j) = (1, 2), (2, 1)$. Clearly $(\overline{\mathfrak{C}}^n)_{n=0}^{\infty}$ is compatible and $\bigcup_{n=1}^{\infty} \overline{\mathfrak{C}}^n$ is equal to the coupled system generated by $\mathfrak{C}$ with respect to $\mathfrak{C}'$.

Let $\mathfrak{C} = (S_i, f_i; i = 1, 2)$ be a coupled system. Now we determine a sequence $(\mathfrak{C}^n)_{n=0}^{\infty}$ of coupled systems $\mathfrak{C}^n = (S_i^n, f_i^n; i = 1, 2)$ (this sequence will be called the *free extension chain over* $\mathfrak{C}$) as follows: Set $\mathfrak{C}^0 = \mathfrak{C}$. If $\mathfrak{C}^n$ is already determined, form $\mathfrak{C}^{n+1}$ in such a way that, for $(i, j) = (1, 2), (2,1)$, $\operatorname{Dom} f_i^{n+1} = \{X \subset S_i^n \mid \operatorname{card} X = 2\}$ and $S_i^{n+1} = S_i^n \cup T_i^n$ where T_i^n is a set disjoint to S_i^n and corresponding to $\operatorname{Dom} f_j^{n+1} \setminus \operatorname{Dom} f_j^n$ in some bijection g_j^{n+1} so that $f_j^{n+1}|_{\operatorname{Dom} f^n_j} = f_j^n$ and $f_j^{n+1}|_{\operatorname{Dom} f_j^{n+1} \setminus \operatorname{Dom} f^n_j} = g_j^{n+1}$. Then $(\mathfrak{C}^n)_{n=0}^{\infty}$ is compatible and $\mathbf{T}(\mathfrak{C}) = \bigcup_{n=1}^{\infty} \mathfrak{C}^n$ will be called the *complete free extension* of $\mathfrak{C}$. Thus $\mathbf{F}(\mathfrak{C})$ is determined uniquely up to isomorphisms. If convenient, we shall use also the symbol $\mathbf{F}(\mathfrak{C})$ up to preceding isomorphisms.

Proposition 1. *Let* $\mathfrak{C} = (S_i, f_i, i = 1,2)$ *be a coupled* $\overline{\mathfrak{C}} = (\overline{S}_i, \overline{f}_i;$ $i = 1,2)$ *be some coupled system generated by* $\mathfrak{C}$. *Then there is an isomorphism over* $\mathfrak{C}$ *of* $\overline{\mathfrak{C}}$ *onto* $\mathbf{F}(\mathfrak{C})$ *iff there is, for each coupled system* $\mathfrak{C}'$ *generated by* $\mathfrak{C}$, *and epimorphism over* $\mathfrak{C}$ *between* $\overline{\mathfrak{C}}$ *and* $\mathfrak{C}'$.

Proof. Necessity: It is to show that there is an epimorphism over $\mathfrak{C}$

between $\mathbf{F}(\mathfrak{C})$ and $\mathfrak{C}'$ if $\mathfrak{C}'$ is an arbitrary coupled system generated by $\mathfrak{C}$. We shall use the corresponding extension chains $(\mathfrak{C}^n)_{n=0}^{\infty}$, $(\mathfrak{C}'^n)_{n=0}^{\infty}$, and form for each $n = 0, 1, 2, \ldots$ a mapping $\varphi^n = (\varphi_1^n, \varphi_2^n)$ of $\mathfrak{C}^n$ upon $\mathfrak{C}'^n$.

The prescription is as follows: First, let φ° be the identity mapping upon $\mathfrak{C}$. Secondly, let φ^n be already formed. We require that φ^{n+1} prolongs φ^n in such a way that $\varphi_i^{n+1} f_j^{n+1}\{x, y\}$ is equal to $f_j'^{n+1}\{\varphi_j^n x, \varphi_j^n y\}$ if $\varphi_j^n x \neq \varphi_j^n y$ and to an arbitrary element of $S_i'^{n+1}$ if $\varphi_j^n x = \varphi_j^n y$. By induction it follows that each $\varphi^n (n = 0,1, \ldots)$ is an epimorphism over $\mathfrak{C}$ between $\mathfrak{C}^n$ and $\mathfrak{C}'^n$. Now there is exactly one epimorphism φ over $\mathfrak{C}$ between $\mathbf{F}(\mathfrak{C})$ and $\mathfrak{C}'$ which prolongs all φ^n. Sufficiency: For given $\mathfrak{C}$, $\overline{\mathfrak{C}}$ suppose that to every coupled system $\mathfrak{C}'$ generated by $\mathfrak{C}$ there is an epimorphism over $\mathfrak{C}$ between $\overline{\mathfrak{C}}$ and $\mathfrak{C}'$. In particular there must exist an epimorphism ψ over $\mathfrak{C}$ between $\overline{\mathfrak{C}}$ and $\mathbf{F}(\mathfrak{C})$. Further we use the epimorphism φ between $\mathbf{F}(\mathfrak{C})$ and $\overline{\mathfrak{C}}$ constructed as above. We shall prove that for $\varphi^n = \varphi|_{\mathfrak{C}^n}$, $\psi^n = \psi|_{\overline{\mathfrak{C}}^n}$, it holds $\varphi^n \circ \psi^n = id_{\mathfrak{C}^n}$, $\psi^n \circ \varphi^n = id_{\overline{\mathfrak{C}}^n}$ $(n = 0, 1, \ldots)$. In fact, for $n = 0$, the assertion holds. Let it hold for some n. Then for any $z \in \overline{T}_j^n$ there is a pair $\{x, y\} \in \text{Dom}\, \overline{f}_i^{n+1}$ such that $\overline{f}_i\{x, y\} = z$. As $z \neq y$, $\psi_i x \neq \psi_i y$, it must be $\psi_j z = (\bigcup_{n=0}^{\infty} f_i^n)\{\psi_i x, \psi_i y\}$ and $\varphi_j(\psi_j z) = \overline{f}_i\{\varphi_i(\psi_i x), \varphi_i(\psi_i y)\} = \overline{f}_i\{x, y\}$; $(i, j) = (1,2), (2,1)$. Thus $\varphi^{n+1} \circ \psi^{n+1} = id_{\overline{\mathfrak{C}}^{n+1}}$. (Similarly for the remaining relation $\psi^n \circ \varphi^n = id_{\overline{\mathfrak{C}}^n}$). Consequently ψ is an isomorphism over $\mathfrak{C}$ between $\overline{\mathfrak{C}}$ and $\mathbf{F}(\mathfrak{C})$. Q.E.D.

Let $\mathfrak{C} = (S_i, f_i; i = 1, 2)$ and $\mathfrak{C}' = (S_i', f_i'; i = 1,2)$ be coupled systems such that $\mathfrak{C} \langle \mathfrak{C}' \langle \mathbf{F}(\mathfrak{C})$. $\mathfrak{C}'$ is said to be a *free extension* of $\mathfrak{C}$ (and this relation will be denoted by $\mathfrak{C} \lhd \mathfrak{C}'$ if, for $(i, j) = (1, 2), (2, 1)$, $z \in S_j'$, $z \in T_j^n$, $z = f_i^{n+1}\{x, y\}$ implies $x, y \in S_i'$.

Proposition 2. Let $\mathfrak{C} = (S_i, f_i; i = 1, 2)$, $\mathfrak{C}' = (S_i', f_i'; i = 1, 2)$, $\mathfrak{C}'' = (S_i'', f_i''; i = 1, 2)$ *be coupled systems such that* $\mathfrak{C} \lhd \mathfrak{C}' \langle \mathfrak{C}''$. *Then* $\mathfrak{C} \lhd \mathfrak{C}''$ *iff* $\mathfrak{C}' \lhd \mathfrak{C}''$.

Proof. Let $(\mathfrak{C}^n)_{n=0}^{\infty}$ and $(\mathfrak{C}'^n)_{n=0}^{\infty}$ be the free extension chains of $\mathbf{F}(\mathfrak{C})$ and $\mathbf{F}(\mathfrak{C}')$ respectively. Clearly $\mathfrak{C}'^0 \langle \mathbf{F}(\mathfrak{C})$. Let $\mathfrak{C}'^n \langle \mathbf{F}(\mathfrak{C})$ be fulfilled. Then, for $(i, j) = (1, 2), (2, 1)$, each $z \in S_j'^{n+1}$ determines the minimal index ν such that $z = f_i^{n+1}\{x, y \Rightarrow x, y\} \in S_i^\nu$ and consequently $f_i^{\nu+1}$ $\{x, y\} \in S_j^{\nu+1}$. Thus, by induction, $\mathfrak{C}'^n \langle \mathbf{F}(\mathfrak{C})$ for all $n = 0, 1, \ldots$ and $\mathbf{F}(\mathfrak{C}') = \mathbf{F}(\mathfrak{C})$.

Now let $\mathfrak{C}' \lhd \mathfrak{C}''$. Let $(i, j) = (1, 2), (2, 1)$. If $z \in S_j''$ and $z \in T_j'^m$ for some m then there is an index k such that $z \in T_j^k$. Thus $f_i^{k+1}\{x, y\} = z \Rightarrow$ $\Rightarrow x, y \in S_i''$ and we have $\mathfrak{C} \lhd \mathfrak{C}''$.

Let $\mathfrak{C} \lhd \mathfrak{C}''$. Let $(i, j) = (1, 2), (2, 1)$. If $z \in S_j''$ and $z \in T_j^m$ for some m then either $z \in S_j'$ or there is h such that $z \in T_j'^h$. By the assumptions about $\mathfrak{C}'$ and $\mathfrak{C}''$ it holds $z = f_i'^{h+1}\{x, y\} \Rightarrow x, y \in S_i''$ so that $\mathfrak{C}' \lhd \mathfrak{C}''$. Q.E.D.

A coupled system $\mathfrak{C} = (S_i, f_i; i = 1, 2)$ is said to be *closed* if, for $(i, j) = (1, 2), (2, 1)$, to every $z \in S_j$ there exist distinct pairs $\{x_1, y_1\}$, $\{x_2, y_2\} \in \mathrm{Dom}\, f_i$ such that $f_i\{x_1, y_1\} = f_i\{x_2, y_2\} = z$.

Proposition 3. *Let* $\mathfrak{C} = (S_i, f_i; i = 1, 2)$ *be a coupled system and* $\mathfrak{C}' = (S_i', f_i'; i = 1, 2)$ *a finite closed coupled system. If* $\mathfrak{C}' \lhd \mathbf{F}(\mathfrak{C})$ *then* $\mathfrak{C}' \lhd \mathfrak{C}$.

Proof. Let $(\mathfrak{C}^n)_{n=0}^{\infty}$ be the free extension chain of $\mathfrak{C}$. Since $\mathfrak{C}'$ is supposed to be finite there is $z \in S_1' \cup S_2'$ with maximal index ν such that $z \in S_1^\nu \cup S_2^\nu$. If $\nu > 0$ then $z = (\bigcup_{n=0}^{\infty} f_j^n)\{x, y\}$ for precisely one $\{x, y\} \subset S_i^{\nu-1}$; here, (i, j) is equal to $(1, 2)$ or to $(2, 1)$ according to the nature of z. Because of the maximality of ν it must be $\{x, y\} \in \mathrm{Dom}\, f_i'$ which contradicts to the assumption that $\mathfrak{C}'$ is closed. Thus $\nu < 0$ and consequently $\mathfrak{C}' \lhd \mathfrak{C}$. Q.E.D.

Proposition 4. *Let* $\mathfrak{C} = (S_i, f_i; i = 1, 2)$, $\mathfrak{C}' = (S_i', f_i'; i = 1, 2)$ *be finite coupled systems. If* $\mathbf{F}(\mathfrak{C})$, $\mathbf{F}(\mathfrak{C}')$ *are isomorphic then* $\mathfrak{C}'$ *has the common free extension with an isomorphic image of* $\mathfrak{C}$.

Proof. Let there exist an isomorphism $\sigma = (\sigma_1, \sigma_2)$ of $\mathbf{F}(\mathfrak{C})$ onto $\mathbf{F}(\mathfrak{C}')$. As $\mathfrak{C}$ and $\mathfrak{C}'$ are finite, there is a coupled system $\mathfrak{C}^* = (S_i^*, f_i^*; i = 1, 2 \langle \mathbf{F}(\mathfrak{C}')$ such that $\mathfrak{C}' \langle \mathfrak{C}^*$, $\sigma\mathfrak{C} \langle \mathfrak{C}^*$ and that for $(i, j) = (1, 2)$, $(2, 1)$, if $z \in T_j'^m$ or $z \in \sigma_j T_j^m$ respectively then $z = f_j'^{m+1}\{x, y\}$ implies $x, y \in S_j^*$. Thus $\mathfrak{C}' \lhd \mathfrak{C}^*$ and $\mathfrak{C} \lhd \sigma^{-1}\mathfrak{C}^*$. Q.E.D.

Let $\mathfrak{C} = (S_i, f_i; i = 1, 2)$ be a coupled system with $\mathrm{Dom}\, f_1 = \mathrm{Dom}\, f_2 = \emptyset$. Then $\mathbf{F}(\mathfrak{C})$ will be called a *free coupled system*.

Proposition 5. *To every complete coupled system* $\mathfrak{C} = (S_i, f_i; i = 1, 2)$ *there is an epimorphism of a free coupled system onto* $\mathfrak{C}$.

Proof. Let $\mathfrak{C}' = (S_i', f_i'; i = 1, 2)$ with $S_1' = S_1$, $S_2' = S_2$ and $\mathrm{Dom}\, f_1' = \mathrm{Dom}\, f_2' = \emptyset$. Let $(\mathfrak{C}'^n)_{n=0}^{\infty}$ be the free extension chain of $\mathfrak{C}'$. Construct a mapping $\sigma = (\sigma_1, \sigma_2)$ of $\mathbf{F}(\mathfrak{C}')$ onto $\mathfrak{C}$ as follows: For all $x \in S_i$, set $\sigma_i^0 x = x$; $i = 1, 2$. Let a mapping $\sigma^n = (\sigma_1^n, \sigma_2^n)$ of $\mathfrak{C}'^n$ onto some coupled subsystem of $\mathfrak{C}$ be already determined. For $(i, j) = (1, 2), (2, 1)$, if $z \in T_j'^n$, $z = f_i'^{n+1}\{x, y\}$ and $\sigma_i^n x \neq \sigma_i^n y$ or $\sigma_i^n x = \sigma_i^n y$ respectively, then set $\sigma_j^{n+1} z = f_i\{\sigma x, \sigma y\}$ or take for $\sigma_j^{n+1} z$ an arbitrary element of $S_j'^{n+1}$. The mapping σ which prolongs simultaneously all σ^n; $n = 0, 1, \ldots$, presents the required epimorphism of $\mathbf{F}(\mathfrak{C}')$ onto $\mathfrak{C}$. Q.E.D.