František Neuman

# Extremal property of the equation $y'' = -k^2 y$

*Archivum Mathematicum*, Vol. 3 (1967), No. 4, 161--164

Persistent URL: http://dml.cz/dmlcz/104641

# EXTREMAL PROPERTY OF THE EQUATION $y'' = -k^2y$

F. Neuman, Brno

Received April 2, 1967

---

**1.** We shall study the equations

$$(q) \qquad y'' = q(t)\, y$$

where $q(t) \in C^\circ(-\infty, \infty)$. At the same time, let $C_i^n$ denote the set of all functions on an interval $i$ having continuous derivatives up to the order $n$, inclusive. Each function $y(t) \in C^2(-\infty, \infty)$ satisfying the equation $(q)$ is a solution of this equation. A function, not defined at a point but which can be extended to be continuous at this point, is being understood as a function extended in this way.

Denote by $Q_\pi$ the set of all functions $q \in C^\circ(-\infty, \infty)$ for which each non-trivial solution of the equation $(q)$ has roots distributed in equidistances $\pi$.

In this paper, the integral $\int_0^\pi q(t)\,dt$ for $q \in Q_\pi$ is considered and it is proved:

**Theorem 1:** *It holds* $\min \int_0^\pi q(t)\,dt = -\pi$ *for* $q \in Q_\pi$, *the minimum being reached only for* $q \equiv -1$.

This result may be easy extended to the class $Q_d$ of all the functions $q \in C^\circ(-\infty, \infty)$ such that the corresponding differential equations $(q)$ have every non-trivial solution with roots in equidistances equal to $d$ ($d > 0$, const.). Then we obtain

**Theorem 2:** $\min \int_0^d q(t)\,dt = -\pi^2/d$ *for* $q \in Q_d$, *the minimum being reached only for* $q \equiv -\pi^2/d^2$.

At the same time, we are going to show that the mentioned integrals are unbounded from above on the classes $Q_\pi$ and $Q_d$.

In this paper there is proved further

**Theorem 3:** *For* $q \leqq 0$, $q \in Q_d$, $0 < \alpha \leqq 1$, *it holds* $\int_0^d |q(t)|^\alpha\,dt \leqq d(\pi/d)^{2\alpha}$; *for any* $\alpha$, *this estimate cannot be improved for any of classes* $Q_d$, *because the equality sets in only for* $q \equiv -\pi^2/d^2 \in Q_d$ *for every* $\alpha$.

**2.** Let $q(t) \in Q_\pi$. In [1] (or similarly see [2] for a general case) there is proved that all the functions $q(t)$ are given by the relation

$$q(t) = f''(t) + f'^2(t) + 2f'(t) \operatorname{cotg} t - 1, \tag{1}$$

where $f(t) \in C^2(-\infty, \infty)$, $f(t + \pi) = f(t)$, $f(0) = f'(0) = 0$, $\int\limits_0^\pi \frac{\exp[-2f(t)] - 1}{\sin^2 t} \, dt = 0$; the solution $y(t)$ of the equation $(q)$ determined by the conditions $y(0) = 0$, $y'(0) = 1$ can be then expressed as

$$y(t) = e^{f(t)} \sin t. \tag{2}$$

**3.** Let $q \in Q_\pi$. Then $\int\limits_0^\pi q(t) \, dt = \int\limits_0^\pi [f''(t) + f'^2(t) + 2f'(t) \operatorname{cotg} t - 1] \, dt =$ $= -\pi + \int\limits_0^\pi [f'^2(t) + 2f'(t) \operatorname{cotg} t] \, dt \geqq -\pi + \int\limits_0^\pi 2f'(t) \operatorname{cotg} t \, dt$. Since $\lim\limits_{t \to 0, \pi} f(t) \operatorname{cotg} t = \lim\limits_{t \to 0, \pi} f'(t)/\cos t = 0$, it holds $2 \int\limits_0^\pi f'(t) \operatorname{cotg} t \, dt =$ $= 2 \int\limits_0^\pi f(t)/\sin^2 t \, dt$.

Further on, for any $t$, there holds the inequality $e^{-2f(t)} - 1 \geqq -2f(t)$, where the equality is reached just for these $t$ for which $f(t) = 0$. Then it is

$$\int\limits_0^\pi \frac{\exp[-2f(t)] - 1}{\sin^2 t} \, dt \geqq -2 \int\limits_0^\pi f(t)/\sin^2 t \, dt$$

and the equality is reached just for $f(t) \equiv 0$. The left-hand side of the last inequality equals zero, and therefore $\int\limits_0^\pi q(t) \, dt \geqq -\pi +$ $+ 2 \int\limits_0^\pi f'(t) \operatorname{cotg} t \, dt = -\pi + 2 \int\limits_0^\pi f(t)/\sin^2 t \, dt \geqq -\pi$, the sign of equality in the last inequality holds just for $f(t) \equiv 0$. Because the first inequality of the last relation is becoming an equality again for $f(t) \equiv 0$ [see the relation (1)], there is $\min\limits_{q \in Q_\pi} \int\limits_0^\pi q(t) \, dt = -\pi$ and this minimum sets in only for $q(t) \equiv -1$. Thus, theorem 1 is being proved.

Let us introduce the immediate

**Corrollary 1:** *For $q \in Q_\pi$ and $q \neq -1$, it holds $\int\limits_0^\pi q(t) \, dt > -\pi$.*

4. There exists a $1 - 1$ correspondence between the elements of the set $Q_\pi$ and the elements of $Q_d$:

"The function $\frac{\pi^2}{d^2} q\left(\frac{\pi}{d} t\right) \in Q_d$ corresponds to the function $q(t) \in Q_\pi$."

And then

$$\min_{\bar{q} \in Q_d} \int_0^d \bar{q}(t)\, dt = \min_{q \in Q_\pi} \frac{\pi^2}{d^2} \int_0^d q\left(\frac{\pi}{d} t\right) dt = \frac{\pi}{d} \min_{q \in Q_\pi} \int^\pi q(t)\, dt = -\pi^2/d$$

and this minimum is reached only for $\bar{q}(t) \equiv \frac{\pi^2}{d^2}(-1) = -\pi^2/d^2$. Thus, theorem 2 is being proved.

5. **Note:** Let us show that $\int_0^\pi q(t)\, dt$ *is not bounded from above on the set* $Q_\pi$. It is enough to take account of $\int_0^\pi (f'^2 + 2f' \operatorname{cotg} t - 1)\, dt = \int_0^\pi (f'^2 - 1)\, dt$, as for the function $f(t)$, in addition being symmetrical with regard to the straight line $t = \pi/2$. Let $M > 0$ be an arbitrary constant. On some interval $[a, b]$, $0 < a < b < \pi/4$, let $f(t) \in C^2[a, b]$ be chosen so that $|f'(t)| > \sqrt{M + \pi}/\sqrt{2(b - a)}$. Further on, let $f(t)$ be defined on an interval $(-\infty, \infty)$ so that $f \in C^2(-\infty, \infty)$, $f(0) = f'(0) = 0$, $f(t)$ be symmetric regarding to the straight line $t = \pi/2$ and periodic with period $\pi$, and especially such that $\int_0^\pi \{\exp[-2f(t)] - 1\}/\sin^2 t\, dt = 0$. It can be satisfied, e.g., in that way that the definition of $f(t)$ is extended on the interval $[0, \pi/4]$ so as $f \in C^2[0, \pi/4]$, $f(0) = f'(0) = 0$. Furthermore, on the interval $[\pi/4, \pi/2]$, let $f$ be chosen so that $f \in C^2[0, \pi/2]$, $f'(\pi/2) = 0$, and $\int_{\pi/4}^{\pi/2} \{\exp[-2f(t)] - 1\}/\sin^2 t\, dt = -\int_0^{\pi/4} \{\exp[-2f(t)] - 1\}/\sin^2 t\, dt$. Then, with regard to the symmetry to $t = \pi/2$, and the periodicity, the function $f(t)$ is determined having the required properties. Besides it holds

$$\int_0^\pi q(t)\, dt = 2 \int_0^{\pi/2} (f'^2 - 1)\, dt > 2 \int_a^b f'^2\, dt - \pi > M.$$

Analogically to this procedure, or that in item 4, it is possible to show $\int_0^d q(t)\, dt$ to be unbounded from above on $Q_d$, as well.

**6.** Now, let $q(t) \leqq 0$, $q \in Q_d$. Since $a^\alpha \leqq a\alpha + 1 - \alpha$ for each $a \geqq 0$, $0 < \alpha \leqq 1$, we can estimate (using theorem 2):

$$\int_0^d \left[ -q(t) \frac{d^2}{\pi^2} \right]^\alpha \mathrm{d}t \leqq -\frac{d^2}{\pi^2} \alpha \int_0^d q(t)\, \mathrm{d}t + d(1 - \alpha) \leqq d$$

where the last inequality bein changed into the equality just for $q \equiv -\pi^2/d^2$. Thus, $\int_0^d [-q(t)]^\alpha \,\mathrm{d}t = \int_0^d | q(t) |^\alpha \,\mathrm{d}t \leqq d(\pi/d)^{2\alpha}$, and the equality may set in and really sets in only for $q \equiv -\pi^2/d^2$. Then, theorem 3 is being proved.

As immediate consequences be mentioned, e.g.:

**Corollary 2:** *For* $q \in Q_\pi$, $q \leqq 0$, $p \geqq 1$, *there is* $\int_0^\pi \sqrt[p]{-q(t)}\, \mathrm{d}t \leqq \pi$.

And especially

**Corollary 3:** *For* $q \in Q_\pi$, $q \leqq 0$, $q \not\equiv -1$, *there is* $\int_0^\pi \sqrt{-q(t)}\, \mathrm{d}t < \pi$.

Concluding this paper, I beg to thank Prof. M. Greguš for his valuable remarks.

## REFERENCES

[1] Neuman F., Sur les équations différentielles linéaires oscillatoires du deuxième ordre avec la dispersion fondamentale $\varphi(t) = t + \pi$, Bull. Inst. Polyt. (Iași), T. X (XIV), 1964, 37—42.

[2] Neuman F., Criterion of periodicity of solutions of a certain differential equation with a periodic coefficient, Annali di Mat. p. ed. appl., S. IV. N., T. LXXV, (1967), 385—396.