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ON THE EXISTENCE OF SOLUTIONS OF CERTAIN NONLINEAR EQUATIONS OCCURRING IN THE TRANSFORMATION THEORY OF A LINEAR SECOND ORDER DIFFERENTIAL EQUATION WITH COMPLEX VALUED COEFFICIENTS

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In the paper [1], J. Barret generalized the modified Prüfer transformation to the complex differential equation

$$[p(x)\,y']' + q(x)\,y = 0, \tag{1}$$

where $p(x) = p_1(x) + ip_2(x) \neq 0$, $q(x) = q_1(x) + iq_2(x)$ and each of the functions p_1, p_2, q_1, q_2 is a continuous real function on the half-line $I = \langle a, \infty)$. The role of trigonometric functions was played here by the functions $s = s(x)$ and $c = c(x)$ which form a solution of a system of differential equations

$$\begin{aligned} s' &= h(x)\,\bar{c}, \\ c' &= -h(x)\,\bar{s}, \quad s(a) = 0, \quad c(a) = 1, \end{aligned} \tag{2}$$

where $h(x) \in C^\circ(I)$ is a suitable complex function and the bar denotes a complex conjugate function. Note that for $h(x)$ real, the solution of (2) is $s = \sin \int_a^x h(t)\,\mathrm{d}t$, $c = \cos \int_a^x h(t)\,\mathrm{d}t$. This suggests an investigation of the corresponding complex solution $(s(x), c(x))$ of (2) when h is complex; these functions satisfy identities and inequalities analogous to those of the real sines and cosines, for example, the sum of the squares of the magnitudes is identically one. This boundedness property is useful in the applications.

Now, suppose $y(x)$ to be a nontrivial solution of (1) such that $y(a) = 0$ and there exist complex functions $\varrho(x) \neq 0$, $w(x) \neq 0$ of class C^1 and $h(x)$ continuous, such that

$$\begin{aligned} y(x) &= \varrho(x)\,s(x), \\ p(x)\,y'(x) &= \bar{w}(x)\,\bar{\varrho}(x)\,\bar{c}(x). \end{aligned} \tag{3}$$

Then the functions ϱ, w and h satisfy

$$(4) \qquad \varrho' = \bar{s}\bar{c}\left(\frac{\bar{w}}{p} - \frac{\bar{q}}{w}\right) - \frac{w'}{w} c\bar{c}\varrho, \qquad \varrho(a) = \frac{\bar{p}(a)\,\bar{y}'(a)}{w(a)} \neq 0,$$

$$(5) \qquad h = \frac{\bar{\varrho}}{\varrho}\left(\frac{\bar{w}}{p} c\bar{c} + \frac{\bar{q}}{w} s\bar{s}\right) + \frac{w'}{w} sc.$$

In the paper [1], the existence theorem for this system is proved for each non-zero function w of class C^1. The method is that of successive approximation and in several lemmas establishes a Lipschitz condition for the system. The main result guarantees that every solution $y(x)$ of (1), $y(a) = 0$, $y'(a) \neq 0$ can be considered in the form (3) where ϱ and h are suitable solutions of (4), (5).

In a similar manner it can be shown that every solution $y(x)$ of (1), $y(a) \neq 0$, $y'(a) = 0$ is

$$(6) \qquad \begin{aligned} y(x) &= \sigma(x)\, c(x), \\ p(x)\, y'(x) &= \bar{\sigma}(x)\, \bar{v}(x)\, \bar{s}(x), \end{aligned}$$

$\sigma(x) \neq 0$, $v(x) \neq 0$; σ, $v \in C^1(I)$ if and only if the functions σ and h satisfy the equations

$$(7) \qquad \sigma' = \bar{\sigma}\bar{s}\bar{c}\left(\frac{\bar{v}}{p} - \frac{\bar{q}}{v}\right) - \frac{v'}{v}\sigma s\bar{s}, \qquad \sigma(a) = y(a) \neq 0,$$

$$(8) \qquad h = -\frac{\bar{\sigma}}{\sigma}\left(\frac{\bar{q}}{v} c\bar{c} + \frac{\bar{v}}{p} s\bar{s}\right) - \frac{v'}{v} sc,$$

for all $x \geqq a$.

Furthermore, by analogy with the real case it can be easily verified that the fundamental system of solutions of (1) is

$$(9) \qquad y_1 = r(x)\, s(x), \qquad y_2 = r(x)\, c(x),$$

$r(x) \neq 0$, $r \in C^1(I)$, $pr' \in C^1(I)$ **if and only if the functions r and h satisfy the equations**

$$(10) \qquad (pr')'r + q(x)\, r^2 = \frac{1}{\bar{p}\bar{r}^2},$$

$$(11) \qquad h = \frac{1}{pr^2}$$

for all $x \geqslant a$.

The following theorems guarantee the existence of solutions of the systems (4), (5); (7), (8); (10), (11). The proofs are simple and will be omitted here.

Theorem 1. *Let $w(x) \neq 0$ be an arbitrary complex-valued function of class $C^1(I)$ and let ϱ_0 be any non-zero constant. Let $y = y(x)$ be a solution of (1) defined on I and satisfying initial conditions*

$$y(a) = 0, \qquad y'(a) = \frac{\bar{w}(a)\,\bar{\varrho}_0}{p(a)}.$$

Define

$$\varrho(x) = \frac{\varrho_0 w(a)}{w(x)} \exp\left\{\int_a^x \frac{(wy)'\bar{w}\bar{y} - \bar{q}\bar{y}py'}{w\bar{w}y\bar{y} + p\bar{p}y'\bar{y}'}\right\},$$

$$s(x) = \frac{y(x)}{\varrho(x)},$$

$$c(x) = \frac{\bar{p}(x)\,\bar{y}'(x)}{w(x)\,\varrho(x)}.$$

Then the function $\varrho(x)$ is of class C^1, satisfies (4), $\varrho(a) = \varrho_0$ and (s, c) is a solution of the system (2), where $h(x)$ is defined by (5).

Note that

$$w\bar{w}\varrho\bar{\varrho} = w\bar{w}y\bar{y} + p\bar{p}y'\bar{y}',$$

so that

$$|\varrho|^2 = |y|^2 + \left|\frac{py'}{w}\right|^2.$$

Theorem 2. *Let $v(x) \neq 0$ be an arbitrary complex-valued function of class $C^1(I)$ and let σ_0 be any non-zero constant. Let $y = y(x)$ be a solution of (1) defined on I and satisfying initial conditions*

$$y(a) = \sigma_0, \qquad y'(a) = 0.$$

Define

$$\sigma(x) = \frac{\sigma_0 v(a)}{v(x)} \exp\left\{\int_a^x \frac{(vy)'\,\bar{v}\bar{y} - \bar{q}\bar{y}py'}{v\bar{v}y\bar{y} + p\bar{p}y'\bar{y}'}\right\},$$

$$s(x) = \frac{\bar{p}(x)\,\bar{y}'(x)}{v(x)\,\sigma(x)},$$

$$c(x) = \frac{y(x)}{\sigma(x)}.$$

Then the function $\sigma(x)$ is of class C^1, satisfies (7), $\sigma(a) = \sigma_0$ and (s, c) is a solution of the system (2) where $h(x)$ is defined by (8).

Note. The real generalized Prüfer transformation is investigated in [2], [4].

Theorem 3. *Let $r_0 \neq 0$, r_0', $\varkappa \neq 0$ be arbitrary constants. Let u, v be the solutions of* (1) *satisfying initial conditions*

$$u(a) = 0, \qquad u'(a) = \frac{\varkappa}{r_0 p(a)},$$

$$v(a) = r_0, \qquad v'(a) = r_0'.$$

Then the function

$$r(x) = r_0 \exp\left\{\int_a^x \frac{u'\bar{u} + v'\bar{v}}{u\bar{u} + v\bar{v}}\right\} \tag{12}$$

is a solution of the differential equation (10) *on I satisfying initial conditions*

$$r(a) = r_0, \qquad r'(a) = r_0'.$$

The pair of functions $s = \dfrac{u(x)}{r(x)}$, $c(x) = \dfrac{v(x)}{r(x)}$ *represents a solution of* (2) *where* $h(x) = \dfrac{\varkappa}{pr^2}$.

Note. If the functions p, q are real, the existence of the solution of the equation (10) defined for all $x \geqslant a$ is proved in [3], [5]. In this case, the formula (12) simplifies to

$$r(x) = r_0 \sqrt{u^2(x) + v^2(x)}.$$

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