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ON THE THREE-POINT BOUNDARY-VALUE PROBLEM FOR A NON-LINEAR THIRD ORDER ORDINARY DIFFERENTIAL EQUATION

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INTRODUCTION

The theory of multi-point boundary value problems for ordinary differential equations has been remarkably developed. The case of linear boundary value problems involving the fundamental properties of the Green's function was investigated by J. Tamarkin, [1], and M. Greguš, [2], [3]. Theorems on the uniqueness which in the linear case means also the existence, of a solution to the interpolation problem by means of a Lipschitz condition were established by Ch. J. de la Vallée Poussin in [4]. His result for the third order non-linear differential equation was improved by G. Casadei in [5]. A method how to apply the uniqueness results to the existence of a solution to the non-linear interpolation problem has been developed by A. Lasota, Z. Opial in [6]. The same authors touched the mentioned problem in a series of papers. Here only [7] will be mentioned. The continuous dependence on the initial conditions connected with an approximation method was used in the proof of the existence of a solution to a non-linear multi-point boundary value problem in [8] by S. Cinquini.

As to the related topics, many papers deal with the multi-point boundary value problems for differential systems. First of all it is necessary to mention the paper [9] by R. Conti containing some general theorems on the existence of a solution to such problems and distinguishing by a comprehensive bibliography. A uniqueness result was given by M. Švec in [10]. The papers [11], [12] by M. Urabe deal with the mentioned problems from a numerical analysts standpoint of view. In a series of papers, I. T. Kiguradze has been interested in the investigation of the singular boundary value problems. Here only the paper [13] is mentioned.

The present paper will deal with the three-point interpolation problem for a third order non-linear differential equation. It will be based on the theory presented in [3] and will contain inquiry into the properties of the Green's function connected with that problem. By means of the Schauder and Banach fixed point theorems an existence theorem as well as a theorem guaranteeing the existence and uniqueness of a solution to such a boundary value problem will be given. These theorems do not follow from the results in [6] by means of [5] nor are contained in the paper [8]. The result will be applied to a disconjugacy criterion for the third order linear differential equation which is independent on the criterion given in [14].

1. First, the following problem will be considered. Given three numbers $a_1 < < a_2 < a_3$ and a function $r \in C_0(\langle a_1, a_3\rangle)$ (throughout the paper only real functions will be taken into consideration), to find the solution of

$$y''' = r(x),\ y(a_1) = y(a_2) = y(a_3) = 0. \tag{1}$$

By the results of [3], the following lemmas hold.

Lemma 1. (See Theorem 1, [3], p. 50). *For an arbitrary point $t \in (a_k, a_{k+1})$, $k = 1, 2$, there exists a function $G_k = G_k(x, t)$ with the following properties:*

1. $G_k, \dfrac{\partial G_k}{\partial x} = G_{kx}$ *are continuous functions of x in $\langle a_1, a_3\rangle$.*

2. $\dfrac{\partial^2 G_k}{\partial x^2} = G_{kxx}$ *is continuous in x everywhere in $\langle a_1, a_3\rangle$ with the exception of t where $G_{kxx}(t + 0, t) - G_{kxx}(t - 0, t) = 1$.*

3. *G_k as a function of x is a solution of $y''' = 0$ in the intervals $\langle a_1, t)$, $(t, a_3\rangle$ and satisfies the boundary conditons from* (1).

4. *The function G_k is uniquely determined by the properties* 1., 2., *and* 3.

Lemma 2 (See Theorem 2, [3], p. 52). *The solution y of the problem* (1) *is given by the formula $y(x) = \sum_{k=1}^{2} \int_{a_k}^{a_{k+1}} G_k(x, t)\, r(t)\, dt$, $a_1 \leqq x \leqq a_3$.*

Lemma 1 yields the explicit form of the functions G_k, $k = 1, 2$. When $a_1 < t < a_2$,

$$G_1(x, t) = \begin{cases} (x - a_1)\,[(c_2(t)\,x + c_3(t)], & a_1 \leqq x \leqq t \\ c_1(t)\,(x - a_2)\,(x - a_3), & t < x \leqq a_3, \end{cases} \tag{2}$$

where

$$c_1(t) = \frac{(t - a_1)^2}{2(a_3 - a_1)\,(a_2 - a_1)}, \tag{3}$$

$$c_2(t) = c_1(t) - \frac{1}{2},$$

$$c_3(t) = \frac{1}{2(a_3 - a_1)\,(a_2 - a_1)}\,[(a_1 - a_2 - a_3)\,t^2 + 2a_2a_3t - a_1a_2a_3].$$

Further if $a_2 < t < a_3$,

$$G_2(x, t) = \begin{cases} c_4(t)\,(x - a_1)\,(x - a_2), & a_1 \leqq x < t \\ (x - a_3)[c_5(t)\,x + c_6(t)], & t \leqq x \leqq a_3. \end{cases} \tag{4}$$

Here

$$c_4(t) = -\frac{(t - a_3)^2}{2(a_3 - a_1)\,(a_3 - a_2)}, \tag{5}$$

$$c_5(t) = c_4(t) + \frac{1}{2},$$

$$c_6(t) = \frac{1}{2(a_3 - a_1)\,(a_3 - a_2)}\,[(a_1 + a_2 - a_3)\,t^2 - 2a_1a_2t + a_1a_2a_3].$$

From (2), (4) the equalities

$$(6)\qquad G_{1x}(x,t)=\begin{cases}2c_2(t)\,x+[c_3(t)-a_1c_2(t)], & a_1\leqq x\leqq t\\ c_1(t)\,(2x-a_2-a_3), & t<x\leqq a_3,\end{cases}$$

$$G_{1xx}(x,t)=\begin{cases}2c_2(t), & a_1\leqq x<t\\ 2c_1(t), & t<x\leqq a_3,\ a_1<t<a_2\end{cases}$$

and

$$(7)\qquad G_{2x}(x,t)=\begin{cases}c_4(t)\,(2x-a_1-a_2), & a_1\leqq x<t\\ 2c_5(t)\,x+[c_6(t)-a_3c_5(t)], & t\leqq x\leqq a_3,\end{cases}$$

$$G_{2xx}(x,t)=\begin{cases}2c_4(t), & a_1\leqq x<t\\ 2c_5(t), & t\leqq x\leqq a_3,\ a_2<t<a_3\end{cases}$$

follow.

Now, the properties of the functions G_1, G_2 and of their derivatives will be studied. From (2), (3), (4), and (5) it follows that for any x, $a_1\leqq x\leqq a_2$,

$$(8)\qquad \lim_{\substack{\xi\to x\\ t\to a_2^-}} G_1(\xi,t)=-\frac{1}{2}\frac{a_3-a_2}{a_3-a_1}(x-a_1)(x-a_2)=\lim_{\substack{\xi\to x\\ x\to a_2^+}} G_2(\xi,t)$$

and

$$(9)\quad \lim_{\substack{\xi\to x\\ t\to a_2^-}} G_1(\xi,t)=\frac{1}{2}\frac{a_2-a_1}{a_3-a_1}(x-a_2)(x-a_3)=\lim_{\substack{\xi\to x\\ t\to a_2^+}} G_2(\xi,t)\ \text{if}\ a_2<x\leqq a_3.$$

Further

$$(10)\qquad \lim_{\substack{\xi\to x\\ t\to a_1^+}} G_1(\xi,t)=0,\ \lim_{\substack{\xi\to x\\ t\to a_3^-}} G_2(\xi,t)=0\ \text{for all}\ x,\ a_1\leqq x\leqq a_3.$$

Similarly, from (6), (7), with the help of (3) and (5), we get the equalities

$$(11)\ \lim_{\substack{\xi\to x\\ t\to a_2^-}} G_{1x}(\xi,t)=-\frac{1}{2}\frac{a_3-a_2}{a_3-a_1}(2x-a_1-a_2)=\lim_{\substack{\xi\to x\\ t\to a_2^+}} G_{2x}(\xi,t),\ a_1\leqq x\leqq a_2$$

$$(12)\ \lim_{\substack{\xi\to x\\ t\to a_2^-}} G_{1x}(\xi,t)=\frac{1}{2}\frac{a_2-a_1}{a_3-a_1}(2x-a_2-2a_3)=\lim_{\substack{\xi\to a\\ t\to a_2^+}} G_{2x}(\xi,t)\ \text{when}\ a_2<x\leqq a_3.$$

Also for all x, $a_1\leqq x\leqq a_3$,

$$(13)\qquad \lim_{\substack{\xi\to x\\ t\to a_1^+}} G_{1x}(\xi,t)=0,\ \lim_{\substack{\xi\to x\\ t\to a_3^-}} G_{2x}(\xi,t)=0.$$

Finally, for $a_1\leqq x<a_2$

$$(14)\qquad \lim_{\substack{\xi\to x\\ t\to a_2^-}} G_{1xx}(\xi,t)=-\frac{a_3-a_2}{a_3-a_1}=\lim_{\substack{\xi\to x\\ t\to a_2^+}} G_{2xx}(\xi,t),$$

while

(15) $$\lim_{\substack{\xi \to x \\ t \to a_2^-}} G_{1xx}(\xi, t) = \frac{a_2 - a_1}{a_3 - a_1} = \lim_{\substack{\xi \to x \\ t \to a_2^+}} G_{2xx}(\xi, t), \ a_2 < x \leqq a_3.$$

The relations

(16) $$\lim_{\substack{\xi \to x \\ t \to a_1^+}} G_{1xx}(\xi, t) = 0 \text{ for } a_1 < x \leqq a_3, \ \lim_{\substack{\xi \to x \\ t \to a_3^-}} G_{2xx}(\xi, t) = 0, \ a_1 \leqq x < a_3$$

can be also verified.

Thus, by putting

(17) $$G(x, t) = \begin{cases} G_1(x, t), & a_1 \leqq x \leqq a_3, \ a_1 < t < a_2 \\ G_2(x, t), & a_1 \leqq x \leqq a_3, \ a_2 < t < a_3 \end{cases}$$

and defining $G(x, t)$ at the points (x, a_2), (x, a_1), (x, a_3) by means of the limits (8), (9), (10) we get that the function $G = G(x, t)$ is continuous on $D = \langle a_1, a_3 \rangle \times \times \langle a_1, a_3 \rangle$. The equalities (6), (7), together with (11), (12), (13), show that $\frac{\partial G}{\partial x}$ is also continuous on D. As to $\frac{\partial^2 G}{\partial x^2}$, from (6), (7), as well as (14), (15), (16), it follows that this function is continuous on D with the exception of the points (t, t), $a_1 \leqq t \leqq a_3$, whereby $\frac{\partial^2 G(t + 0, t)}{\partial x^2} - \frac{\partial^2 G(t - 0, t)}{\partial x^2} = 1$ $(a_1 < t < a_3)$.

Consider the sign of the functions $G, \frac{\partial G}{\partial x}, \frac{\partial^2 G}{\partial x^2}$. First, $c_1(t) > 0$, $a_1 < t < a_2$, gives that $G_1(x, t) < 0$ for $a_2 < x < a_3$, $a_1 < t < a_2$, and $G_1(x, t) > 0$ if $t < x < a_2$, $a_1 < t < a_2$. By a continuity argument $c_2(t) < 0$ $(a_1 < t < a_2)$ implies that $G_1(x, t) > > 0$ for $a_1 < x \leqq t < a_2$. Similar statements hold for G_2. Thus

(18) $G(x, t) > 0$ when $a_1 < x < a_2$, and $G(x, t) < 0$ for $a_2 < x < a_3$, $a_1 < t < a_3$.

The above inequalities for c_1, c_2 together with $c_4(t) < 0$, $c_5(t) > 0$ $(a_2 < t < a_3)$ lead to the inequalities

(19) $$\frac{\partial^2 G(x, t)}{\partial x^2} < 0, \ a_1 \leqq x < t \text{ and } \frac{\partial^2 G(x, t)}{\partial x^2} > 0, \ t < x \leqq a_3, \ a_1 < t < a_3.$$

The situation with the sign of $\frac{\partial G}{\partial x}$ is more complicated. Considering the sign of c_1 and c_4, we get

(20) $$\frac{\partial G(x, t)}{\partial x} < 0 \text{ for } t \leqq x < \frac{a_2 + a_3}{2}, \ a_1 < t \leqq a_2 \text{ and } \frac{a_1 + a_2}{2} < x \leqq t, a_2 \leqq t < a_3,$$

$$\frac{\partial G(x, t)}{\partial x} > 0 \text{ for } \frac{a_2 + a_3}{2} < x \leqq a_3, \ a_1 < t \leqq a_2 \text{ and } a_1 \leqq x < \frac{a_1 + a_2}{2}, a_2 \leqq t < a_3.$$

As to the behaviour of $\frac{\partial G}{\partial x}$ in the domain $a_1 \leqq x < t$, $a_1 < t \leqq a_2$, from the inequalities $\frac{\partial G(a_1, t)}{\partial x} > 0$, $\frac{\partial G(t, t)}{\partial x} < 0$ for $a_1 < t \leqq a_2$, with regard to (19), it follows that for any t, $a_1 < t \leqq a_2$, there is exactly one $x_1 = x_1(t)$, with $a_1 < x_1 < t$ such that $\frac{\partial G(x_1(t), t)}{\partial x} = 0$ and

$$(21) \quad \frac{\partial G(x, t)}{\partial x} < 0 \text{ if } x_1 < x < t, \text{ and } \frac{\partial G(x, t)}{\partial x} > 0 \text{ when } a_1 \leqq x < y_1, a_1 < t \leqq a_2.$$

In the domain $t < x \leqq a_3, a_2 \leqq t < a_3$ $\frac{\partial G}{\partial x}$ shows similar properties. First, $\frac{\partial G(t, t)}{\partial x} < 0$ for $a_2 \leqq t < a_3$. On the other hand, by (20), $\frac{\partial G(a_3, a_2)}{\partial x} > 0$. By a simple calculation we get that $\frac{d^2}{dt^2} \frac{\partial G(a_3, t)}{\partial x} < 0$, $a_2 \leqq t \leqq a_3$, and, since $\frac{\partial G(a_3, a_3)}{\partial x} = 0$, $\frac{\partial G(a_3, t)}{\partial x} > 0$ for all t, $a_2 \leqq t < a_3$. Taking (19) into consideration, we come to the conclusion that for any t, $a_2 \leqq t < a_3$, there exists exactly one $x_2 = x_2(t)$, with $t < x_2 < a_3$, such that $\frac{\partial G(x_2(t), t)}{\partial x} = 0$, and

$$(22) \quad \frac{\partial G(x, t)}{\partial x} < 0 \text{ for } t < x < x_2, \text{ and } \frac{\partial G(x, t)}{\partial x} > 0 \text{ for } x_2 < x \leqq a_3, a_2 \leqq t < a_3.$$

For each t, $a_1 < t \leqq a_2$, from $\frac{\partial G(t, t)}{\partial x} = \int_{x_1(t)}^{t} \frac{\partial^2 G(\xi, t)}{\partial x^2} d\xi$, in view of (3), (6), we get

$$(23) \qquad x_1(t) = \frac{a_2 + a_3}{2} + \frac{1}{2c_2(t)} \left(\frac{a_2 + a_3}{2} - t \right)$$

and $x_1'(t) = \frac{1}{4c_2^2(t)} \cdot \frac{1}{(a_2 - a_1)(a_3 - a_1)} (t - a_2)(t - a_3) > 0$ for $a_1 < t < a_2$. Hence $a_1 = x_1(a_1) \leqq x_1(t) \leqq x_1(a_2) = \frac{a_1 + a_2}{2}$, $a_1 \leqq t \leqq a_2$. Further $x_1''(t) = \varphi(t)\, \psi(t)$, where $\varphi(t) = \frac{2(a_3 - a_1)(a_2 - a_1)}{[(t - a_1)^2 - (a_3 - a_1)(a_2 - a_1)]^3}$ for $a_1 \leqq t \leqq a_2$ and $\psi(t) = \left(t - \frac{a_2 + a_3}{2}\right) [(t - a_1)^2 - (a_3 - a_1)(a_2 - a_1)] - 2(t - a_1)(t - a_2)(t - a_3)$, $a_1 \leqq t \leqq a_2$. Since $\psi''(t) = -6\left(t - \frac{a_2 + a_3}{2}\right) > 0$ and $\psi'(a_2) = 0$, the inequality $\psi'(t) < 0$ must hold for $a_1 \leqq t < a_2$ and finally, using $\psi(a_2) = \frac{(a_3 - a_2)^2}{2}(a_2 - a_1) > 0$, we get that $\psi(t) > 0$ for $a_1 \leqq t \leqq a_2$. Thus $x_1''(t) < 0$ and if $t_1 = t_1(x)$ is the inverse function of x_1, we have

(24) $$0 \leqq t_1(x) - x \leqq x - a_1,\ a_1 \leqq x \leqq \frac{a_1 + a_2}{2}.$$

Similarly, we can find that $x_2(t) = \frac{a_1 + a_2}{2} + \frac{1}{2\,c_4(t) + 1}\left(t - \frac{a_1 + a_2}{2}\right)$, $a_2 \leqq t < a_3$. Hence $x_2'(t) = \frac{1}{[2c_4(t) + 1]^2}\frac{(t - a_1)(t - a_2)}{(a_3 - a_1)(a_3 - a_2)} > 0$ for $a_2 < t < a_3$. Thus $x_2(a_2) = \frac{a_2 + a_3}{2} \leqq x_2(t) \leqq x_2(a_3) = a_3$ for $a_2 \leqq t \leqq a_3$. After some calculations we obtain $x_2''(t) = \omega(t)\,\varrho(t)$, where $\omega(t) = \frac{2(a_3 - a_1)(a_2 - a_1)}{[(t - a_3)^2 - (a_3 - a_1)(a_3 - a_2)]^3} < 0$ for $a_2 \leqq t \leqq a_3$ and $\varrho(t) = \left(t - \frac{a_1 + a_2}{2}\right)[(t - a_3)^2 - (a_3 - a_1)(a_3 - a_2)] - 2(t - a_1)\,.\,(t - a_2)(t - a_3)$, $a_2 \leqq t \leqq a_3$. $\varrho''(t) = -6\left(t - \frac{a_1 - a_2}{2}\right) < 0$ and $\varrho'(a_2) = 0$ imply that $\varrho'(t) < 0$ for $a_2 < t \leqq a_3$. This, together with $\varrho(a_2) < 0$, gives that $\varrho(t) < 0$ for $a_2 \leqq t \leqq a_3$ and so, $x_2''(t) > 0$ in the same interval. When $t_2 = t_2(x)$ is the inverse function of x_2, by the last inequality and $x_2'(a_3) = 1$, we have $0 \leqq x - t_2(x) \leqq a_3 - x$, $\frac{a_2 + a_3}{2} \leqq x \leqq a_3$.

Before going to the next lemma, let us consider the solution y_0 of $y''' = 1$, $y(a_1) = y(a_2) = y(a_3) = 0$. By a straightforward calculation we get $y_0(x) = \frac{1}{6}(x - a_1)\,.\,(x - a_2)(x - a_3)$, $y_0'(x) = \frac{1}{2}x^2 - \frac{1}{3}(a_1 + a_2 + a_3)\,x + \frac{1}{6}(a_1a_2 + a_1a_3 + a_2a_3)$, $y_0''(x) = x - \frac{1}{3}(a_1 + a_2 + a_3)$. Since for $a_1 \leqq x \leqq a_3$ $-\frac{4}{27}(a_3 - a_1)^3 \leqq (x - a_1)^2\,.\,(x - a_3) \leqq \Phi(x, a_2) \equiv (x - a_1)(x - a_2)(x - a_3) \leqq (x - a_1)(x - a_3)^2 \leqq \frac{4}{27}(a_3 - a_1)^3$, the inequality

(25) $$|\,y_0(x)\,| \leqq \frac{2}{81}(a_3 - a_1)^3,\ a_1 \leqq x \leqq a_3$$

is true. From the inequality $y_0''(x) < 0$ ($y_0''(x) > 0$) which is valid for $a_1 \leqq x < \frac{1}{3}(a_1 + a_2 + a_3)$ $\left(\frac{1}{3}(a_1 + a_2 + a_3) < x \leqq a_3\right)$ it follows that it is sufficient to estimate $|\,y_0'(x)\,|$ only at the points $x = a_1$, $x = \frac{1}{3}(a_1 + a_2 + a_3)$, $x = a_3$. But $y_0'(a_1) = \frac{1}{6}(a_3 - a_1)(a_2 - a_1) \leqq \frac{1}{6}(a_3 - a_1)^2$, $y_0'(a_3) = \frac{1}{6}(a_3 - a_1)\,.\,(a_3 - a_2) \leqq \frac{1}{6}(a_3 - a_1)^2$ and $0 > y_0'\left(\frac{a_1 + a_2 + a_3}{3}\right) = y_0'\left(\frac{a_1 + a_2}{2}\right) + \int_{\frac{a_1 + a_2}{2}}^{\frac{a_1 + a_2 + a_3}{3}} y_0''(x)\,dx > -\frac{1}{24}(a_2 - a_1)^2 - \frac{1}{9}(a_3 - a_1)^2 > -\frac{1}{6}(a_3 - a_1)^2$. Thus

$$(26) \qquad | y_0'(x) | \leqq \frac{1}{6} (a_3 - a_1)^2, \ a_1 \leqq x \leqq a_3.$$

Lemma 3. *For the function G, given by* (17), *the following estimations hold:*

$$(27) \qquad \int_{a_1}^{a_3} | G(x, t) | \, dt \leqq \frac{2}{81} (a_3 - a_1)^3,$$

$$(28) \qquad \int_{a_1}^{a_3} \left| \frac{\partial G(x, t)}{\partial x} \right| dt \leqq \frac{1}{6} (a_3 - a_1)^2,$$

$$(29) \qquad \int_{a_1}^{a_3} \left| \frac{\partial^2 G(x, t)}{\partial x^2} \right| dt \leqq \frac{2}{3} (a_3 - a_1), \ a_1 \leqq x \leqq a_3.$$

Proof. On the basis of (18), $\int_{a_1}^{a_3} | G(x, t) | \, dt = \varepsilon \int_{a_1}^{a_3} G(x, t) \, dt$ where $\varepsilon = 1$ or -1, according as $a_1 < x < a_2$, or $a_2 < x < a_3$, respectively. By Lemma 2, $\int_{a_1}^{a_3} G(x, t) \, dt = y_0(x)$, $a_1 \leqq x \leqq a_3$. Thus (27) follows from (25).

As to the estimation of the left-hand side of (28), by the inequalities (20), (21), (22), considering the properties of the functions x_1, x_2, we have to investigate the following 3 cases:

a. $\frac{a_1 + a_2}{2} \leqq x \leqq \frac{a_2 + a_3}{2}$. Then $\int_{a_1}^{a_3} \left| \frac{\partial G(x, t)}{\partial x} \right| dt = -y_0'(x)$ and (26) implies (28).

b. $a_1 \leqq x < \frac{a_1 + a_2}{2}$. By the inequalities (20), (21), (22), $\int_{a_1}^{a_3} \left| \frac{\partial G(x, t)}{\partial x} \right| dt = y_0'(x) - 2 \int_{a_1}^{t_1(x)} G_{1x}(x, t) \, dt$. Using (6), we have $-2 \int_{a_1}^{x} G_{1x}(x, t) \, dt = 4 \left(\frac{a_2 + a_3}{2} - x \right) \int_{a_1}^{x} c_1(t) \, dt = \frac{2}{3} \frac{(x - a_1)^3}{(a_2 - a_1)(a_3 - a_1)} \left(\frac{a_2 + a_3}{2} - x \right)$. From the equality $G_{1x}(x, t) = \int_{x_1(t)}^{x} G_{1xx}(\xi, t) \, d\xi = 2c_2(t)[x - x_1(t)]$, by means of (3), (23) we obtain $-2 \int_{x}^{t_1(x)} G_{1x}(x, t) \, dt = \frac{2}{(a_3 - a_1)(a_2 - a_1)} \int_{x}^{t_1(x)} [(a_3 - a_1)(a_2 - a_1) - (t - a_1)^2] \, . \, [x -$

$- x_1(t)]\,dt \leqq \frac{2}{(a_3 - a_1)(a_2 - a_1)} [(a_3 - a_1)(a_2 - a_1) - (x - a_1)^2] \cdot [x - x_1(x)] \cdot$ $\cdot [t_1(x) - x] = \frac{2}{(a_3 - a_1)(a_2 - a_1)} \cdot \left(\frac{a_2 + a_3}{2} - x\right)(x - a_1)^2 [t_1(x) - x]$ what is, by (24), $\leqq \frac{2}{(a_3 - a_1)(a_2 - a_1)} \cdot \left(\frac{a_2 + a_3}{2} - x\right)(x - a_1)^3$.

Thus $\int_{a_1}^{a_3} \left| \frac{\partial G(x, t)}{\partial x} \right| dt \leqq h(x) \equiv \frac{1}{2} x^2 - \frac{1}{3}(a_1 + a_2 + a_3)\, x + \frac{1}{6}(a_1a_2 + a_1a_3 + a_2a_3) + \frac{8}{3} \frac{1}{(a_2 - a_1)(a_3 - a_1)} \left(\frac{a_2 + a_3}{2} - x\right)(x - a_1)^3$. After some calculations we get that $h''(x) = 1 - \frac{32}{(a_2 - a_1)(a_3 - a_1)}(x - a_1)\left(x - \frac{6a_1 + 3a_2 + 3a_3}{12}\right) > 0$ for $a_1 \leqq x < \frac{a_1 + a_2}{2}$. Therefore $h(x) \leqq \max \left[h(a_1),\, h\left(\frac{a_1 + a_2}{2}\right)\right]$. But $h(a_1) = \frac{1}{6}(a_3 - a_1)(a_2 - a_1)$, $h\left(\frac{a_1 + a_2}{2}\right) = \frac{1}{8}(a_2 - a_1)^2$. Hence (28) is true also in this case.

c. $\frac{a_2 + a_3}{2} < x \leqq a_3$. Using similar considerations as in the ferogoing case we come to the inequality $\int_{a_1}^{a_3} \left| \frac{\partial G(x, t)}{\partial x} \right| dt = y_0'(x) - 2 \int_{t_2(x)}^{a_3} G_{2x}(x, t)\,dt \leqq k(x) \equiv \frac{1}{2} x^2 - \frac{1}{3}(a_1 + a_2 + a_3)\, x + \frac{1}{6}(a_1a_2 + a_1a_3 + a_2a_3) + \frac{8}{3} \frac{1}{(a_3 - a_1)(a_3 - a_2)} \left(x - \frac{a_1 + a_2}{2}\right)(a_3 - x)^3$. Since $k''(x) = 1 + \frac{32}{(a_3 - a_1)(a_3 - a_2)}(a_3 - x)\left(x - \frac{3a_1 + 3a_2 + 6a_3}{12}\right) > 0$ for $\frac{a_2 + a_3}{2} < x \leqq a_3$, $k(x) \leqq \max \left[k\left(\frac{a_2 + a_3}{2}\right),\, k(a_3)\right]$. The result then follows from the relations $k(a_3) = \frac{1}{6}(a_3 - a_1)(a_3 - a_2)$, $k\left(\frac{a_2 + a_3}{2}\right) = \frac{1}{8}(a_3 - a_2)^2$.

Now we shall derive (29). When $a_1 \leqq x \leqq a_2$, by (19), (6), (3) $\int_{a_1}^{a_3} \left| \frac{\partial^2 G(x, t)}{\partial x^2} \right| dt = -y_0''(x) + 4 \int_{a_1} c_1(t)\,dt = \frac{1}{3}(a_1 + a_2 + a_3) - x + \frac{2}{3} \frac{(x - a_1)^3}{(a_2 - a_1)(a_3 - a_1)} \equiv l(x)$. Since $l''(x) = \frac{4(x - a_1)}{(a_2 - a_1)(a_3 - a_1)} \geqq 0$, $l(x) \leqq \max\, [(la_1),\, l(a_2)]$. $l(a_1) < l(a_2)$ is equivalent to $1 < \frac{2}{3} \frac{a_2 - a_1}{a_3 - a_1}$ and, thus, cannot occur. Therefore $l(x) \leqq l(a_1) \leqq \frac{2}{3}(a_3 - a_1)$, what implies (29).

If $a_2 < x \leqq a_3$, again by (19), (7), (5), $\int_{a_1}^{a_3} \left| \frac{\partial^2 G(x, t)}{\partial x^2} \right| dt = y_0''(x) - 4 \int_x^{a_3} c_4(t)\, dt \leqq$ $\leqq \frac{2}{3}(a_3 - a_1)$. Thus, in both cases (29) is true.

Let b_i, $i = 1, 2, 3$, be real numbers. Then the Lagrange interpolation polynomial

$$(30)\quad y_L(x) = \frac{b_1}{(a_1 - a_2)(a_1 - a_3)}(x - a_2)(x - a_3) + \frac{b_2}{(a_2 - a_1)(a_2 - a_3)}(x - a_1)\cdot (x - a_3) + \frac{b_3}{(a_3 - a_1)(a_3 - a_2)}(x - a_1)(x - a_2)$$

represents the unique solution of the boundary value problem $y''' = 0$, $a_1 \leqq x \leqq a_3$, $y(a_i) = b_i$, $i = 1, 2, 3$. The following inequalities for y_L hold.

Lemma 4. *Let there exist constants $M > 0$, $N > 0$, $P > 0$ such that*

$$(31)\quad |b_1| \leqq M, \left|\frac{b_3 - b_1}{a_3 - a_1}\right| \leqq N, \left|\frac{b_1(a_3 - a_2) - b_2(a_3 - a_1) + b_3(a_2 - a_1)}{(a_2 - a_1)(a_3 - a_1)(a_3 - a_2)}\right| \leqq P.$$

Then

$$(32)\quad |y_L(x)| \leqq M + N(a_3 - a_1) + \frac{P}{2}(a_3 - a_1)^2,$$

$$|y_L'(x)| \leqq N + P(a_3 - a_1),$$

$$|y_L''(x)| \leqq 2P, \qquad a_1 \leqq x \leqq a_3.$$

Proof. The last inequality in (32) follows from

$$y_L''(x) = 2\,\frac{b_1(a_3 - a_2) - b_2(a_3 - a_1) + b_3(a_2 - a_1)}{(a_2 - a_1)(a_3 - a_1)(a_3 - a_2)}.$$

Since $y_L'\left(\frac{a_1 + a_3}{2}\right) = \frac{b_3 - b_1}{a_3 - a_1}$, by the mean value theorem

$$(33)\quad |y_L'(x)| \leqq N + 2P\left|x - \frac{a_1 + a_3}{2}\right|$$

is true and thus we get the second inequality in (32). Using (33), we arrive to the inequality $|y_L(x)| \leqq \left|y_L\left(\frac{a_1 + a_3}{2}\right)\right| + N\left|x - \frac{a_1 + a_3}{2}\right| + P\left(x - \frac{a_1 + a_3}{2}\right)^2$ which together with $\left|y_L\left(\frac{a_1 + a_3}{2}\right)\right| \leqq M + \frac{N}{2}(a_3 - a_1) + \frac{P}{4}(a_3 - a_1)^2$ gives the first inequality (32).

Now we are able to prove an existence theorem for the three-point boundary value problem. In what follows $a < A$ will mean real numbers, M, N, P non-negative, R_0, R_1, R_2 positive numbers.

Theorem 1. *Let $f = f(x, y, z, p)$ be a continuous function on the set $B = \{(x, y, z, p) : a \leqq x \leqq A, |y| \leqq R_0, |z| \leqq R_1, |p| \leqq R_2\}$. Let a_i, b_i, $i = 1, 2, 3$, be real numbers such that $a \leqq a_1 < a_2 < a_3 \leqq A$, and the conditions* (31) *are fulfilled. Let $|f(x, y, z, p, q)| \leqq q$ on B and let*

$$(34) \qquad M + N(a_3 - a_1) + \frac{P}{2}(a_3 - a_1)^2 + \frac{2}{81} q(a_3 - a_1)^3 \leqq R_0$$

$$N + P(a_3 - a_1) + \frac{1}{6} q(a_3 - a_1)^2 \leqq R_1,$$

$$2P + \frac{2}{3} q(a_3 - a_1) \leqq R_2.$$

Then the boundary value problem

$$(35) \qquad y''' = f(x, y, y', y''), \quad y(a_i) = b_i, \quad i = 1, 2, 3,$$

has at least one solution y. Furthermore, if y_L is the function (30), *the following inequalities hold:*

$$| y(x) - y_L(x)| \leqq \frac{2}{81} q(a_3 - a_1)^3, \ |y'(x) - y_L'(x)| \leqq \frac{1}{6} q(a_3 - a_1)^2,$$

$$| y''(x) - y_L''(x)| \leqq \frac{2}{3} q(a_3 - a_1), \ a_1 \leqq x \leqq a_3.$$

Proof. Consider the Banach space $C_2(\langle a_1, a_3 \rangle)$, with the norm $\| y \|_2 = \max (\| y \|, \| y' \|, \| y'' \|)$, where $\| y \| = \max\limits_{a_1 \leqq x \leqq a_3} | y(x) |$. The set $E = \{y \in C_2(\langle a_1, a_3 \rangle) : \| y \| \leqq R_0, \| y' \| \leqq R_1, \| y'' \| \leqq R_2\}$ is a closed and convex subset of $C_2(\langle a_1, a_3 \rangle)$. Consider the mapping T of E into $C_2(\langle a_1, a_3 \rangle)$ defined by

$$(36) \qquad (Ty)(x) = y_L(x) + \int_{a_1}^{a_3} G(x, t) f(t, y(t), y'(t), y''(t)) \, dt,$$

where G is the function (17). By Lemmas 3 and 4, for a $y \in E$ we have $|(Ty)(x)| \leqq \leqq M + N(a_3 - a_1) + \frac{P}{2}(a_3 - a_1)^2 + \frac{2}{81} q (a_3 - a_1)^3$, $|(Ty)'(x)| \leqq N + P(a_3 - - a_1) + \frac{1}{6} q(a_3 - a_1)^2$, and $|(Ty)''(x)| \leqq 2P + \frac{2}{3} q(a_3 - a_1)$. Thus, (34) implies that T maps E into itself. Also we see that the families of all functions Ty, $(Ty)'$, $(Ty)''$ satisfy the conditions of the Ascoli's Lemma and, hence, the set TE is relatively compact. Further T is continuous. By the Schauder Fixed-Point theorem T has a fixed point in E. The fixed point is a solution of the problem (35). From (36), on the basis of Lemma 3, the last assertion of the theorem follows.

Corollary. *Assume that $f = f(x, y, z, p)$ is a continuous function for $a \leqq x \leqq A$, $-\infty < y, z, p < +\infty$ and that there exist non-negative constants $h, k_i, \alpha_i, i = 0, 1, 2$, s uch that*

$$(37) \qquad | f(x, y, z, p)| \leqq h + k_0 \, | y |^{\alpha_0} + k_1 \, | z |^{\alpha_1} + k_2 \, | p |^{\alpha_2}$$

in the domain of f. Then the following statements hold:

1. *If all $\alpha_i < 1$, $i = 0, 1, 2$, then any boundary value problem* (35) *with $a \leqq a_1 < a_2 < a_3 \leqq A$, b_i, $i = 1, 2, 3$, being arbitrary, has a solution.*

2. *If $\alpha_0 = \alpha_1 = \alpha_2 = 1$, the last statement remains in power if*

(38) $$\frac{2}{81} k_0(a_3 - a_1)^3 + \frac{1}{6} k_1(a_3 - a_1)^2 + \frac{2}{3} k_2(a_3 - a_1) < 1.$$

3. *If at least one of α_i, $i = 0, 1, 2$, is greater than 1, then for given boundary conditions $y(a_i) = b_i$, $i = 1, 2, 3$, $a \leqq a_1 < a_2 < a_3 \leqq A$ and a given h there exists a $\delta > 0$, $\delta = \delta(\alpha_0, \alpha_1, \alpha_2, h, M, N, P)$ such that if $k_i < \delta$, $i = 0, 1, 2$, the boundary value problem* (35) *has at least one solution.*

Proof. In view of (37), in the system (34) we can put $q = h + k_0R_0^{\alpha_0} + k_1R_1^{\alpha_1} + k_2R_2^{\alpha_2}$. Then in the first case the inequalities (34) are true for all sufficiently great R_0, R_1, R_2. In the third case, taking a fixed triple (R_0, R_1, R_2) such that $R_0 > M + N \cdot (a_3 - a_1) + \frac{P}{2}(a_3 - a_1)^2 + \frac{2}{81} h(a_3 - a_1)^3$, $R_1 > N + P(a_3 - a_1) + \frac{1}{6} h(a_3 - a_1)^2$, $R_2 > 2P + \frac{2}{3} h(a_3 - a_1)$, we see that (34) is satisfied for $k_0 = k_1 = k_2 = 0$. By a continuity argument the assertion 3. follows.

Now we shall consider the second case. Replacing the zero values of k_i, $i = 0, 1, 2$, by sufficiently small positive numbers, we may assume that all k_i in (38) are positive. The inequality (37) with regard to $\alpha_0 = \alpha_1 = \alpha_2 = 1$, implies that the system (34) is of the form

(39) $$\frac{2}{81}(a_3 - a_1)^3 [k_0R_0 + k_1R_1 + k_2R_2] \leqq R_0 - [M + N(a_3 - a_1) + \frac{P}{2}(a_3 - a_1)^2 + \frac{2}{81}(a_3 - a_1)^3 h]$$

$$\frac{1}{6}(a_3 - a_1)^2 [k_0R_0 + k_1R_1 + k_2R_2] \leqq R_1 - \left[N + P(a_3 - a_1) + \frac{1}{6}(a_3 - a_1)^2 h\right]$$

$$\frac{2}{3}(a_3 - a_1) [k_0R_0 + k_1R_1 + k_2R_2] \leqq R_2 - \left[2P + \frac{2}{3}(a_3 - a_1) h\right]$$

The characteristic equation of the matrix S of coefficients of that system is $\det(S - \lambda U) = -\lambda^2(\lambda - \gamma) = 0$, where $\gamma = \frac{2}{81}(a_3 - a_1)^3 \cdot k_0 + \frac{1}{6}(a_3 - a_1)^2 k_1 + \frac{2}{3}(a_3 - a_1) k_2$ and U is the identity matrix. By Theorem 27, [15], p. 114, there exists a vector $\bar{\mathbf{R}} = (R_0, R_1, R_2)$ with positive components such that $S\bar{\mathbf{R}} \leqq \gamma\bar{\mathbf{R}}$. Taking a suitable multiple $c\bar{\mathbf{R}}$, $c > 0$, we can reach that $\gamma\bar{\mathbf{R}} \leqq \bar{\mathbf{R}} - \bar{\mathbf{b}}$, where the components of the vector $\bar{\mathbf{R}} - \bar{\mathbf{b}}$ are given by the right-hand side of the system (39). Hence (39) has a positive solution (R_0, R_1, R_2) and the statement 2. is valid.

Theorem 2. *If all assumptions of Theorem 1 are satisfied and furthermore f fulfils in B a Lipschitz condition*

(40) $$|f(x, y_1, z_1, p_1) - f(x, y_2, z_2, p_2)| \leqq \vartheta_0 |y_1 - y_2| + \vartheta_1 |z_1 - z_2| + \vartheta_2 |p_1 - p_2|$$

where the Lipschitz constants satisfy the condition

(41) $$\frac{2}{81}(a_3 - a_1)^3 \vartheta_0 + \frac{1}{6}(a_3 - a_1)^2 \vartheta_1 + \frac{2}{3}(a_3 - a_1) \vartheta_2 < 1,$$

then the boundary value problem (35) *has one and only one solution y.*

Proof. Keeping the notations from the proof of the previous theorem with the only exception that the Banach space $C_2(\langle a_1, a_3\rangle)$ will be provided with the norm $\| y \|_{2a} = \max \left(\| y \|, \frac{4}{27} (a_3 - a_1) \| y' \|, \frac{1}{27} (a_3 - a_1)^2 \| y'' \| \right)$, we shall have that E is closed, and, by (34), $TE \subset E$. Further if $y_i \in E$, $Ty_i = Y_i$, $i = 1, 2$, then, in view of (40), (41) and Lemma 3, after denoting the left-hand side of (41) by μ, we shall obtain that each of the expressions $\| Y_1 - Y_2 \|$, $\frac{4}{27} (a_3 - a_1) \| Y_1' - Y_2' \|$, $\frac{1}{27} (a_3 - a_1)^2 . \| Y_1'' - Y_2'' \|$ is less or equal to $\frac{2}{81} (a_3 - a_1)^3 \vartheta_0 \| y_1 - y_2 \| + + \frac{1}{6} (a_3 - a_1)^2 \vartheta_1 \left[\frac{4}{27} (a_3 - a_1) \| y_1' - y_2' \| \right] + \frac{2}{3} (a_3 - a_1) \vartheta_2 \left[\frac{1}{27} (a_3 - a_1)^2 . \| y_1'' - y_2'' \| \right]$, and thus, $\| Ty_1 - Ty_2 \|_{2a} \leqq \mu \| y_1 - y_2 \|_{2a}$. By the Banach fixed point theorem the statement of the theorem follows.

Corollary 1. *When the functions* $p_i \in C_0(\langle a, A\rangle)$, $i = 0, 1, 2, 3$, *and* $| p_i(x)| \leqq k_i$, $i = 0, 1, 2$, $| p_3(x)| \leqq h$, $a \leqq x \leqq A$, *then* (38) *is a sufficient condition for the existence and the uniqueness of the solution of the boundary value problem*

(42) $$y''' = p_0(x) y + p_1(x) y' + p_2(x) y'' + p_3(x), \; y(a_i) = b_i, \; i = 1, 2, 3,$$

where $a \leqq a_1 < a_2 < a_3 \leqq A$, b_i, $i = 1, 2, 3$, *are arbitrary numbers.*

The proof follows from the fact that for the linear differential equation in (42) k_i can be taken as the Lipschitz constants ϑ_i, $i = 0, 1, 2$. Then (41) and (38) are of the same meaning.

By the special choice $p_3(x) = 0$ and $b_i = 0$, $i = 1, 2, 3$, we get

Corollary 2. *The equation*

(43) $$y''' = p_0(x) y + p_1(x) y' + p_2(x) y''$$

is disconjugate on each interval $\langle a_1, a_3\rangle \subset \langle a, A\rangle$ *for which* (38) *is true.*

Proof. By the previous reasonings there is no non-trivial solution of (43) having three zero-points in $\langle a_1, a_3\rangle$. In order to exclude the existence of a non-trivial solution with two zero points in $\langle a_1, a_3\rangle$ one of them being a double zero we shall apply the following remark.

2. Remark. So far we have considered the boundary value problem (35) at three distinct points $a_1 < a_2 < a_3$. The corresponding Green's function (17) $G = G(x, t; a_1, a_2, a_3)$. Now let us fix a_1, a_3 and let us consider the boundary value problem

(35′) $$y''' = f(x, y, y', y''), \; y(a_1) = b_1, \; y'(a_1) = b_1', \; y(a_3) = b_3$$

or

(35″) $$y''' = f(x, y, y', y''), \; y(a_1) = b_1, \; y(a_3) = b_3, \; y'(a_3) = b_3',$$

where b_1', b_3' are real numbers. After some calculations we get that the Green's functions for the corresponding homogeneous problems are

$$
(17') \qquad G(x, t; a_1, a_1, a_3) = \begin{cases} -\dfrac{1}{2(a_3-a_1)^2}(x-a_1)^2(t-a_3)^2, & a_1 \leqq x < t \\ (x-a_3)\left[\left(\dfrac{1}{2}-\dfrac{1}{2}\dfrac{(t-a_3)^2}{(a_3-a_1)^2}\right)x - \right. \\ \left. -\dfrac{(a_3-2a_1)t^2+2a_1^2t-a_3a_1^2}{2(a_3-a_1)^2}\right], & t \leqq x \leqq a_3 \end{cases}
$$

and

$$
(17'') \qquad G(x, t; a_1, a_3, a_3) = \begin{cases} (x-a_1)\left[\left(\dfrac{(t-a_1)^2}{2(a_3-a_1)^2}-\dfrac{1}{2}\right)x + \right. \\ \left. +\dfrac{(a_1-2a_3)t^2+2a_3^2t-a_1a_3^2}{2(a_3-a_1)^2}\right], & a_1 \leqq x \leqq t \\ \dfrac{(t-a_1)^2}{2(a_3-a_1)^2}(x-a_3)^2, & t < x \leqq a_3. \end{cases}
$$

From (17′), (4), (5) as well as from (17″), (2), (3) it follows that $\lim\limits_{a_2 \to a_1^+} G(x, t; a_1, a_2, a_3) = G(x, t; a_1, a_1, a_3)$ and $\lim\limits_{a_2 \to a_3^-} G(x, t; a_1, a_2, a_3) = G(x, t; a_1, a_3, a_3)$, respectively, uniformly in x, t in each closed region $\langle a_1, a_3\rangle \times \langle a_1 + \delta, a_3\rangle$ and in each closed region $\langle a_1, a_3\rangle \times \langle a_1, a_3 - \delta\rangle$, respectively, for $0 < \delta < a_3 - a_1$. Similar statements are true for the first and the second derivatives with respect to x of the considered functions. This implies that Lemma 3 is true also for the functions (17′), (17″). The Lagrange interpolation polynomial of the second degree $y_{L'}$ satisfying the boundary conditions in (35′) can be obtained as a uniform limit in $\langle a_1, a_3\rangle$ for $a_2 \to a_1 +$ of y_L satisfying the boundary conditions $y_L(a_1) = b_1$, $y_L(a_2) = b_1 + b_1'(a_2 - a_1)$, $y_L(a_3) = b_3$. Similar statements are true for the first and the second derivatives of $y_{L'}$. Thus Lemma 4 remains in power for $y_{L'}$, when the condition (31) is replaced by

$$
(31') \qquad |b_1| \leqq M, \left|\frac{b_3-b_1}{a_3-a_1}\right| \leqq N, \left|\frac{-b_1-b_1'(a_3-a_1)+b_3}{(a_3-a_1)^2}\right| \leqq P.
$$

In the same way, the Langrange interpolation polynomial of the second degree $y_{L''}$ satisfying the conditions in (35″) is the uniform limit in $\langle a_1, a_3\rangle$ for $a_2 \to a_3 -$ of y_L which fulfils the conditions $y_L(a_1) = b_1$, $y_L(a_2) = b_3 - b_3'(a_3 - a_2)$, $y_L(a_3) = b_3$. Since $y_{L''}' = \lim\limits_{a_2 \to a_3^-} y_L'$, $y_{L''}'' = \lim\limits_{a_2 \to a_3^-} y_L''$, Lemma 4 is valid for $y_{L''}$ when instead of (31) we assume

$$
(31'') \qquad |b_1| \leqq M, \left|\frac{b_3-b_1}{a_3-a_1}\right| \leqq N, \left|\frac{b_1-b_3+b_3'(a_3-a_1)}{(a_3-a_1)^2}\right| \leqq P
$$

Then Theorems 1 and 2 remain in power for the boundary value problems (35′) and (35″), respectively, if instead of the condition (31), the inequality (31′) and (31″), respectively, is supposed. The same is true for Corollary to Theorem 1 and Corollary 1 to Theorem 2. This completes the proof of Corollary 2 to Theorem 2.

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