Miroslav Katětov

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ON THE SPACE OF IRRATIONAL NUMBERS

Miroslav KATĚTOV, Praha

In the present note, a problem stated in the author's article " Remarks on characters and pseudocharacters " (abbreviated " Characters " in the sequel), these Commentationes, vol. 1, fasc. 1, pp. 20 - 25, is solved ; it is shown that the external character (" Characters ", 1.5) of the space $\mathcal{J}$ of irrational numbers is equal to $\mathfrak{b}$ (the least cardinal of a cofinal set in N^N).

The terminology of J. Kelley, General Topology, 1955, is used (with slight differences), as well as some notions and symbols introduced in " Characters ".

1.

We shall need some notions and lemmas concerning ordered sets. If ordered sets X , Y are isomorphic, $X \cong Y$ is written.

1.1. Definition. Let $\{X_\alpha\}_{\alpha \in A}$ be a (non-void) indexed system of ordered sets. We denote $\mathcal{P}_\alpha X_\alpha$ the cartesian product defined as usually. If every X_α has a least element u_α , then $\mathcal{P}^*_\alpha X_\alpha$, called the restricted product, is the (ordered) subset of $\mathcal{P}_\alpha X_\alpha$ consisting of $x = \{x_\alpha\}$ such that $x_\alpha = u_\alpha$ for almost all α (i.e. all $\alpha \in A - B$ with B finite).

1.2. Space always means a completely regular topological space. If P is a space, $S \subset P$, then $\mathcal{G}(S, P)$ denotes the family, ordered by inverse inclusion, of all open $H \subset P$ containing S .

1.3. Let P_α , $\alpha \in A$, be disjoint spaces. Then $\bigcup_\alpha P_\alpha$ denotes, as usually, the set $P = \bigcup_\alpha P_\alpha$ with the topology such that $G \subset P$ is open if and only if every $G \cap P_\alpha$ is so (in P_α) ; $\bigcup^*_\alpha P_\alpha$ denotes P augmented by a point ξ with defining neighborhoods of the form

$$(\xi) \cup \bigcup_{\alpha \in B} P_\alpha \quad , \quad B \subset A \quad , \quad A - B \text{ finite.}$$

If $S_\alpha \subset P_\alpha$, then $\bigcup^*_\alpha S_\alpha$ denotes, of course, the subspace $(\xi) \cup \bigcup S_\alpha$ of $\bigcup^*_\alpha P_\alpha$.

1.4. Let spaces P_α , $\alpha \in A$, be disjoint, $S_\alpha \subset P_\alpha$. Then $\mathcal{P}_\alpha \mathcal{G}(S_\alpha, P_\alpha) \cong \mathcal{G}(\bigcup_\alpha S_\alpha, \bigcup_\alpha P_\alpha)$, $\mathcal{P}^*_\alpha \mathcal{G}(S_\alpha, P_\alpha) \cong \mathcal{G}(\bigcup^*_\alpha S_\alpha, \bigcup^*_\alpha P_\alpha)$.

1.5. Let X be an additive family of sets ordered by inclusion. If $X = \bigcup_\alpha X_\alpha$, and every X_α contains the void set, then X is an image of $\mathcal{P}^*_\alpha X_\alpha$ under an isotone (i.e. order - preserving) mapping.

Proof. If $x = \{x_\alpha\} \in \mathcal{P}^*_\alpha X_\alpha$, put $\varphi(x) = \bigcup_\alpha x_\alpha$; then $\varphi(x) \in X$ since X is additive and almost all x_α are void. Clearly, φ is isotone, $\varphi(\mathcal{P}^*_\alpha X_\alpha) = X$.

1.6. We shall denote Φ the class of all ordered sets X such that for some countable metrizable P and some $S \subset P$ there exists an isotone mapping of $\mathcal{G}(S, P)$ onto X .

1.7. Lemma. If $X \in \Phi$, Y is an ordered set and there is an isotone mapping of X onto Y , then $Y \in \Phi$. If $X_n \in \Phi$, $n = 1, 2, \dots$, then $\mathcal{P} X_n \in \Phi$, $\mathcal{P}^* X_n \in \Phi$.

Proof. The first assertion is clear. If $X_n \in \Phi$, $n = 1, 2, \dots$, let P_n be disjoint countable metrizable spaces, $S_n \subset P_n$, and let φ_n be, for $n = 1, 2, \dots$, an isotone mapping of $Y_n = \mathcal{G}(S_n, P_n)$ onto X_n . Clearly, $\mathcal{P} X_n$ (respectively, $\mathcal{P}^* X_n$) is an isotone image of $\mathcal{P} Y_n$ (respectively, $\mathcal{P}^* Y_n$). Put $P = \bigcup_n P_n$, $S = \bigcup_n S_n$, $P^* = \bigcup^*_n P_n$, $S^* = \bigcup^*_n S_n$. Then P^* , P are countable metrizable and, by 1.4, $\mathcal{P}_n Y_n \cong \mathcal{G}(S, P)$, $\mathcal{P}^*_n Y_n \cong \mathcal{G}(S^*, P^*)$ which implies $\mathcal{P} X_n \in \Phi$, $\mathcal{P}^* X_n \in \Phi$.

1.8. Lemma. Let X be an additive family of sets ordered by inclusion. Suppose that $X = \bigcup_{\alpha \in A} X_\alpha$; A countable, and every X_α contains the void set. If $X_\alpha \in \Phi$ for every α , then $X \in \Phi$.

This follows directly from 1.5, 1.7.

1.9. Lemma. The cofinality character of an ordered set $X \in \Phi$ does not exceed $\mathfrak{b}$.

Proof. Let P be a countable metrizable space, $S \subset P$, φ an isotone mapping of $\mathcal{G}(S,P)$ onto X. By " Characters ", 2.2, there exists a cofinal set $Y \subset \mathcal{G}(S,P)$ with card $Y \leqq \mathfrak{b}$. Clearly, $\varphi(Y)$ is cofinal in X.

2.

2.1. " Derivatives " of a set S (in a space P) are defined in the well known way : $S^{(0)} = S$; for an ordinal $\alpha > 0$, $S^{(\alpha)}$ is the set of $x \in P$ such that, for any neighborhood U of x, all $U \cap S^{(\beta)}$, $\beta < \alpha$, are infinite.

2.2. If P is a space, $K \subset S$ is compact and dispersed (i.e. contains no dense-in-itself subset), let $h(K)$ denote the least ordinal β such that $K^{(\beta)}$ is finite. For any ordinal α, let $\mathfrak{K}(\alpha, P)$ denote the family of all compact dispersed subspaces $K \subset P$ such that $h(K) < \alpha$, and let $\mathfrak{K}(B, \alpha, P)$, $B \subset P$ finite, denote the family of those compact dispersed $K \subset P$ for which $K^{(\alpha)} \subset B \subset K$ holds. If P is the space R of rational numbers, $\mathfrak{K}(\alpha)$, $\mathfrak{K}(B,\alpha)$ is written instead of $\mathfrak{K}(\alpha, P)$, $\mathfrak{K}(B, \alpha, P)$.

2.3. For any ordinal α, $\mathfrak{K}(\alpha+1) = \bigcup_B \mathfrak{K}(B,\alpha)$, B running over all finite subsets of R ; for any limit ordinal α, $\mathfrak{K}(\alpha) = \bigcup_{\beta<\alpha} \mathfrak{K}(\beta)$.

2.4. Lemma. If $B \subset R$ is finite non-void, $0 < \alpha < \omega_1$, then $\mathfrak{K}(B, \alpha) \cong \prod_{n=1}^{\infty} X_n$ where $X_n \cong \mathfrak{K}(\alpha)$, $n = 1, 2, \ldots$.

Proof. Choose $H_n \subset R$, $n = 1, 2, \ldots$, so that $H_1 = R$, $H_n \supset H_{n+1}$, $H_n \neq H_{n+1}$, H_n are closed and open, $\bigcap_{n=1}^{\infty} H_n = B$, and every neighborhood of $\mathcal{B}$ contains some H_n ; put $M_n = H_n - H_{n+1}$. If $K \in \mathcal{K}(B, \alpha)$, i.e. if $K^{(\alpha)} \subset B \subset K$, put $\varphi(K) = \{M_n \cap K\}_{n=1}^{\infty}$;

then, clearly, $M_n \cap K \in X_n = \mathcal{K}(\alpha, M_n)$, φ is an isotone mapping of $\mathcal{K}(B, \alpha)$ into $\mathcal{P} X_n$. If $K \in \mathcal{K}(B, \alpha)$, $L \in \mathcal{K}(B, \alpha)$, $\varphi(K) = \varphi(L)$, then clearly $K - B = L - B$, hence $K = L$; thus φ is one-to-one.

If $K_n \in X_n$, $n = 1, 2, \dots$, i.e. K_n are compact, $K_n^{(\alpha)}$ are finite, $K_n \subset M_n$, put $K = \left(\bigcup_{n=1}^{\infty} K_n\right) \cup B$;

it is easy to see that $K \in \mathcal{K}(B, \alpha)$, $\varphi(K) = \{K_n\}$; thus φ maps $\mathcal{K}(B, \alpha)$ onto $\mathcal{P} X_n$. This proves the lemma since every M_n is homeomorphic with R and therefore $X_n \cong \mathcal{K}(\alpha)$.

2.5. Lemma. For any ordinal α, $0 < \alpha < \omega_1$, $\mathcal{K}(\alpha)$ belongs to the class Φ.

Proof. This is clear for $\alpha = 1$ since $\mathcal{K}(1)$ consists exactly of all finite subsets of R. Suppose that $1 < \alpha < \omega_1$ and $\mathcal{K}(\beta) \in \Phi$ for $0 < \beta < \alpha$. If α is a limit number, then $\mathcal{K}(\alpha) \in \Phi$ by 2.3 and 1.8. If $\alpha = \gamma + 1$, then, for any finite non-void $B \subset R$, $\mathcal{K}(B, \gamma) \in \Phi$ by 2.4 and 1.7, and moreover, $\mathcal{K}(\emptyset, \gamma) = \mathcal{K}(\gamma) \in \Phi$; hence, by 2.3 and 1.8, $\mathcal{K}(\alpha) = \mathcal{K}(\gamma + 1) \in \Phi$.

2.6. Theorem. The external character of the space of irrational numbers and the k-character of the space of rational numbers are both equal to b, the cofinality character of N^N.

Proof. By 2.5 and 1.9, the cofinality character of $\mathcal{K}(\alpha)$, $0 < \alpha < \omega_1$, does not exceed b. For $0 < \alpha < \omega_1$ let $\mathfrak{M}_\alpha \subset \mathcal{K}(\alpha)$ be cofinal in $\mathcal{K}(\alpha)$, card $\mathfrak{M}_\alpha \leq b$. Then $\mathfrak{M} = \bigcup_{\alpha < \omega_1} \mathfrak{M}_\alpha$ is a k-base

(see " Characters ", 1.1) of R (since every compact

$K \subset R$ belongs to some $\mathcal{R}(\alpha)$, $\alpha < \omega_1$).
Clearly, card $\mathfrak{M} \leqq \mathfrak{b}$, hence $k\chi(R) \leqq \mathfrak{b}$.
On the other hand, by " Characters ", 2.3, $\chi(\mathcal{J}, E_1) \geqq \mathfrak{b}$
and therefore $e\chi(\mathcal{J}) \geqq \mathfrak{b}$. This proves the theorem
for (" Characters ", 1.5) R, $\mathcal{J}$ are associated,
$k\chi(R) = e\chi(\mathcal{J})$.