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ON CONVERGENCE OF SEQUENCES OF FUNCTIONS

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If (Q_λ, u_λ) $(\lambda \in \Lambda)$ are spaces, we can define on a cartesian product Q of sets Q_λ a convergence of sequences by a well known way: For $x^n, x \in Q$ $(n = 1, 2, \dots)$, $x^n \xrightarrow[n]{} x$ if and only if $x^n_\lambda \xrightarrow[n]{} x_\lambda$ in the space (Q_λ, u_λ) for every $\lambda \in \Lambda$ (x_λ denotes the λ-th coordinate of the point x). This convergence defines a topology u on Q in the well known way (for $A \subset Q$ uA consists of all $x \in Q$ such that $x^n \xrightarrow[n]{} x$ for some $x^n \in A$. Following J. Novák [3], we call (Q, u) an $\mathcal{L}$-product of spaces (Q_λ, u_λ) and denote $(Q, u) = \mathcal{L}_{\lambda \in \Lambda} (Q_\lambda, u_\lambda)$. Let us point out that, following E. Čech [1] a topology u on the set Q is defined as a mapping u , which to every $M \subset Q$ assigns a set $uM \subset Q$ and satisfies the following axioms: $u\emptyset = \emptyset$, $u(x) = (x)$, $u(M_1 \cup M_2) = uM_1 \cup uM_2$. The condition $u(uM) = uM$, called axiom F by E. Čech, is not required in general; if it is satisfied, then u is called an F -topology and (Q, u) an F -space; if it does not hold, then u is called a non - F - topology and (Q, u) a non - F -space.

For any topology u on Q two further F -topologies are defined: $\tilde{u}$, the F -reduction of u ,which has an open base consisting of all $Q - uA$, $A \subset Q$; u^* , the F -modification of u , which is the finest of all F -topologies, coarser than u . Clearly $\tilde{u} = u$ or $u = u^*$ if and only if u is an F -topology.

In this note an $\mathcal{L}$-product of two-point spaces is studied. The smallest cardinal number $\aleph$ is found, for which an $\mathcal{L}$ -product of $\aleph$ two-point spaces is non - F -space, event. it is not countably compact.

It is shown, that the $\mathcal{L}$-product (Q, u) of uncountable number of two-point spaces and (Q, u^*) are not regular. Several criteria are given, when the space $(Q, \tilde{u})$ is discrete (I. - III.).

In IV.- VIII. similar questions for subspaces of the space of real-valued functions on some F-space are studied.

In the whole note proofs are omitted.

In this note, N denotes the set of all natural numbers. If A, B are sets, A^B denotes the set of all mappings of B into A. If $\alpha \in A^B$, $x \in B$ then the element, corresponding to x in the mapping α, is denoted $\alpha(x)$ or α_x.
If $\alpha \in A^B$, $C \subset B$, then $\alpha | C$ denotes the mapping of C into A, for which $\alpha | C(x) = \alpha(x)$ for all $x \in C$. $\aleph$ denotes an arbitrary cardinal number.

I. Countable compactness.

Definition:

The space (Q, u) is called countably compact, if every infinite subset of Q has a cluster point.

Theorem 1, 1:

Let (Q, u) be a space. The following properties are equivalent:

(1) (Q, u) is countably compact.

(2) (Q, u^*) is countably compact.

Theorem 1,1 does not hold for compactness only.

Definition.

Let σ be the smallest power of a system $\mathcal{A}$ of subsets of N, which has the following property:
if $S \subset N$ is infinite, then there exists $A \in \mathcal{A}$ such that the sets $S \cap A$, $S - A$ are infinite.

Theorem 1, 2:

Let (Q, u) be an $\mathcal{L}$-product of $\aleph$ two-point spaces. The following properties are equivalent:

(1) (Q, u) is not countably compact.

(2) $\aleph \geqq \sigma$.

II. F-axiom, order and regularity.

Definitions:

Let A be an infinite countable set, $\alpha, \beta \in N^A$. We write $\alpha \succ \beta$ if $\alpha(x) > \beta(x)$ for all $x \in A$, except a finite number. If $\alpha \succ \beta$ does not hold, we write $\alpha \not\succ \beta$.

We say that $\mathcal{A} \subset N^A$ is an unbounded system in N^A if for every $\gamma \in N^A$ there exists some $\alpha \in \mathcal{A}$ such that $\gamma \not\succ \alpha$.

We say that $\mathcal{A} = \{\alpha^\lambda\} \subset N^N$ is a hereditary unbounded system, if the system $\{\alpha^\lambda | A\}$ is unbounded in N^A for every infinite $A \subset N$.

We say that a set $\mathcal{A} \subset N^N$ is a chain, if it is linearly ordered by the relation $\succ$.

An unbounded system, which is also a chain, is called an unbounded chain. The existence of an unbounded chain follows from Zorn's lemma.

Definition:

Let τ_1 be the smallest power of unbounded chain. Let τ_2 be the smallest power of hereditary unbounded system.

It is clear that $\aleph_1 \leq \tau_1 \leq \tau_2 \leq 2^{\aleph_0}$.

Theorem 2,1:

Let (Q, u) be an $\mathcal{L}$-product of $\aleph$ two-point spaces. The following properties are equivalent:

(1) (Q, u) is a non - F-space.

(2) $\aleph \geq \tau_2$

Let (Q, u) be a space, $A \subset Q$. We put $u^0 A = A$, and for ordinal number α $u^\alpha A = u\left(\bigcup_{\beta < \alpha} u^\beta A\right)$.

Theorem 2,2:

Let an $\mathcal{L}$-product (Q, u) of two-point spaces be a non - F-space and not countably compact. Then there exists $A \subset Q$ such that $u^\alpha A \neq u^{\alpha+1} A$ for all countable ordinal numbers α.

Definition:

We call a space (Q, u) countably regular at a point x, if x is an R-point x) of every subspace P, of (Q, u) such that $P = T \cup A \cup (x)$, $x \notin uT$, A is countable.

We call a space (Q, u) countably regular, if it is countably regular at each of its points.

Theorem 2,3 xx):

Let (Q, u) be an $\mathcal{L}$-product of $\aleph$ two-point spaces, $\aleph \geqq \tau_1$. Then (Q, u) and (Q, u^*) are not countably regular.

Theorem 2,4:

Let (Q, u) be an $\mathcal{L}$-product of $\aleph$ two-point spaces. The following properties are equivalent:

(1) For every $x \in Q$ there exists a closed set $T \subset Q$ such that $x \notin T$, $\operatorname{card} T = \aleph_1$ and if $\mathcal{U}$ is a neighborhood of T in (Q, u), then $x \in u\,\mathcal{U}$.

(2) (Q, u) is not regular.

(3) (Q, u^*) is not regular.

(4) $\aleph \geqq \aleph_1$.

Problem: I do not know if τ_1 is the smallest cardinal number, satisfying the Theorem 2,3.

III. F - reduction.

Definition:

Let (Q, u) be a space, $\aleph$ a cardinal number. We denote by a symbol $\mathcal{O}_\aleph(Q, u)$ every collection $\{x_{\lambda, n}; \lambda \in \Lambda, n \in N\}$ of elements of Q such that

(1) $\operatorname{card} \Lambda = \aleph$

(2) There exists a point $x \in Q$ such that $x_{\lambda, n} \xrightarrow[n]{} x$ for every $\lambda \in \Lambda$.

(3) If $\{n_i\} \in N^N$, $\{\lambda_i\} \in \Lambda^N$, $\lambda_i \neq \lambda_j$ for $i \neq j$, then $x_{\lambda_i, n_i} \not\xrightarrow[i]{} x$.

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x) x is an R-point of a space (Q, u), if for every neighborhood $\mathcal{U}$ of x there exists its neighborhood $\mathcal{V}$ such that $u\,\mathcal{V} \subset \mathcal{U}$.

xx) cf [6], Theorem 1,1.

Theorem 3,1:

Let $(Q, \mathfrak{u})$ be an $\mathcal{L}$-product of $\aleph$ two-point spaces. Then $(Q, \hat{\mathfrak{u}})$ is a discrete space if and only if there exists some $C_\kappa\,(Q, \mathfrak{u})$.

Theorem 3,2:

Let $(Q, \mathfrak{u})$ be an $\mathcal{L}$-product of $\aleph$ two-point spaces. Let $(P, \mathfrak{v}) = \mathcal{L}_{\lambda \in \Lambda} (P_\lambda, \mathfrak{v}_\lambda)$, $\operatorname{card} \Lambda = \aleph$, $\operatorname{card} P_\lambda \leqq \aleph$ and every P_λ contain at least two points.
If $(Q, \hat{\mathfrak{u}})$ is a discrete space, $(P, \hat{\mathfrak{v}})$ is also a discrete space.

Theorem 3,3:

Let $(Q, \mathfrak{u})$ be an $\mathcal{L}$-product of $\aleph_1$ two-point spaces. The following properties are equivalent:

(1) $(Q, \hat{\mathfrak{u}})$ is discrete.

(2) $\aleph_1 = \tau_c$

Theorem 3,4 x):

Let $(P, \mathfrak{v}) = \mathcal{L}_{\lambda \in \Lambda} (P_\lambda, \mathfrak{v}_\lambda)$, $\operatorname{card} \Lambda = \aleph_1^{\aleph_0}$, $\operatorname{card} P_\lambda \leqq 2^{\aleph}$, every P_λ contain at least two points. Then $(P, \hat{\mathfrak{v}})$ is a discrete space.

IV. The space of continuous functions.

Now we consider an $\mathcal{L}$-product $(Q, \mathfrak{u})$ and its subspaces, where $(Q, \mathfrak{u}) = \mathcal{L}_{\lambda \in P} (Q_\lambda, \mathfrak{u}_\lambda)$ and all $(Q_\lambda, \mathfrak{u}_\lambda)$ are the spaces of real numbers E_1 (with a usual topology). We suppose that P is also a topological space and consider the space of real continuous functions on P.
In the following theorems $C(P)$ denotes the set of all real continuous functions on P, or the set of all real continuous and bounded functions on P, or the set of all mappings of P into $\langle 0, 1 \rangle$; $D(P)$ denotes any system of real functions on P. $\mathfrak{u}$ denotes a topology on $C(P)$ (event. $D(P)$) such that $(C(P), \mathfrak{u})$ (event. $(D(P), \mathfrak{u})$) is a subspace of a given $\mathcal{L}$-product.

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x) cf [6] Theorem 2,2.

Theorem 4,1:

Let $\aleph$ be a cardinal number. Let card $f(P) \leqq \aleph$ for every $f \in C(P)$. Then card $F(P) \leqq \aleph$ for every continuous mapping F of P into any separable metric space.

This Theorem implies easily:

Theorem 4,2:

Let a set $f(P)$ be countable for every $f \in C(P)$. Then $(C(P), u)$ is an F-space.

Proposition II,3 in [6] implies easily the following Theorem:

Theorem 4,3:

Let $D(P)$ satisfy the following conditions:

1) If $g \in C(E_1)$, $f \in D(P)$, then $g \circ f \in D(P)$ ($g \circ f$ denotes the composition of f and g)

2) There exists a function $f \in D(P)$ such that $f(P)$ contains a closed subset which is dense-in-itself and non-meager. Then $(D(P), u)$. is a non - F -space.

This Theorem implies easily:

Theorem 4,4:

Let P be a compact space, containing an infinite discrete normally imbedded x) subset. Then $(C(P), u)$ is a non - F -space.

V. F-reduction of a space of continuous functions.

Definition:

Let $D(P)$ be a system of real functions on P, $R(P)$ the system of all real functions on P, let $\aleph$ be a cardinal number. The symbol $j_\aleph(D(P))$ denotes every collection $\{h_{\xi,m}; \xi \in \Xi, m \in N\}$ of elements of $D(P)$ such that

1) card $\Xi = \aleph$.

2) $h_{\xi,m} \xrightarrow[m]{} 1$ for all $\xi \in \Xi$.

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x) A set R is said to be normally imbedded in a space P, if $R \subset P$ and every bounded continuous function on Q can be extended continuously to P. By discrete subset we mean simply a subset which, as a subspace, contains isolated points only.

3) if $\{m_i\} \in N^N$, $\{h_i\} \in (R(P))^N$, $\{\xi_i\} \in \Xi^N$, $\xi_i \neq \xi_j$ for $i \neq j$, then $g_i \cdot h_{\xi_i m_i} \not\to 1$.

Theorem 5,1:

Let $(C(P), \tilde{u})$ be discrete, card $C(P) = \aleph$. Then there exists $\sigma_{\aleph}(C(P))$. Let P contain a d subset X, card $X = \aleph$, and let $j_{\aleph}(C(P))$ exist there. Then $(C(P), \tilde{u})$ is discrete.

Theorem 5,2:

Let $1 \in D(P)$. Let $D(P)$ satisfy

a) $f \in D(P) \Longrightarrow \frac{1}{1+f^2} \in D(P)$

b) if $g \in C(E_1)$, g has all derivations, $g(0) = 0$, $g(1) = 1$, and $f \in D(P)$, then $g \circ f \in D(P)$.

Then the following propositions are equivalent:

(1) there exists $\sigma_{\aleph_0}(D(P))$.

(2) there exists $j_{\aleph_0}(D(P))$.

Theorems 5,1 and 5,2 imply easily the Theorem II, 1 in [6].

Theorem 5,3:

Let P be a space, containing a dense countable metrisable subset. Let every neighborhood of every point $x \in P$ contain a neighborhood of x, which is a dense-in-itself non-meager normal space.

Let $D(P) \subset C(P)$ such that:

1) $u\, D(P) = C(P)$ (i.e.: for every $f \in C(P)$ there exist $f_n \in D(P)$, $n = 1, 2, \dots$ such that $f_n \xrightarrow[n]{} f$).

2) if $A \subset P$ is closed, $y \notin A$, then there is a function $f \in D(P)$ with $f(y) = 0$, $f(x) = 1$ for all $x \in A$.

3) if $f, g \in D(P)$, then $f \cdot g \in D(P)$.

Then for any $f \in C(P) - (0)$ there exists a set $H_f \subset D(P)$ such that $u\, H_f = C(P) - (f)$ x).

If more

4) there exists a function $h \in C(P)$ such that $h \neq \theta$, and $h - f \in D(P)$ for all $f \in D(P)$,

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x) The closure $u\, H_f$ of H_f we certainly consider in the space $(C(P), u)$ only. Some non-continuous functions are the limits of sequences of points of H_f, too.

then there exists $H_0 \subset D(P)$ such that

$uH_0 = C(P) - (0)$.

This Theorem may be applied, for example, for the set of all real functions of real variables, having all derivations.

VI. Some results about the space $(C(P), u)$.

Theorem 6,1:

If a normal space P contains a locally finite disjoint system, the power of which is $\aleph_1$ (event. τ_1, event. $\omega \cdot \tau_2$), then $(C(P), u)$ and $(C(P), u^*)$ are not regular (event. $(C(P), u)$ and $(C(P), u^*)$ are not countably regular, event. $(C(P), u)$ contains a set H such that $u^{\alpha}H \neq u^{\alpha+1}H$ for all countable ordinal numbers α).

Theorem 6,2:

Let ω be the smallest ordinal number, the power of which is a regular cardinal number $\aleph$. Let every subspace of some space (Q, u) contain a dense subset, the power of which is $< \aleph$.
Then there exists an ordinal number α for every $R \subset Q$ such that $\alpha < \omega$ and $u^{\alpha}R = u^{\alpha+1}R$

Theorem 6,3:

Let P be a union of the countable number of compact metric spaces. Then every subspace of $(C(P), u)$ contains a dense countable subset.

Theorem 6,4 which is a strengthening of Theorem 1,2 in [2], follows immediately from the Theorems 6,2 and 6,3:

Theorem 6,4:

Let P be a union of the countable number of compact metric spaces.
Then, for every $H \subset C(P)$, $u^{\alpha}H = u^{\alpha+1}H$ for some countable α.

and consisting from open sets,

VII. Countable compactness of $(C(P), u)$.

$C(P)$ denotes the set of all continuous mappings of P into $\langle 0, 1 \rangle$ in this section.

Theorem 7,1:

Let a space P contain a normally imbedded discrete set of the power $\mathfrak{S}$. Then $(C(P), u)$ is not countably compact.

Theorem 7,2:

Let a normal space P contain a closed G_δ subset which is not open. Then $(C(P), u)$ is not countably compact.

Theorem 7,3:

Let $\mathfrak{S} = \aleph_1$.

a) If P is a perfectly normal space, then $(C(P), u)$ is countably compact only for a countable discrete P.

b) If P is a normal space and $(C(P), u)$ is a non-F-space, then $(C(P), u)$ is not countably compact.

VIII. Borel functions.

Let P be a perfectly normal space. $B(P)$ denotes the set of all real Borel functions on P, or the set of all real bounded Borel functions (or bounded by a certain constant), or the set of all characteristic functions of Borel subsets of P. A definition of the topology u on $B(P)$ is evident.

Theorem 8,1:

Let us suppose that a perfectly normal space P contains a normally imbedded discrete subset, the power of which is $\mathfrak{n} = \aleph^{\aleph_0}$, let and $\bar{P} \leq 2^{\mathfrak{n}}$.
Then $(B(P), \tilde{u})$ is a discrete space.

Theorem 8,2:

If a perfectly normal space P contains a Borel subset, which may be mapped continuously on a topological product of $\mathfrak{n}$ two-point spaces, and $\bar{P} \leq 2^{\mathfrak{n}}$, then $(B(P), \tilde{u})$ is a discrete space.

It is clear that the Theorems 6,1; 6,3; 6,4; 7,1 hold also for $(B(P), u)$.

The problem, raised by J. Novák, whether $(B(E_1), u)$ is regular, remains unsolved. It may be shown only, that there exists a subspace P of E_1 such that $(B(P), u)$, and $(b(P), u^*)$ are not regular (neither are they countably regular).

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