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OPEN MAPPING THEOREM AND SOLUTION OF NONLINEAR EQUATIONS IN LINEAR NORMED SPACES

(Preliminary communication)

Josef KOLOMÝ , Praha

In this note some theorems about the solution of nonlinear functional equations in linear normed spaces are given. These theorems are based on local approximation of nonlinear mappings by linear continuous mappings and on some open mapping theorems. Proofs are omitted and they will be published with some further theorems in Čas.pěst.mat.

First of all we introduce some well-known notation and definitions. Let X, Y be linear normed spaces and let $f : X \to Y$, where $f : X \to Y$ denote a mapping f from X into Y. We define $m(f)$ on $V \subset X$ as the infimum, and $M(f)$ as the supremum, of $\| f(u_1) - f(u_2) \| / \| u_1 - u_2 \|$ taken over all $u_1, u_2 \in V$ with $u_1 \neq u_2$. We shall say that Y is complete for f if, for each Cauchy sequence $\{ u_n \} \in X$, the sequence $\{ f(u_n) \}$ has a limit in Y .

<u>Definition</u>. We shall say that the mapping $\varphi : X \to X_1$, where X, X_1 are linear normed spaces, is open, if $\varphi(G)$ is open in $\varphi(X)$ for each open set $G \subset X$.

<u>Lemma 1</u>. Let X, X_1 be linear normed spaces. Let $\varphi : X \to X_1$ be a linear mapping. Then φ is open if and only if there exists a positive constant M with the following property: If $y \in \varphi(X)$, then there exist $x \in X$

such that $\varphi(x) = y$ and $\|x\| \leq M\|y\|$.

Lemma 2. (Open Mapping Theorem.) Let X, X_1 be linear normed spaces, X complete. Let $\varphi : X \to X_1$ be a linear continuous mapping. Let $\varphi(X)$ be a set of the second category in X_1. Then φ is open and $\varphi(X) = X_1$.

Let us consider the equation

$$F(x) = 0 . \tag{1}$$

Theorem 1. Let F be a mapping of X into Y, where X, Y are linear normed spaces. Let Z be a Banach space and f, g mappings such that $f : Y \to Z$, $g : Z \to X$. Let φ be a linear continuous mapping of Z onto Z and E a closed subset of Z. Furthermore, let the following conditions be fulfilled: 1) For every $z_1, z_2 \in E$ the inequality

$$\| f F(g(z_1)) - f F(g(z_2)) - \varphi(z_1 - z_2) \| \leq \alpha \| z_1 - z_2 \| \tag{2}$$

holds, where the mappings F, f are such that $m(F) = b > 0$ on $g(E) \subset X$ and $m(f) = a > 0$ on $F(g(E)) \subset Y$, $f(o) = 0$. 2) The closed ball $D = \{ z \in Z ; \| z - z_1 \| \leq r \}$, is contained in E, where z_1 is defined by the equality $y_0 = \varphi(z_1 - z_0)$, z_0 being an arbitrary element of E, y_0 being defined by $y_0 = f F(g(z_0))$, $r \geq \beta (1 - \beta)^{-1}$. $\| x_1 - x_0 \|$, $\beta = \alpha M < 1$ (M being a constant from lemma 1). Then the equation (1) has a unique solution x^* in $\overline{g(D)} \subset X$. The sequence $\{x_n\}$ defined by $x_n = g(z_n)$, $y_{n-1} = \varphi(z_n - z_{n-1})$, $y_n = y_{n-1} - f F(g(z_{n-1})) + f F(g(z_n))$ converges in the norm topology of X to x^* and

$$\| x^* - x_n \| \leq \beta^n (\| \varphi \| + \alpha) \, [ab(1 - \beta)]^{-1} \| z_1 - z_0 \| . \tag{3}$$

Theorem 2. Let X, X, Z be linear normed spaces, $F : X \to Y$, $f : Y \to Z$, $g : Z \to X$. Let φ be a linear continuous mapping of Z into Z having a continuous inverse φ^{-1} . Let $z_o \in Z$ be such that (2) holds for every $z_1, z_2 \in D$, where $D = \{z \in Z ; \|z - z_o\| \leq r\}$, $r \geq (1-\beta)^{-1} \|\varphi^{-1}\| \|y_o\|$, $\beta = \alpha \|\varphi^{-1}\| < 1$, $y_o = f F(g(z_o))$, $m(F) = b > 0$ on $\overline{g(D)} \subset X$ and $m(f) = a > 0$ on the set $\overline{F(g(D))} \subset Y$, $f(o) = 0$. If either a) Z is complete, or b) X is complete for g and $f F$ is closed, then the equation (1) has a unique solution x^* in the set $\overline{g(D)} \subset X$. The sequence $\{x_n\}$ defined by $x_n = g(z_n)$, where $z_{n+1} = z_n - \varphi^{-1} f F(g(z_n))$, converges in the norm topology of X to x^* and the inequality (3) holds with $\beta = \alpha \|\varphi^{-1}\|$.

Theorem 3. Let F be mapping defined on the bounded set $D(F) \subset X$, $F : D(F) \rightarrow Y$, $f : Y \rightarrow Z$, $g : Z \rightarrow X$, $M(g) < +\infty$, $\varphi : Z \rightarrow Z$, where X, Y, Z are linear normed spaces. Let f, φ be linear mappings, φ continuous, and such that there exist inverses f^{-1} , φ^{-1}; φ^{-1} continuous. Let $z_o \in Z$ be such that the inequality (2) holds for every $z_1, z_2 \in D$, where $D = \{z \in Z ; \|z - z_o\| \leq r\}$, $r \geq (1-\beta)^{-1} \|\varphi^{-1}\| \|f F(g(z_o))\|$, $\beta = \alpha \|\varphi^{-1}\| < 1$. Let $\overline{g(D)} \subset D(F)$. If either a) Z is complete, or b) X is complete for g and $f F$ is closed, then the conclusions of theorem 2 remain valid. Instead of (3), the error $\|x^* - x_n\|$ satisfies $\|x^* - x_n\| \leq M(g) \beta^n (1-\beta)^{-1} \|z_1 - z_o\|$.

On taking X, Y Banach spaces, $Z = X$, $g = I$ (I is identity mapping) we obtain the following

Corollary. Let X, Y be Banach spaces, $F : X \rightarrow Y$, φ a linear continuous mapping from X onto X ,

$f : Y \to X$ linear having the inverse f^{-1}. Let E be a closed subset of X. Furthermore, let the following conditions be fulfilled: 1) For every $u, v \in E$ the inequality $\| f(F(u)) - f(F(v)) - \varphi(u - v) \| \leq \alpha \| u - v \|$ holds. 2) The closed ball $D = \{x \in X ; \| x - x_1 \| \leq r\}$, where x_1 is defined by the equality $y_0 = \varphi(x_1 - x_0)$, x_0 is an arbitrary element from E, $y_0 = f(F(x_0))$, $r \geq \beta(1 - \beta)^{-1} \cdot \| x_1 - x_0 \|$, $\beta = \alpha M < 1$ (M is a constant from lemma 1), is contained in E. Then the equation (1) has a unique solution x^* in D. The sequence $\{x_n\}$ defined by $y_{n-1} = \varphi(x_n - x_{n-1})$, $y_n = y_{n-1} - f(F(x_n) - F(x_{n-1}))$ converges in the norm topology of X to x^* and $\| x^* - x_n \| \leq \beta^n (1 - \beta)^{-1} \| x_1 - x_0 \|$.

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