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Bohumil Cenkl

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THE ELLIPTIC DIFFERENTIAL OPERATORS

Bohumil CENKL, Praha

1. Sufficient conditions for vanishing of the cohomology groups, of a complex compact manifold M, with values in the sheaf of germs of holomorphic sections of a complex line bundle over M were given by K.Kodaira [4]. The conditions are formulated in terms of the characteristic class of a complex line bundle over M. In this paper a generalization of this problem is solved for a regular elliptic system of linear partial differential equations on a compact differentiable manifold M. The condition for vanishing of the cohomology groups $\mathbb{H}^{r}(M, \Theta), r > 0$, Θ being the sheaf of germs of solutions of a homogeneous regular elliptic system $\mathcal{D}$ is stated in terms of sufficient positivity of the curvature of the operator $\mathcal{D}$.

The Spencer's resolution of Θ by sheaves of germs of jet forms C^{r}, exactness of which is assumed, can be simplified in some sense. We get the so-called " β -resolution

$$0 \longrightarrow \Theta \longrightarrow \underline{B}^{0} \xrightarrow{\beta D_0} \underline{B}^{1} \xrightarrow{\beta D_0} \dots \xrightarrow{\beta D_0} \underline{B}^{n} \longrightarrow 0 ,$$

where B are differential forms and βD_0 is a first order differential operator. This resolution is equivalent to the original Spencer's resolution. Sufficient conditions for vanishing of $\mathbb{H}^{r}(M, \Theta)$, $r > 0$ are then given in terms of the β-sequence.

Considering the manifold M and the respective complex analytic vector bundles we get, on the basis of the Atiyah-Singer index theorem and the vanishing theorem, relation between the dimension of the space of global solutions of the homogeneous system and the topological index of some elliptic differential operator associated to the original operator.

The exactness of the resolutions of Θ is closely related to the existence problem for over-determined systems of elliptic differential equations, $\mathcal{D}s = f$ (see [6]). The exactness itself can be studied in connection with a local D-Neumann problem [8],[9]. The fundamental estimate for the Dirichlet integral $Q(u,u) = \|Du\|^2 + \|D^*u\|^2 + \|u\|^2$ (u is a section of C^{r} , $r > 0$) is required for the solvability of the D-Neumann problem for a finite submanifold M of a C^∞-manifold M' . The curvature of the operator $\mathcal{D}$ allows to give an explicit expression for the Dirichlet integral, and also some sufficient conditions for vanishing of the cohomology groups $\mathbf{H}^{r} = Z(\mathbb{C}^{r}) / D(\mathbb{C}^{r+1})$ in positive degrees. Here $Z(\mathbb{C}^{r})$ is the kernel of the map D : $\mathbb{C}^{r} \longrightarrow \mathbb{C}^{r+1}$, where $\mathbb{C}^{r}$ denotes the space of sections of C^{r} over M which are smooth up to the boundary of M .

Details and complete proofs will be given in the paper "Vanishing theorem for an elliptic differential operator" in the Pacific J.Math. - This work was done during the author's stay at Stanford University. -

2. We consider only manifolds, vector bundles and maps of these objects which are "smooth", i.e. C^∞ . The sheaf of

germs of smooth sections of a vector bundle E will be denoted by $\underline{E}$. Let M be an n-dimensional manifold and E , F be vector bundles of dimensions m, ℓ respectively, over M . We denote by $J_\mu(E) \to M$ the bundle of μ-jets of E . There is the natural map $j_\mu : \underline{E} \longrightarrow \underline{J_\mu(E)}$, which to a section s of E and a point $x \in M$ associates the μ-jet of s at x . Denoting by $S^\mu(T^*)$ the μ-th symmetric product we see that $E \otimes S^{\mu+1}(T^*)$ is the kernel of the natural projection $\pi : J_{\mu+1}(E) \to J_\mu(E)$. Let us denote $E^k = E \otimes \Lambda^k T^*$ for any vector bundle E . There exists a first order differential operator $D : J_{\mu+1}(E) \longrightarrow J^1_\mu(E)$ such that for any smooth function f on M and $\sigma \in \underline{J_{\mu+1}(E)}$ holds

$$D f \sigma = df \otimes \pi\sigma + f D\sigma .$$

And $D\sigma = 0$ if and only if $\sigma = j_\mu s$ for some $s \in \underline{E}$. The operator D extends uniquely to a "derivation" $D: J^k_{\mu+1}(E) \to J^{k+1}_\mu(E)$. Then $D^2 = 0$. The operator D gives rise to the formal differential δ . The operator δ is defined in such a way that the following diagram is commutative.

$$(1)\qquad \begin{array}{ccccccccc} 0 \longrightarrow & E\otimes S^{\mu+1}(T^*)\otimes\Lambda^k T^* & \longrightarrow & J_{\mu+1}(E)\otimes\Lambda^k T^* & \xrightarrow{\pi} & J_\mu(E)\otimes\Lambda^k T^* & \longrightarrow 0 \\ & \downarrow{\scriptstyle -\delta} & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D} & \\ 0 \longrightarrow & E\otimes S^{\mu}(T^*)\otimes\Lambda^{k+1} T^* & \longrightarrow & J_{\mu}(E)\otimes\Lambda^{k+1} T^* & \xrightarrow{\pi} & J_{\mu-1}(E)\otimes\Lambda^{k+1} T^* & \longrightarrow 0 \end{array}$$

Locally in the coordinate neighborhood U on M , with coordinates $x = (x^1, \dots, x^n)$ a local section $\sigma \in J^k_{\mu+1}(E)$ over U can be expressed as follows: $\sigma = \{ \sigma_q \,|\, |q| \leq \mu + 1 \}$, where $q = (q_1, \dots, q_k, \dots, q_n)$ is an ordered n-tuple of non-negative integers q_k, $|q| = q_1 + \dots + q_n$; $\sigma_q = \{ \sigma^j_q \mid 1 \leq j \leq m \}$, where

$$\sigma_q^j = \frac{1}{n!} \sum \sigma^j_{q, i_1 \dots i_n}.$$

The formal differential δ of σ is given by

$$(\delta\sigma)_q = \sum_{j=1}^{n} dx^j \wedge \sigma_{q+1_j} \,,$$

and the operator D applied to σ has the form $D\sigma =$
$= d\pi\sigma - \delta\sigma$.

A sheaf map $\mathcal{D} : \underline{E} \longrightarrow \underline{F}$, which is in trivializations of the bundles E and F over the coordinate neighborhood U given by the formula

$$\mathcal{D}s = \sum_{|\alpha| \leqq \mu_0} a_\alpha D^\alpha s \,, \quad \alpha = (\alpha_1, \dots, \alpha_m) \,,$$

$D^\alpha = \dfrac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1} \dots (\partial x^n)^{\alpha_n}}$, $s \in \underline{E|U}$, a_α are $(\ell \times m)$-matrix-valued functions, is said to be a differential operator of order μ_0 from E to F. There exists a unique bundle map $\rho_{\mu_0+\nu}$, for each $\nu \geqq 0$, which makes the diagram

(2)
$$\begin{tikzcd}
\underline{J_{\mu_0+\nu}(E)} \arrow[r, "\rho_{\mu_0+\nu}"] & \underline{J_\nu(F)} \\
\underline{E} \arrow[u, "j_{\mu_0+\nu}"] \arrow[ur, "\mathcal{D}_\nu"] \arrow[r, "\mathcal{D}"'] & \underline{F} \arrow[u, "j_\nu"']
\end{tikzcd}$$

commutative. The operator $\mathcal{D}_\nu$ is the so-called ν-th prolongation of $\mathcal{D}$.

<u>Definition 1</u>. A regular system of partial differential equations of order μ_0 given by $\mathcal{D}$ on M is the kernel R_{μ_0} of the map ρ_{μ_0} in the exact sequence

$$0 \longrightarrow R_{\mu_0} \longrightarrow J_{\mu_0}(E) \xrightarrow{\rho_{\mu_0}} F$$

if

(i) the R_μ are vector bundles over M,

(ii) the map $\pi : R_{\mu+1} \longrightarrow R_\mu$ is surjective for $\mu \geqq \mu_0$.

A smooth section $f \in \underline{E}$ such that $j_{\mu_0} f \in \underline{R_{\mu_0}}$ is called a solution of the system R_{μ_0}.

If the conditions (i) and (ii) are satisfied we speak about the regular operator $\mathcal{D}$, and we shall study such operators only.

To the differential operator $\mathcal{D}$ and any $\xi \in T^*_x$, $x \in M$, $\xi \neq 0$ there is defined the symbol $\sigma(\mathcal{D}, \xi) : E_x \to F_x$ of $\mathcal{D}$ as the composed map

$$E_x \longrightarrow E_x \otimes S^{\mu_0}(T^*_x) \longrightarrow J_{\mu_0}(E)_x \xrightarrow{\rho_{\mu_0}} F_x .$$

Locally for some $e \in E_x$. We have

$$\sigma(\mathcal{D}, \xi)e = \sum_{|\alpha| = \mu_0} a_\alpha(x)\xi^\alpha e .$$

The exact sequence

$$0 \longrightarrow g^k_\mu \longrightarrow R^k_\mu \xrightarrow{\pi} J^k_{\mu-1}(E) \longrightarrow 0$$

defines for $\mu \geqq \mu_0$ the vector bundles g^k_μ, $k = 0, 1, \dots, n$; $\mu \geqq \mu_0$.

The sequence

$$(3)\ 0 \to g_{\nu+n} \xrightarrow{\delta} g^1_{\nu+n-1} \xrightarrow{\delta} \dots \xrightarrow{\delta} g^n_\nu \longrightarrow 0, \quad \nu \geqq \mu_0$$

is not exact in general; but we still have $\delta^2 = 0$. The corresponding cohomology $H^*(g)$ is called δ-cohomology. The cohomology groups $H^{\mu,k}(g)$ vanish for $\nu \geqq \mu_0 + 1$, $k = 0, 1, 2, \dots$ if and only if the sequence

(3) is exact.

The system of regular partial differential equations R_{μ_0} (or the operator $\mathcal{D}_{\mu_0}$) is said to be involutive if the sequence (3) is exact. The invalutioness defined in this way is that one used by E.Cartan, as was proved by J.P.Serre [7].

3. Assume that $\mu \geqq \mu_1$ i.e. $H^{\mu, r}(g) = 0$, and define the vector bundles

$$A^r_\mu = \{ \xi \in g^r_\mu \mid \delta \xi = 0 \}, \quad C^r_\mu = R^r_{\mu+1} / A^r_{\mu+1}$$

for all non-negative integers r . The sections of C^r_μ over M are called jet-forms. The commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \underline{g^r_{\mu+2}} \arrow[r] \arrow[d, "-\delta"] & \underline{R^r_{\mu+2}} \arrow[r] \arrow[d, "D"] & \underline{R^r_{\mu+1}} \arrow[r] \arrow[d, "D'"] & 0 \\
0 \arrow[r] & A^{r+1}_{\mu+1} \arrow[r] & \underline{R^{r+1}_{\mu+1}} \arrow[r, "p"] & \underline{C^{r+1}_\mu} \arrow[r] & 0
\end{tikzcd}$$

defines the first-order operator $D' : R^r_{\mu+1} \longrightarrow C^{r+1}_\mu$ which factors through C^r_μ , so that we have the first order differential operator $D : \underline{C^r_\mu} \longrightarrow \underline{C^{r+1}_\mu}$ such that $D' = Dp$. We get then the Spencer's resolution

$$(4) \quad 0 \longrightarrow \Theta \xrightarrow{\iota} \underline{C^0_\mu} \xrightarrow{D} \underline{C^1_\mu} \xrightarrow{D} \dots \xrightarrow{D} \underline{C^n_\mu} \longrightarrow 0$$

the exactness of which is a non-trivial problem discussed later. Straightforward calculation gives an explicit description of C^r_μ .

<u>Proposition 1</u>. Each element $u \in C^r_\mu$, $\mu \geqq \mu_1$, $r \geqq 0$ can be represented as a pair $(\sigma, \xi) \in R^r_\mu \oplus A^{r+1}_\mu$ such that $\sigma = \pi \varphi$, $\xi = \delta \varphi$ for some element $\varphi \in R^r_{\mu+1}$, and

$$\mathcal{D}u = (d\sigma - \xi, -d\xi); \quad \mathcal{D}^2 = 0.$$

If the adjoint of the operator D (with respect to some metric) is to be considered it is useful to give representation of the elements of C^k_μ by pairs of independent elements.

Proposition 2. To a given splitting $\lambda : R^k_\mu \longrightarrow R^k_{\mu+1}$ of the exact sequence

$$0 \longrightarrow g^k_{\mu+1} \longrightarrow R^k_{\mu+1} \longrightarrow R^k_\mu \longrightarrow 0$$

there corresponds an isomorphism $C^k_\mu \cong R^k_\mu \oplus A^{k+1}_\mu$, and $\mathcal{D}u = (\mathcal{D}_0\sigma - \xi, \mathcal{D}_0(\mathcal{D}_0\sigma - \xi))$, where $u = (\sigma, \xi) \in R^k_\mu \oplus A^{k+1}_\mu$, $\mathcal{D}_0 = d - \sigma\lambda$, $\mathcal{D}^2 = 0$.

Let us introduce a riemannian metric along the fibres of the vector bundle R_μ and also some riemannian metric along the fibres of $T(M)$.

We have then on R^k_μ, $k = 0, 1, \dots, n$ the inner product $\langle \; , \; \rangle_x$, $x \in M$ and the corresponding norm $\| \; \|_x$, and the orthogonal decomposition $R^k_\mu = A^k_\mu \oplus B^k_\mu$. Let us denote by α and β the orthogonal projections of R^k_μ onto A^k_μ and B^k_μ. We have the " β -resolution"

$$(5) \quad 0 \longrightarrow \Theta \longrightarrow \underline{B^0_\mu} \xrightarrow{\beta\mathcal{D}_0} \underline{B^1_\mu} \xrightarrow{\beta\mathcal{D}_0} \dots \xrightarrow{\beta\mathcal{D}_0} \underline{B^n_\mu} \longrightarrow 0$$

which is equivalent to the Spencer's resolution in a sense of the following

Theorem 1. The diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
 & 0 \arrow[r] & \Lambda^0_\mu \arrow[r, "D"] \arrow[d] & \Lambda^1_\mu \arrow[r, "D"] \arrow[d] & \cdots \arrow[r, "D"] & \Lambda^n_\mu \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Theta \arrow[r] & C^0_\mu \arrow[r, "D"] \arrow[d] & C^1_\mu \arrow[r, "D"] \arrow[d] & \cdots \arrow[r, "D"] & C^n_\mu \arrow[d] & \\
0 \arrow[r] & \Theta \arrow[r] & B^0_\mu \arrow[r, "\beta D_0"] \arrow[d] & B^1_\mu \arrow[r, "\beta D_0"] \arrow[d] & \cdots \arrow[r, "\beta D_0"] & B^n_\mu \arrow[d] & \\
 & & 0 & 0 & & 0 &
\end{tikzcd}
$$

($\Lambda^r_\mu = A^r_\mu \oplus A^{r+1}_\mu$) is commutative, and the last row is exact if and only if the middle one is. The first row is always exact.

<u>Definition 2</u>. The differential operator $\mathcal{D} : \underline{E} \longrightarrow \underline{F}$ is said to be elliptic if for any nonzero cotangent vector $\xi \in T^*_x$ the symbol map

$$\sigma(\mathcal{D}, \xi) : E_x \longrightarrow F_x$$

is injective at each point $x \in M$.

We can characterize the ellipticity of the operator (which is understood to be regular) $\mathcal{D}$ in somewhat more convenient ways for our purposes.

<u>Proposition 3</u>. The ellipticity of an operator $\mathcal{D}$ is equivalent to any one of this following properties:

(i) the symbol of ω in the following sequence is injective,

$$0 \longrightarrow R_\mu \longrightarrow J_\mu(E) \xrightarrow{\omega} J_\mu(E)/R_\mu \longrightarrow 0 ,$$

i.e. for every $\xi \in T^*_x$, $\xi \neq 0$, $x \in M$, the composition

$$E_x \xrightarrow{i(\xi)} E_x \otimes S^{\mu}(T^*_x) \xrightarrow{\gamma} J_{\mu}(E) \xrightarrow{\omega} J_{\mu}(E)/R_{\mu}$$

is injective, where $i(\xi) : e \longrightarrow e \otimes \xi^{\mu}$;

(ii) the composition

$$g_{\mu} \longrightarrow E \otimes S^{\mu}(T^*) \longrightarrow E \otimes S^{\mu}(T^*)/i(\xi)E$$

is injective for each ξ ;

(iii) for each ξ the composition

$$g_{\mu} \xrightarrow{\sigma} g_{\mu-1} \otimes T^* \xrightarrow{\xi} g_{\mu-1} \otimes \Lambda^2 T^*$$

is injective;

(iv) the sequence

$$0 \longrightarrow \sigma(g_{\mu}) \xrightarrow{\xi} \sigma(g_{\mu} \otimes T^*)$$

is exact for each nonzero ξ .

<u>Proof</u>. (i) follows easily from the Definition 2.

(i) $\longleftrightarrow$ (ii) The fact that $\gamma \circ i(\xi) E_x \cap R_{\mu} = \emptyset$ implies that $i(\xi) E_x \cap g_{\mu} = \emptyset$. If the composition map in (ii) is not injective for some $\xi \neq 0$, and some $x \in M$, then $\gamma \circ i(\xi) e \in R_{\mu}|_x$ for some $e \in E_x$ and then $i(\xi) e \in g_{\mu}|_x$ which gives the contradiction.

(ii) $\longleftrightarrow$ (iii) is essentially proved by the fact that the kernel of the composition map $\xi \wedge \sigma$,

$$E_x \otimes S^{\mu}(T^*_x) \xrightarrow{\sigma} E_x \otimes S^{\mu-1}(T^*_x) \otimes T^*_x \xrightarrow{\xi}$$
$$\xrightarrow{\xi} E_x \otimes S^{\mu-1}(T^*_x) \otimes \Lambda^2 T^*_x$$

is $i(\xi) E_x$.

And the last statement (iv) follows from (iii).

Ellipticity of an involutive operator is characterized by

Proposition 4. The prolongation $\mathcal{D}_\mu$, $\mu \geq \mu_1$ of the operator $\mathcal{D}$ of order μ_0 is an involutive operator, and it is an elliptic operator if and only if the map

$$\rho \circ \sigma : \tilde{\omega}^* g_{\mu+1} \longrightarrow \tilde{\omega}^* A^2_\mu$$

as a composition of

$$\rho : \tilde{\omega}^* A^1_\mu \longrightarrow \tilde{\omega}^* A^2_\mu \quad \text{and} \quad \sigma : \tilde{\omega}^* g_{\mu+1} \longrightarrow \tilde{\omega}^* A^1_\mu$$

(where $\tilde{\omega}^* : S^*(M) \longrightarrow M$ is the unit cotangent sphere bundle) is injective.

This statement is a consequence of Proposition 3.

Let us denote by d^* the (formal) adjoint to d with respect to the scalar product

$$(\;,\;) = \int \langle \;,\; \rangle \, dM .$$

We can now state

Lemma 1. ([18]) The ellipticity of an operator $\mathcal{D}$ is equivalent to the existence of a positive number c such that for each section $\xi \in A^r_\mu$, $\mu \geq \mu_1$, $1 \leq r \leq n$, over any coordinate neighborhood $U \subset M$ with compact support, the following inequality holds:

$$(6) \quad (\{ d(\alpha d^*) + (\alpha d^*) d \} \xi , \xi) \geq c \, \| \xi \|_{(1)} - \| \xi \|^2 .$$

Proof. From Proposition 3 follows that the ellipticity of an operator $\mathcal{D}$ is equivalent to the exactness of the short sequence

$$(7) \qquad 0 \longrightarrow A^1_\mu \xrightarrow{\xi} A^2_\mu .$$

But D.G.Quillen ([6]) proved that the exactness of $0 \longrightarrow$ $\longrightarrow C^0_\mu \xrightarrow{\sigma(D,\xi)} C^1_\mu$ is equivalent to the exactness of the last row of the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & 0 \arrow[d] & \\
 & 0 \arrow[r] & A^1_\mu \arrow[r, "-\xi"] \arrow[d] & {} \arrow[r, "-\xi"] & A^n_\mu \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R^0_\mu \arrow[r, "\xi"] & R^1_\mu \arrow[r, "\xi"] \arrow[d] & \cdots \arrow[r, "\xi"] & R^n_\mu \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^0_\mu \arrow[r, "\sigma(D,\xi)"] & C^1_\mu \arrow[r, "\sigma(D,\xi)"] \arrow[d] & \cdots \arrow[r, "\sigma(D,\xi)"] & C^n_\mu \arrow[r] \arrow[d] & 0 \\
 & & 0 & & 0 &
\end{tikzcd} \tag{8}$$

in stable range. This proves that the exactness of (7) is equivalent to the exactness of the first row in (8) for all $\mu \geq \mu_1$. Then there exists a positive constant c' such that for each nonzero $\xi \in T^*_x$ and any $\lambda \in A^r_\mu |_x$, $0 \leq r \leq$ $\leq n$, the inequality

$$|\xi \lambda|^2_x + |(\alpha \xi^*)\lambda|^2_x \geq c' |\xi|^2_x |\lambda|^2_x \tag{9}$$

holds at each point $x \in M$. Using Fourier transform we obtain the equivalence of (9) and (6).

On the basis of the " β -sequence" we get

<u>Lemma 2</u>. The ellipticity of an operator $\mathcal{D}$ is equivalent to the existence of a constant c, $0 < c < 1$, such that

$$\| (\beta d^*)\xi \|^2 \leq (1-c) \sum_j \| d_j \xi \|^2 \tag{10}$$

for any $\xi \in A^r_\mu$, $\mu \geq \mu_1$, $r \geq 1$ with compact support in U, $\sum dx^j \wedge d_j \xi = d\xi$.

<u>Proof</u>. Let us notice first of all that for $\mu \geq \mu_1$ the exactness of the sequence

$$0 \longrightarrow B^0_\mu \xrightarrow{\beta\xi} B^1_\mu \xrightarrow{\beta\xi} B^2_\mu \xrightarrow{\beta\xi} \dots \xrightarrow{\beta\xi} B^n_\mu \longrightarrow 0$$

is equivalent to the ellipticity of the operator $\mathcal{D}$. But from the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & g^0_\mu \arrow[r] \arrow[d] & \beta(g^1_\mu) \arrow[r] \arrow[d] & \dots \arrow[r] & \beta(g^n_\mu) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B^0_\mu \arrow[r] \arrow[d] & B^1_\mu \arrow[r] \arrow[d] & \dots \arrow[r] & B^n_\mu \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R^0_{\mu-1} \arrow[r] \arrow[d] & R^1_{\mu-1} \arrow[r] \arrow[d] & \dots \arrow[r] & R^n_{\mu-1} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & & 0 &
\end{tikzcd}$$

follows that the first row is exact if and only if $\mathcal{D}$ is elliptic. And now similar argument as in the proof of Lemma 1 gives the statement.

4. Before we are able to give the proof of the main result we have to consider that differential geometric aspect of the regular differential operator $\mathcal{D}$ of order μ on the compact n-dimensional C^∞-manifold M .

Let $\{\omega^1, \dots, \omega^n\}$ be the orthogonal coframe in the coordinate neighborhood $U \subset M$ with respect to the chosen riemannian metric on M . This means that $\omega^i \cdot \omega^k = \delta^i_k$; $i, k = 1, 2, \dots, n$. Let $\omega^i = a^i_k \, dx^k$, $dx^i = b^i_k \, \omega^k$, where for the C^∞-functions holds $a^k_j \, b^i_k = \delta^i_j$. We shall use the symbol $\frac{\partial}{\partial \omega^i}$ for the cotangent vector dual to ω^i and also for the derivation in the exterior algebra of differential forms on M . Let us denote by ∇ , $\nabla = \omega^i \wedge \nabla_i$, the covariant differential with respect to the metric connection on M . We have then from the fact $\nabla \omega^i = 0$, $i = 1, 2, \dots, m$ the identity $d\omega^i = \pi^i_{jk}\, \omega^k \wedge \omega^j$. The curvature form of the riemannian metric on M is the 2-form $R = \nabla(\pi)$; $\pi = (\pi^i_{jk}\, \omega^k)$ being the connection form. We shall use the usual star operation " $*$ " on forms with values in a vector bundle, the symbol π for interior product. The volume element of M will be denoted by $*(1) = \omega^1 \wedge \dots \wedge \omega^n$.

Let $\{ e_1(x), \dots, e_k(x) \}$ be a frame in the fibre of A^o_μ over $x \in U$, and $\{ e_{k+1}(x), \dots, e_m(x) \}$ a frame of B^o_μ at x . Then the choice of the frame $\{ e_1(x), \dots, e_m(x) \}$ in $R^r_\mu |_x$ gives a trivialization of R^r_μ over U for all r , and any section $\sigma \in \underline{R^r_\mu | U}$ can be written in the form

$$\sigma = \frac{1}{r!} \sum_{i_1, \dots, i_r} \sum_\alpha \sigma^\alpha_{i_1 \dots i_r}\, \omega^{i_1} \wedge \dots \wedge \omega^{i_r} \otimes e_\alpha \ .$$

The matrix $a = a(x) = (a_{\alpha\beta}(x))$ is the matrix of the metric tensor, with components $a_{\alpha\beta}(x) = \langle e_\alpha(x), e_\beta(x) \rangle$. Remember that the global product on M is given by the formula $(\sigma, \rho) = \int_M \sigma \wedge a * \rho$ for any sections σ, ρ of R^r_μ . The 1-form $\Theta = \frac{1}{2} a^{-1} da$ is the connection form of the metric connection uniquely associated to the riemannian metric along the fibres of R_μ . The curvature form of this connection is given by the formula $\Pi = d\Theta + \Theta \wedge \Theta$.

We have seen already that Λ is the 1-form of connection on R_μ , which is given by the covariant differential $D_o = d + e(\Lambda)$. Let us define further the operators

$\mathcal{D}_\Gamma : \underline{R^k_\mu} \longrightarrow \underline{R^{k+1}_\mu}$ by the formula

$$\mathcal{D}_\Gamma = \mathcal{D}_0 + e(\Gamma), \quad \Gamma = a^{-1}\mathcal{D}_0 a .$$

The operator D_Γ is again the covariant differential of some connection on R_μ. Considering the commutator of D_0 and D_Γ, we define a local 2-form $\Omega = \mathcal{D}_0 \mathcal{D}_\Gamma + \mathcal{D}_\Gamma \mathcal{D}_0$. Using the orthogonal coframe introduced above, we have

$$\mathcal{D}_\Gamma = \omega^j \wedge \mathscr{D}_j, \quad \mathcal{D}_0 = \omega^j \wedge D_j .$$

Then for each $\sigma \in \underline{R^k_\mu}$

$$(\mathscr{D}_j D_k - D_k \mathscr{D}_j)\, \sigma^\alpha_{i_1 \dots i_k} = \Omega^\alpha_{\beta j k}\, \sigma^\beta_{i_1 \dots i_k} .$$

Definition 3. The 2-form Ω is called the curvature form of the operator $\mathscr{D}$ and the corresponding tensor the curvature tensor of $\mathscr{D}$.

Let D^*_0 be the adjoint operator to D_0 with respect to the global product $(\, , \,)$. Then for $\sigma \in \underline{R^k_\mu}$ the Weitzenböck formula has the form

(11)
$$\{(\mathcal{D}_0 D^*_0 + D^*_0 \mathcal{D}_0)\,\sigma\}^\alpha_{i_1 \dots i_k} = -\sum_{j=1}^{n} \mathscr{D}_j D_j \sigma^\alpha_{i_1 \dots i_k} +$$
$$+ \sum_{s=1}^{k} (\Omega^\alpha_{\beta j i_s} - \sigma^\alpha_\beta R^j_{i_s})\, \sigma^\beta_{i_1 \dots i_{s-1} j i_{s+1} \dots i_k} -$$
$$- \frac{1}{2} \sum_{i_\mu, i_\nu, j} R^e_{i_\mu i_\nu j}\, \sigma^\alpha_{i_1 \dots i_{\mu-1} e i_{\mu+1} \dots i_{\nu-1} j i_{\nu+1} \dots i_k}) ,$$
$$R^j_{i_s} = \sum_{k=1}^{n} R^j_{k i_s k} ;$$

or symbolically

$$\square_0 \sigma = -\sum \mathscr{D}_j D_j \sigma - \frac{1}{2} R\sigma - \hat{R}\sigma + \Omega\sigma .$$

We shall use the notation $\Omega(\sigma, \sigma) = (\Omega\sigma, \sigma)$, $\hat{R}(\sigma, \sigma) = (\hat{R}\sigma, \sigma)$, $R(\sigma, \sigma) = (R\sigma, \sigma)$.

Lemma 3. For each element $\sigma \in R^n_\mu$ the folllowing identity holds:

$$(12)\quad \sum_{j,k=1}^{n}(\{\mathcal{D}_j D_k - D_k \mathcal{D}_j\}(\sigma \pi \frac{\partial}{\partial \omega^j}), \sigma \pi \frac{\partial}{\partial \omega^k}) = \Omega(\sigma,\sigma) - \hat{R}(\sigma,\sigma) - \frac{1}{2} R(\sigma,\sigma).$$

Proof. For any $\sigma \in R^n_\mu$ we get in U the formulas

$$\{\mathcal{D}_j D_k(\sigma \pi \frac{\partial}{\partial \omega^j})\}^\alpha = \frac{1}{(n-1)!} \sum \{\sigma^\alpha_{j i_1 \dots i_{n-1};k;j} + \sigma^\beta_{j i_1 \dots i_{n-1};k} \Gamma^\alpha_{\beta j}\} \cdot$$
$$\cdot\, \omega^{i_1} \wedge \dots \wedge \omega^{i_{n-1}},$$

$$\{D_k \mathcal{D}_j(\sigma \pi \frac{\partial}{\partial \omega^j})\}^\alpha = \frac{1}{(n-1)!} \sum \{\sigma^\alpha_{j i_1 \dots i_{n-1};j;k} + \Gamma^\alpha_{\beta j;k} \sigma^\beta_{j i_1 \dots i_{n-1}} +$$
$$+ \Gamma^\alpha_{\beta j} \sigma^\beta_{j i_1 \dots i_{n-1};k} - \Lambda^\beta_{\gamma k} \Gamma^\alpha_{\beta j} \sigma^\gamma_{j i_1 \dots i_{n-1}}\} \omega^{i_1} \wedge \dots \wedge \omega^{i_{n-1}} .$$

This proves the statement.

If M is a finite submanifold of the C^∞-manifold M', the closure $\overline{M}$ is compact, and the boundary bM of M is regularly imbedded C^∞-submanifold of M' of codimension 1. Let r(x) be the distance function, then we have the formulas $dr \wedge * dr = dr \cdot dr * (1) = * (1)$, $dr = \frac{\partial r}{\partial \omega^k} \omega^k$.

We have for each $\sigma \in R^n_\mu | U$ the decomposition $\sigma = t\sigma + n\sigma$, where $t\sigma$ is tangential and $n\sigma$ normal at each $x \in bM$. On the basis of the formula

$$t(\omega^1 \wedge \dots \wedge \omega^l \wedge \dots \wedge \omega^n) = (-1)^{l-1} \frac{\partial r}{\partial \omega^l} * (dr)$$

we get the "integration by parts".

Proposition 5. For any C^∞-functions a, f, g holds on a finite manifold M the identity

$$\int \frac{\partial f}{\partial \omega^j} a \frac{\partial g}{\partial \omega^k} * (1) = \int_M \frac{\partial f}{\partial \omega^k} a \frac{\partial g}{\partial \omega^j} * (1) + \int_{bM} fa [\frac{\partial g}{\partial \omega^k} \frac{\partial r}{\partial \omega^j} - \frac{\partial g}{\partial \omega^j} \frac{\partial r}{\partial \omega^k}] * (dr)$$
$$+ \int_M fa (c^k_{jk} - c^k_{kj}) \frac{\partial g}{\partial \omega^k} * (1) + \int_M fa (\frac{\partial g}{\partial \omega^j} S_k - \frac{\partial g}{\partial \omega^k} S_j) * (1) +$$
$$+ \int_M (\frac{\partial f}{\partial \omega^k} \frac{\partial a}{\partial \omega^j} g - f \frac{\partial a}{\partial \omega^j} \frac{\partial g}{\partial \omega^k}) * (1) .$$

5. The exactness of the sequence (4) or (5) is a consequence of solvability of so-called Neumann problem, which will be discussed in this part.

Let $\mathcal{D}$ be an elliptic operator and M a finite submanifold as above. D^* be the formal adjoint to the operator D. We use the following notation:

$\mathbb{C}$: the restriction of the space of sections of $C_\mu = \oplus_n C^n_\mu$ to $\overline{M}$, elements which are smooth up to and including the boundary;

$\mathbb{C}_0$: the completion of $\mathbb{C}$ in the norm $\| \ \|$;

$\mathcal{D}, \mathcal{D}^*$: the extension to $\mathbb{C}_0$ of the operators D, D^* in Spencer's sequence;

$\mathbb{N}$: the elements $u \in \mathbb{C}_0$ such that Du lies in the domain of D^*, and D^*u lies in the domain of D;

$\mathbb{H}$: the subspace of $\mathbb{N}$ composed of the elements of $\mathbb{N}$ which are annihilated by the Fridrichs extension L of $\square = \mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D}$ on $\mathbb{N}$.

<u>Definition 4</u>. We say that the D-Neumann problem is solvable for a finite manifold M, and the elliptic differential operator $\mathcal{D}$, if $L\,\mathbb{N}$ is closed in $\mathbb{C}_0$ and the Neumann operator

(13) $$N : \mathbb{C}_0 \longrightarrow \mathbb{N}$$

commutes with D.

<u>Remark</u>. If $L\,\mathbb{N}$ is closed in $\mathbb{C}_0$ we have the orthogonal decomposition $\mathbb{C}_0 = \mathcal{D}\mathcal{D}^*\,\mathbb{N} \oplus \mathcal{D}^*\mathcal{D}\,\mathbb{N} \oplus \mathbb{H}$, and we define the mapping (13) by the relation $Nu = w - Hw$, where $H : \mathbb{C}_0 \longrightarrow \mathbb{H}$ is an orthogonal projection and

$u = Lw + Hw, \quad w \in N.$

Proposition 6. The D-Neumann problem is solvable if and only if the operators H and N have the following properties: N is a self-adjoint bounded operator satisfying NH = HN = 0 ,

$$u = DD^{*}Nu + D^{*}DNu + Hu$$

for each $u \in \mathbb{C}_0$, and

$$ND = DN \; .$$

Proof. Follows essentially the lines of the proof of Proposition 2.8 ([5]).

From the work of J.J.Kohn and L.Nirenberg follows that if for each $u \in \mathbb{C}_0$ which is in the intersection of the domains of D and D* holds for some $\varepsilon > 0$ the inequality

$$(14) \qquad (Du, Du) + (D^{*}u, D^{*}u) + (u, u) \geq \varepsilon \int_{bM} |u|_x^2 * (dr)$$

then the D-Neumann problem is solvable and

$$\mathbb{H} \cong Z(\mathbb{C}_0)/D(\mathbb{C}_0) \; .$$

The solvability of the D-Neumann problem implies the isomorphism $\mathbf{H}^r \cong \mathbb{H}^r$, where $\mathbf{H} = \oplus_r \mathbf{H}^r$ is the space of elements $u \in \mathbb{C}$, which satisfy the conditions

$$(15) \qquad \begin{aligned} (D^{*}u, v) &= (u, Dv) \; , \\ (D^{*}Du, v) &= (Du, Dv) \end{aligned}$$

for all $v \in \mathbb{C}$, and which are annihilated by the laplacian $\square$.

Spencer's conjecture. (The local Neumann problem): If $\mathcal{D}_{u_0}$ is an elliptic operator, then the D-Neumann problem is solvable on sufficiently small spherical neighborhood of euclidean n-space and, for these domains $\mathbf{H}^r = 0$ for $r \geq 1$.

The problem then remains, to prove an estimate of the form (14).

6. The harmonic space $\mathbf{H}$ is the subspace of $\mathbb{C}$, which elements satisfy both boundary conditions and which are annihilated by the laplacian $\square$. For the fundamental estimate is essential the explicit expression of the Dirichlet integral, when the elements $u \in \mathbb{C}$ are smooth up to and including the boundary and satisfy only the first boundary condition $(D^*u, v) = (u, Dv)$ for all $v \in \mathbb{C}$.

Let us use the notation

$$d_{kj} = (\Lambda_k + \tilde{\Lambda}_k + \tilde{\Gamma}_k - \tilde{\theta}_k)\mathcal{D}_j - (\tilde{\Lambda}_j + \Lambda_j + \Gamma_j - \tilde{\theta}_j)D_k ,$$

$$\tilde{\Lambda} = a\Lambda a^{-1}, \ \tilde{\Gamma} = a\Gamma a^{-1}, \ \tilde{\theta} = a\,\theta a^{-1}, \ \alpha_k = \alpha D_k - D_k\alpha ,$$

$$\tilde{\alpha}_k = \alpha\mathcal{D}_k - \mathcal{D}_k\alpha , \quad L_{ij} = \nabla_j \nabla_i r .$$

We say that the finite manifold M is "strongly pseudoconvex" if at each point of bM holds

$$(16) \qquad \sum_{i,j=1}^{n} L_{ij} \langle \alpha(\xi\pi\frac{\partial}{\partial\omega^i}), \ \alpha(\xi\pi\frac{\partial}{\partial\omega^j})\rangle > 0 .$$

The quadratic forms $\tilde{\Omega}(\xi,\xi)$, $\hat{\mathcal{R}}(\xi,\xi)$, $\mathcal{R}(\xi,\xi)$ are defined by the formula

$$\tilde{\Omega}(\xi,\xi) - \hat{\mathcal{R}}(\xi,\xi) - \frac{1}{2}\mathcal{R}(\xi,\xi) =$$

$$= \sum_{j,k}(\{\mathcal{D}_j D_k - D_k\mathcal{D}_j\}\alpha(\xi\pi\frac{\partial}{\partial\omega^j}), \alpha(\xi\pi\frac{\partial}{\partial\omega^k})) .$$

Lemma 4. For any $u \in \mathbb{C}$ which satisfies the first boundary condition (15), we have the following identity:

$$\|Du\|^2 + \|D^*u\|^2 + \|u\|^2 = K(u,u) + T(u,u) + \int_{bM} L(u,u) * (dr) ,$$

where

$$K(u,u) = \Omega(\sigma,\sigma) - \hat{R}(\sigma,\sigma) - \frac{1}{2}R(\sigma,\sigma) + \tilde{\Omega}(\xi,\xi) - \hat{\mathcal{R}}(\xi,\xi) - \frac{1}{2}\mathcal{R}(\xi,\xi),$$

$$L(u,u) = \sum_{i,j=1}^{n} L_{ij}\langle \alpha(\xi\pi\frac{\partial}{\partial\omega^i}), \alpha(\xi\pi\frac{\partial}{\partial\omega^j})\rangle + \sum_{i,j=1}^{n} L_{ij}\langle \sigma\pi\frac{\partial}{\partial\omega^i}, \sigma\pi\frac{\partial}{\partial\omega^j}\rangle$$

$$T(u,u) = \sum_{k=1}^{n}\|D_k\sigma\|^2 + \sum_{k=1}^{n}\|D_k\xi\|^2 - (\beta D_j(\xi\pi\frac{\partial}{\partial\omega^k}), \beta D_k(\xi\pi\frac{\partial}{\partial\omega^j})) + (d_{kj}(\sigma\pi\frac{\partial}{\partial\omega^j}), \sigma\pi\frac{\partial}{\partial\omega^k}) + (S_k D_j(\sigma\pi\frac{\partial}{\partial\omega^k}), \sigma\pi\frac{\partial}{\partial\omega^j}) - 2(D_j\sigma\pi\frac{\partial}{\partial\omega^k}, \sum C_{kj}^{\ell}(\sigma\pi\frac{\partial}{\partial\omega^\ell})) - (\mathcal{D}_j(\sigma\pi\frac{\partial}{\partial\omega^j}), S_k(\sigma\pi\frac{\partial}{\partial\omega^k})) + (d_{kj}\,\alpha(\xi\pi\frac{\partial}{\partial\omega^j}), \alpha(\xi\pi\frac{\partial}{\partial\omega^k})) + (S_k\mathcal{D}_j\,\alpha(\xi\pi\frac{\partial}{\partial\omega^k}), \alpha(\xi\pi\frac{\partial}{\partial\omega^j})) - 2(D_j\xi\pi\frac{\partial}{\partial\omega^k}, \sum C_{kj}^{\ell}(\xi\pi\frac{\partial}{\partial\omega^\ell})) - (\mathcal{D}_j\,\alpha(\xi\pi\frac{\partial}{\partial\omega^j}), S_k\alpha(\xi\pi\frac{\partial}{\partial\omega^k})) - 2(D_j\,\alpha(\xi\pi\frac{\partial}{\partial\omega^k}), \alpha_k(\xi\pi\frac{\partial}{\partial\omega^j})) + (\mathcal{D}_k\alpha(\xi\pi\frac{\partial}{\partial\omega^k}), \tilde{\alpha}_j(\xi\pi\frac{\partial}{\partial\omega^j})) - (\sum c_{jk}^{h}(\sigma\pi\frac{\partial}{\partial\omega^h}), \sum c_{kj}^{\ell}(\sigma\pi\frac{\partial}{\partial\omega^\ell})) - (\sum c_{jk}^{h}(\xi\pi\frac{\partial}{\partial\omega^h}), \sum c_{kj}^{\ell}(\xi\pi\frac{\partial}{\partial\omega^\ell})) -$$

$$- (\alpha_j(\xi\pi\frac{\partial}{\partial\omega^k}), \alpha_k(\xi\pi\frac{\partial}{\partial\omega^j})) + (\tilde{\alpha}_j(\xi\pi\frac{\partial}{\partial\omega^j}), \tilde{\alpha}_k(\xi\pi\frac{\partial}{\partial\omega^k})) + \|\sigma\|^2 + 2\|\xi\|^2 + \|D_0^2\sigma\|^2 + \|D_0^{*2}\xi\|^2 + \|\alpha\sigma\|^2 + 2\{(D_0^*\sigma, D_0^{*2}\xi) + (\alpha\sigma, \alpha D_0^*\xi) - (\xi, D_0\sigma) - (D_0\xi, D_0^2\sigma)\} - \int_M \frac{\partial}{\partial\omega^k}\langle \pi_j(\sigma\pi\frac{\partial}{\partial\omega^k}), \sigma\pi\frac{\partial}{\partial\omega^j}\rangle * (1) - \int_M \frac{\partial}{\partial\omega^j}\langle \pi_j\,\alpha(\xi\pi\frac{\partial}{\partial\omega^k}), \alpha(\xi\pi\frac{\partial}{\partial\omega^j})\rangle * (1).$$

Theorem 2. If the curvature of the elliptic operator $\mathcal{D}$ is such that the quadratic form K(u,u) is sufficiently positive for all $u \in \mathbb{C}^n$, $n \geq 1$, and if both conditions

(15) and condition (16) are satisfied, then $H^r = 0,\ r \geq 1$.

__Proof.__ Investigation of the expression

$$K(u,u) + T(u,u) + \int_{\partial M} L(u,u) * (dr) = \|u\|^2$$

gives the statement.

The main result concerns the cohomology groups $\mathbb{H}^r$ for a compact manifold, assuming that $\mathcal{D}$ is an elliptic operator. On the basis of the resolution (4) and (5)(which is assumed to be exact) of the sheaf Θ, and using de Rham's and Hodge's theorem we conclude that it is enough to investigate the harmonic elements of the graded vector space $C = \oplus_r C^r$ of sections of $C_\mu = \oplus_r C^r_\mu$ over the manifold M. We get

__Lemma 5.__ If $\mathcal{D}$ is an elliptic operator, then for any harmonic jet form $u = (\sigma, \xi) \in C^r,\ r \geq 1$, there exist positive constants K_1, K_2 such that

$$K(u,u) \leqq K_1 \|\sigma\|^2 + K_2 \|\xi\|^2 ,$$

where

$$K(u,u) = \Omega(\sigma,\sigma) - \frac{1}{2} R(\sigma,\sigma) + \tilde{\Omega}(\xi,\xi) - \frac{1}{2} \mathcal{R}(\xi,\xi) - \hat{R}(\sigma,\sigma) - \hat{\mathcal{R}}(\xi,\xi) .$$

__Proof.__ Let us notice first of all that for any harmonic jet form $u(\sigma,\xi) \in C^r,\ r \geq 1$, where (σ,ξ) is an obvious representation of u, we get the formulas

$$\Omega(\sigma,\sigma) - \frac{1}{2} R(\sigma,\sigma) - \hat{R}(\sigma,\sigma) = -\sum_j \|D_j \sigma\|^2 + \cdots ,$$

$$\tilde{\Omega}(\xi,\xi) - \frac{1}{2} \mathcal{R}(\xi,\xi) - \hat{\mathcal{R}}(\xi,\xi) = -\sum_j \|D_j \xi\|^2 + \|_\beta D_0^* \xi\|^2 + \cdots ,$$

where . . . denotes the half order terms. From Lemma 2 we have the formula

$$\| \beta D_0^* \xi \|^2 - \sum_j \| D_j \xi \|^2 = - \sum_j \| d_j \xi \|^2 + (\beta d_k (\xi \pi \frac{\partial}{\partial x^k}),$$

$$\beta d_k (\xi \pi \frac{\partial}{\partial x^j})) + \cdots \quad .$$

Applying these identities to the formula for K(u,u) , we get the statement after some investigation.

Theorem 3. Let M be a compact manifold and $\mathcal{D}$ an elliptic differential operator. If the quadratic form K(u,u) is sufficiently positive for all $u \in C^r$, $r \geq 1$, then $\mathbb{H}^r(M, \Theta) = 0$, $r \geq 1$.

Considering the β-resolution of the sheaf Θ we get another form. The cohomology $\mathbb{H}(M, \Theta)$ is isomorphic to the space $B = \oplus_r B^r$ of sections of $B_\mu = \oplus_r B^r_\mu$ over M , which are annihilated by the laplacian $D_0^* \beta D_0 + \beta D_0 D_0^*$. We have analogy of the Weitzenböck formula. Let

$$K_\beta(\varphi, \varphi) = (\{ \mathcal{D}_j D_k - D_k \mathcal{D}_j \} \beta (\varphi \pi \frac{\partial}{\partial \omega^j}), \ \beta (\varphi \pi \frac{\partial}{\partial \omega^k})) \quad ,$$

and

$$K_\beta(\varphi, \varphi) = \Omega_\beta(\varphi, \varphi) - \frac{1}{2} R_\beta(\varphi, \varphi) - \hat{R}_\beta(\varphi, \varphi) \ .$$

Then by similar reasoning to that one we have made earlier the following statement can be proved.

Theorem 4. Let M be a compact manifold and $\mathcal{D}$ an elliptic operator. If the quadratic form $\Omega_\beta(\varphi, \varphi)$ is sufficiently positive in the sense that $K_\beta(\varphi, \varphi)$ is sufficiently positive, for any $\varphi \in B^r$, $r \geq 1$, then for $r \geq 1$

$$H^r(M, \Theta) = 0 \ .$$

It can be proved that the Kodaira's vanishing theorem is a special case of these statements.

Let us assume that E and F are differentiable complex vector bundles over a compact differentiable manifold M ,and the elliptic differential operator $\mathcal{D} : \underline{E} \longrightarrow \underline{F}$ as above. We can state then

Theorem 5. If the quadratic form K(u,u) is sufficiently positive for all $u \in C^{r}$, $r \geq 1$, then $\dim \mathbb{H}^{0}(M, \Theta)$ is equal to the topological index of the differential operator D + D* .

Proof. Let us consider

$$C_0 = \oplus_r C^{2r} , \quad C_1 = \oplus_r C^{2r+1} .$$

Then

$$D + D^* : C_0 \longrightarrow C_1 ,$$

and the adjoint operator D + D* maps C_1 into C_0 . The Euler-Poincaré characteristic $\chi(M, \Theta) = \sum_{r=0}^{m} (-1)^r \cdot \dim H^r(M, \Theta)$ is the analytic index of D + D* . And by the Atiyah-Singer theorem this is equal to the topological index of the operator D + D* . This gives the statement.

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