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THE SOLVABILITY OF NONLINEAR INTEGRAL EQUATIONS

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1. In this remark we continue the investigations [1] on solutions of nonlinear integral equations. In [1] we gave some conditions for the solvability of Hammerstein integral equations in L_2-space. The purpose of this note is to investigate the equation $u - Ahu = 0$, where $A: L_q \to L_p$ $(1 < q < 2,\ p^{-1} + q^{-1} = 1)$ is a linear continuous mapping of L_q into L_p and $h(u) = g(u(x), x)$ is an operator of Nemyckij such that h is a mapping of L_p into L_q. To the end of this note we shall also consider Urysohn integral equations in L_2-space. Some recent works in this subject are cited in [1].

First, I must correct the misprint from [1]. In theorem 7 and remark 3 [1] must be $mes\, G = \infty$ (instead of $mes\, G < \infty$). The Golomb-Vajnberg theorem also holds for the domains G with $mes\, G = \infty$, cf. [2].

The theorems 5,7,8 and corollaries 1,3 [1] hold in more general form. We can suppose only that $N \leqq g'_x(x, t) \leqq M$, $N, M = const$ and $\lambda M \|A\| \leqq 1$ or $\lambda M \|A\| < 1$ (or $M \|A\| \leqq 1$) is satisfied if $M > 0$. If $M < 0$, then these assumptions are unnecessary. Moreover, we can consider in theorems 5,8,9 [1] the following more general equations: $x - \lambda A\phi(x) = f$, $x - A\phi(x) = 0$ $A\phi(x) = 0$ instead of $x(s) - \lambda \int_G K(s,t) g(x(t), t)\,dt = f(s)$, $x(s) -$

$-\int\limits_G K(s,t)g(x(t),t)dt=0, \int\limits_G K(s,t)g(x(t),t)\,dt = 0$, *respectively*.

2. Let X, Y be real Banach spaces. A mapping $F: X \to Y$ is said to be bounded if F transforms bounded sets in X into bounded sets in Y. It is well known that an uniformly continuous (nonlinear) mapping $F: D_R \to Y$, $D_R = \{u \in X : \|u\| \leqq R\}$ is bounded on D_R. A mapping $F: X \to Y$ is said to be quasi-bounded [3] (or linearly upper bounded [4]) if there exist two constants $\alpha > 0$, $\gamma > 0$ such that $\|F(u)\| \leqq \gamma \|u\|$ for all $u \in X$ with $\|u\| \geqq \alpha$. In particular, a mapping $F: X \to Y$ is asymptotic close to zero if $\lim\limits_{\|u\|\to\infty} \frac{\|F(u)\|}{\|u\|} = 0$.

Denote by E_s the euclidean s-space.

<u>Lemma 1</u>. Let $g(u, x)$ be a N-function [5, chapt. VI] ($u \in (-\infty, +\infty)$, $x \in G$, G denotes a measurable subset of E_s with $mes\, G < \infty$) such that an operator of Nemyckij $h(u) = g(u(x), x)$ maps L_p into L_q ($p > 2$, $p^{-1} + q^{-1} = 1$). If $|g(u,x)| \leqq \varphi(x)|u|^{1-\alpha} + \psi(x)$, ($u \in (-\infty, +\infty)$, $x \in G$), where $\varphi(x) \in L_{p/p-2}$, $\psi(x) \in L_q$, $0 < \alpha < 1$, then h is bounded continuous and asymptotic close to zero, i.e. $\lim\limits_{\|u\|_{L_p}\to\infty} \frac{\|h(u)\|_{L_q}}{\|u\|_{L_p}} = 0$.

<u>Proof</u>. In fact

$$(1)\quad \|h(u)\|_{L_q} = \Big(\int\limits_G |g(u(x),x)|^q dx\Big)^{\frac{1}{q}} \leqq$$

$$\leqq \Big(\int\limits_G (\varphi(x)|u(x)|^{1-\alpha} + \psi(x))^q dx\Big)^{\frac{1}{q}} \leqq$$

$$\leqq \Big(\int\limits_G (\varphi(x)|u(x)|^{1-\alpha})^q dx\Big)^{\frac{1}{q}} + \|\psi\|_{L_q}.$$

Applying the Hölder's inequality with $p_1^{-1} = \frac{p-2}{p-1}$, $q_1^{-1} = \frac{q}{p}$

we obtain

(2) $\quad \| \varphi u^{1-\alpha} \|_{L_q} \leq \| \varphi \|_{L_{p/p-2}} \| u^{1-\alpha} \|_{L_p}$.

Using the Hölder's inequality with $p_1 = \alpha^{-1}$, $q_1^{-1} = 1-\alpha$ $(0 < \alpha < 1)$, then

(3) $\quad \| u^{1-\alpha} \|_{L_p} \leq (\operatorname{mes} G)^{\frac{\alpha}{p}} \| u \|_{L_p}^{1-\alpha}$.

From (1),(2),(3) it follows that

(4) $\quad \| h(u) \|_{L_q} \leq C \| \varphi \|_{\frac{L_p}{p-2}} \| u \|_{L_p}^{1-\alpha} + \| \psi \|_{L_q}$,

$C = (\operatorname{mes} G)^{\frac{\alpha}{p}}$. Hence $\lim\limits_{\|u\|_{L_p} \to \infty} \| h(u) \|_{L_q} \| u \|_{L_p}^{-1} = 0$.

From (4) we conclude that h is bounded. Since $h : L_p \to L_q$, h is also continuous [6,chapt.I]. This completes the proof.

Lemma 2. Let X, Y, Z be Banach spaces, $A : X \to Y$ a linear continuous mapping of X into Y. Assume that a mapping $F : Y \to Z$ is nonlinear bounded and asymptotic close to zero. Then the mapping $FA : X \to Z$ is bounded and asymptotic close to zero.

This assertion is a slight generalization of George's result [7].

Theorem 1 [1]. Let $F : X \to X$, $P : X \to X$, $T : X \to X$ be mappings of a Hilbert space X into X, P, T be linear continuous mappings onto X having the inverses P^{-1}, T^{-1}. Let the inequality

$$\| PF(u_1) - PF(u_2) - T(u_1 - u_2) \| \leq \alpha \| u - v \|$$

hold for every $u_1, u_2 \in X$ with $\alpha \| T^{-1} \| \leq 1$.

If there exist two positive constants α, γ, $\gamma < \| T^{-1} \|$

such that $\| T(u) - PF(u) \| \leq \frac{\gamma}{\| T^{-1} \|} \| u \|$ for all $u \in X$ with $\| u \| \geq \alpha$, then the equation $F(u) = y$ has at least one solution $u_0 \in X$ for every $y \in X$.

From theorem 1 it is easy to deduce the following

<u>Corollary 1</u> [1]. Let $F: X \to X$ be a mapping of a Hilbert space X into X which has the Gâteaux derivative $F'(u)$ for every $u \in X$. Let $PF'(u)$ be a normal operator for every $u \in X$ and such that $(PF'(u)v, v) \geq 0$ for every $u \in X$, $v \in X$, where P is a linear mapping of X into X having an inverse P^{-1} and $\| P \| \leq (\sup_{u \in X} \| F'(u) \|)^{-1}$.

If there exist two positive constants α, γ, $\gamma < 1$ such that $\| u - PF(u) \| \leq \gamma \| u \|$ for all $u \in X$ with $\| u \| \geq \alpha$, then F is onto.

An another result concerning the solution of functional equations with quasi-bounded operators has been obtained by W.V. Petryshyn [8]. His assertion is as follows: Suppose that A is P-compact quasi-bounded mapping (with constant γ) of a real Banach space X into itself. If $\mu > \gamma$, then $(A - \mu I)$ is onto.

A linear bounded operator $A: X \to X$ is said to be strictly positive in a Hilbert space X, if $u \neq 0$ implies $(Au, u) > 0$.

<u>Lemma 3</u> [6, chap. I]. Let $K: L_q \to L_p$ be a linear continuous mapping of L_q into L_p $(1 < q_0 < q < 2$, $p^{-1} + q^{-1} = 1)$. Suppose that K acts as a continuous strictly positive self-adjoint mapping from L_2 into L_2. Then K can be represented in the form $K = AA^*$, where

$A = K^{\frac{1}{2}} : L_2 \to L_p$ is continuous and A^* denotes the adjoint of A, so that $A^* : L_q \to L_2$.

In lemma 3 $K^{\frac{1}{2}}$ denotes the positive square root of K. Moreover, it is easy to prove that $\| A \|_{L_2 \to L_p} \leq \| K \|^{\frac{1}{2}}_{L_q \to L_p}$ and $\| A \|_{L_2 \to L_2} \leq \| K \|^{\frac{1}{2}}_{L_2 \to L_2}$, where $\| A \|_{L_2 \to L_p}$ (or $\| K \|_{L_q \to L_p}$) denotes the norm of A (or K) considered as a mapping of L_2 into L_p (or from L_q into L_p).

Under the assumptions of lemma 3, let $h(u) = g(u(x), x)$ be an operator of Nemyckij having the property that $h : L_p \to L_q$. Consider the equation

(5) $$\varphi = K h(\varphi) .$$

Then the equation (1) investigated in L_p is equivalent to

(6) $$u - A^* h(Au) = 0$$

considered in L_2 in the following sense: If u_0 is a solution of (6) in L_2, then $\varphi_0 = A u_0$ is a solution of (5) in L_p. Conversely: if φ_0 is a solution of (5) in L_p, then $u_0 = A^* h(\varphi_0)$ is a solution of (6) in L_2.

Theorem 2. Under the assumptions of lemma 3 let the following conditions be fulfilled:

1° $h'(u) = g'_u(u(x), x)$ is a continuous mapping from L_p into $L_{p/p-2}$, $N \leq g'_u(u, x) \leq M$ for every $u \in (-\infty, +\infty)$ and almost every $x \in G$, where G is a measurable subset of E_s with $mes\, G < \infty$ and

$M \| K \|_{L_2 \to L_2} \leq 1$ if $M > 0$ ($N, M = const$).

2° $|g(u, x)| \leq \varphi(x) |u|^{1-\alpha} + \psi(x)$, ($u \in (-\infty, +\infty)$, $x \in G$),

where $\varphi \in L_{p/p-2}$, $\psi \in L_q$ and $0 < \alpha < 1$.

Then the equation (5) has at least one solution φ_0 in L_p.

Proof. The proof of theorem 2 depends on lemma 1 - 3 and corollary 1. Since $1 < q_0 < q < 2$, $p > 2$. In view of 1^0 and [5,§ 20] the operator h acts from L_p into L_q and has a linear Gâteaux differential

$$Dh(u,v) = g'_u(u(x),x)\,v(x), \quad u, v \in L_p .$$

Since $g'_u(u, x)$ is bounded,

$$\| Dh(u,v)\|_{L_2} \leq \| h'(u)\|_{L_{\frac{p}{p-2}}} \|v\|_{L_p} \leq N_2 \|v\|_{L_p} ,$$

$N_2 = N_1 (\mathrm{mes}\, G)$, $N_1 = \mathrm{Max}(|M|, |N|)$. Thus $Dh(u,v)$ is bounded in L_p and continuous in $u \in L_p$ for an arbitrary (but fixed) $v \in L_p$. Consider the equation (6) in L_2. Using lemma 3, we have that $K = AA^*$, where A is a continuous mapping of L_2 into L_p. Set $Q(u) = A^*h(Au)$. Then the mapping $Q : L_2 \to L_2$ has a linear bounded Gateaux differential

$$DQ(u,v) = A^* g'_u(Au(x),x)Av = Q'(u)v, \quad v, u \in L_2$$

on the space L_2 ($Q'(u)$ denotes the Gâteaux derivative at the point $u \in L_2$). Furthermore, assuming 1^0

$$\| Q'(u)v\|^2 = \| A^* g'_u(Au(x),x)Av(x)\|^2 \leq$$

$$\leq \| A\|^2_{L_2 \to L_p} \int_G | g'_u(Au,x)Av|^2 dx \leq$$

$$\leq N_1^2 \|A\|^2_{L_2 \to L_p} \|A\|^2_{L_2 \to L_2} \|v\|^2_{L_2} \leq N_1^2 \|K\|_{L_2 \to L_p} \|K\|_{L_2 \to L_2} \|v\|^2_{L_2} .$$

Hence $k = \sup_{u \in L_2} \|F'(u)\| \leq 1 + N_1 \|K\|^{1/2}_{L_2 \to L_p} \|K\|^{1/2}_{L_2 \to L_2}$,

where $F(u) = u - Q(u)$. Suppose $M < 0$, then

$$(F'(u)\, v, v) \geq \|v\|^2, \quad u, v \in L_2 .$$

If $M > 0$, we have

$$(Q'(u)v,v)=(A^* g'_u(Au,x)Av,v)=\int_G g'_u(Au,x)(Av)^2 dx$$
$$\leq M\|Av\|^2_{L_2} \leq M\|A\|^2_{L_2\to L_2}\|v\|^2_{L_2} \leq M\|K\|_{L_2\to L_2}\|v\|^2_{L_2} .$$

Thus $(F'(u)v,v) \geq 0$ for every $u \in L_2$ and $v \in L_2$. Moreover, $A^* h(Au) = \operatorname{grad} f(Au)$, where
$$f(u) = f_0 + \int_G dx \int_0^u g(v,x)\,dv .$$
Using theorem 5.1 [5] we see that
$$(Dh(u,v_1),v_2) = (Dh(u,v_2),v_1)$$
for every $v_1, v_2 \in L_2$ and $u \in L_2$. Hence
$$(DQ(u,v_1),v_2) = (A^* Dh(Au,Av_1),v_2) =$$
$$= (Dh(Au,Av_1),Av_2) = (Dh(Au,Av_2),Av_1) =$$
$$= (A^* Dh(Au,Av_2),v_1) = (v_1, DQ(u,v_2)) .$$

Hence $Q'(u)$ is self-adjoint mapping in L_2 for every $u \in L_2$.

According to lemma 1,2 $hA : L_2 \to L_2$ and obviously $A^* h A$ are asymptotic close to zero. Set $P = \vartheta I$, where ϑ is a fixed number satisfying the inequality $0 < \vartheta < (1 + N_1 \|K\|^{\frac{1}{2}}_{L_2\to L_p} \|K\|^{1/2}_{L_2\to L_2})^{-1}$. Taking $0<\varepsilon<\vartheta$, there exists a positive number N_2 such that for every $u \in L_2$ with $\|u\| \geq N_2$, we have $\vartheta\|A^* h(Au)\| < \varepsilon\|u\|$. Clearly, for every $u \in L_2$ with $\|u\| \geq N_2$ there is $\|u - \vartheta F(u)\| \leq \gamma\|u\|$, where $\gamma = 1 - \vartheta + \varepsilon < 1$. Using corollary 1 we see that the euqation (6) has at least one solution u^* in L_2. Hence $\varphi_0 = A^* u^*$ is a solution of (5). This concludes the proof.

Remark 1. Recall that the condition 1° of theorem 2 implies the boundedness of $h : L_p \to L_q$ on L_p.

Moreover, h is Lipschitzian on L_p. Indeed, from the equality $(u, v \in L_p)$

$$h(u) - h(v) = (u(x) - v(x)) \int_0^1 g'_u(v(x) + t(u(x) - v(x)), x) dx$$

it follows [5, § 20] that

$$\| h(u) - h(v) \|_{L_q} \leq \| u - v \|_{L_p} \cdot$$

$$\cdot \left(\int_0^1 dt \int_G | g'_u (v(x) + t(u(x) - v(x)), x) |^{\frac{p}{p-2}} dx \right)^{\frac{p-2}{p}} .$$

Since $g'_u(u, x)$ is bounded,

$$\| h(u) - h(v) \|_{L_q} \leq N_2 \| u - v \|_{L_p} ,$$

where $N_2 = N_1 \, mes \, G$, $N_1 = Max(|M|, |N|)$.

Assume that K is an operator determined by

$$(7) \qquad K(u) = \int_G K(s, t) u(t) dt ,$$

where $K(s, t)$ is defined on $G \times G$, G is a measurable subset of E_s with $mes \, G < \infty$.

Theorem 3. Under the conditions of theorem 2 let K be an operator defined by (7), where the kernel $K(s, t)$ is such that $\operatorname{vrai\,sup}_{s, t \in G} | K(s, t) | = d^2 < \infty$. Then the equation (5) has at least one solution φ_0 such that $\operatorname{vrai\,sup}_{x \in G} | \varphi_0(x) | < \infty$.

Proof. According to theorem 2 the equation (6) has at least one solution $u_0 \in L_2$. Then $\varphi_0 = A u_0$ is a solution of (5). By the Vajnberg-Golomb theorem $(A = K^{\frac{1}{2}})$ we obtain

$\operatorname{vrai\,sup}_{x \in G} | K^{\frac{1}{2}} u_0 | \leq d \, \| u_0 \|_{L_2}$. This concludes the proof.

Theorem 4. Under the assumptions of lemma 3 let the following conditions be fulfilled:

1° K is defined by (7) and $\operatorname{vrai\,sup}_{s,t\in G} |K(s,t)| = d^2 < \infty$;

2° $h'(u) = g'_u(u(x), x)$ is a continuous mapping from L_p into $L_{p/p-2}$, where $g'_u(u, x)$ is such that for every $u \in \langle -c, c\rangle$, $(c > 0)$ and almost every $x \in G$ there is $N \leq g'_u(u, x) \leq M$, $(N, M = const)$.

If either a) $M < 0$, $0 < \lambda < R\|Ah(o)\|^{-1}$, where $R = cd^{-1}$, or b) $M > 0$,

$$|\lambda| < Min\left(\frac{1}{M\|K\|_{L_2\to L_2}}, \frac{Rm}{\|Ah(o)\|}\right),$$

where $m = 1 - |\lambda| M \|K\|_{L_2\to L_2}$, then the equation

$$\varphi(s) - \lambda \int_G K(s,t)\, g(\varphi(t), t)\, dt = 0.$$

has at least one solution $\varphi_o \in A(D_R)$ such that $\operatorname{vrai\,sup}_{x\in G} |\varphi_o(x)| < +\infty$, where $D_R = \{u \in L_2 : \|u\| \leq R, R = cd^{-1}\}$.

Proof. Instead of the equation

(8) $$\varphi - \lambda K h(\varphi) = 0$$

we shall solve the equation

(9) $$u - \lambda A^* h(Au) = 0$$

in L_2. By the Golomb-Vajnberg theorem we have that $\operatorname{vrai\,sup}_{x\in G} |Au| \leq d\|u\|_{L_2}$ for every $u \in L_2$. Thus for every $u \in D_R = \{u \in L_2;\ \|u\| \leq R, R = cd^{-1}\}$ there is $\operatorname{vrai\,sup}_{x\in G} |Au| \leq c$ and

(10) $$N \leq g'_u(Au, x) \leq M.$$

By 2°, (10) and according to [5, § 20] we see that the mapping $Q(u) = A^* h(Au)$, $Q: L_2 \to L_2$, has for every $u \in D_R$, $v \in L_2$, a linear bounded Gâteaux differential $DQ(u, v) =$

$= A^* g'_u (Au, x) Av .$

Suppose a), then $(F'(u)v, v) \geqq \|v\|^2$ for every $u \in D_R$ and $v \in L_2$, where $F(u) = u - \lambda Q(u)$. We shall apply theorem 3 [1] with $E = D_R$, $u_0 = 0$, $m = 1$, $P_1 = I$ (I denotes the identity mapping of L_2) and $k = (1 + |\lambda| N_1 \|K\|^{\frac{1}{2}}_{L_2 \to L_p} \|K\|^{\frac{1}{2}}_{L_2 \to L_2})^2$, $N_1 = Max(|M|, |N|)$.

It remains to prove that $D_{R_\vartheta} = \{u \in L_2 ; \|u - u_1\| \leqq R_\vartheta\} \subset D_R$, where

$$u_1 = \vartheta \lambda A^* h(0), \quad R_\vartheta = \alpha_\vartheta (1 - \alpha_\vartheta)^{-1} \vartheta \|u_1\| ,$$

$$\alpha_\vartheta = \sup_{u \in D_R} \|I - \vartheta F'(u)\| \leqq (1 - 2\vartheta + \vartheta^2 k)^{\frac{1}{2}} < 1 .$$

A number ϑ satisfies

(11) $0 < \vartheta < Min(k^{-1}, 2Rab^{-1})$,

where $a = R - \lambda \|A^* h(0)\|$, $b = R^2 k - \lambda^2 \|A^* h(0)\|^2$.

For the verification of this assertion cf. the proof of theorem 6 [1].

Assuming b) we have $(F'(u)v, v) \geqq m \|v\|^2$ for every $u \in D_R$, $v \in L_2$ with $m = 1 - |\lambda| M \|K\|_{L_2 \to L_2}$. It is easy to show that

$$D^*_{R_\vartheta} = \{u \in L_2 : \|u - u_1\| \leqq R^*_\vartheta\} \subset D_R ,$$

where $u_1 = \vartheta \lambda A^* h(0)$, $R^*_\vartheta = \alpha^*_\vartheta (1 - \alpha^*_\vartheta)^{-1} \vartheta \|u_1\|$, $\alpha^*_\vartheta \leqq (1 - 2m\vartheta + \vartheta^2 k)^{\frac{1}{2}} < 1$. In this case a number satisfies the inequality

(12) $$0 < \vartheta < Min\left(\frac{m}{k}, \frac{2Ra_1}{b_1}\right) ,$$

where $a_1 = Rm - \|\lambda A^* h(0)\|$, $b_1 = R^2 k - \|\lambda A^* h(0)\|^2$.

Therefore, according to theorem 3 [1] the equation (9) has a unique solution u^* in D_{R_ϑ} ($D_{R_\vartheta} \subset D_R \subset L_2$) (or in $D^*_{R_\vartheta}$). Hence $\varphi_0 = Au^* \in A(D_R)$ is a solution of (8) in L_p. Moreover, by Vajnberg-Golomb theorem $\operatorname{vrai}\sup\limits_{x\in G} |\varphi_0| < +\infty$. This completes the proof of theorem 4.

Remark 2. If the conditions of theorem 4 are satisfied, then $\varphi_n \to \varphi_0$ in the norm topology of L_p, where $\varphi_n = Au_n$, $u_{n+1}(1-\vartheta)u_n + \lambda\vartheta A^* h(Au_n)$, $u_0 = 0$ and φ_0 denotes a solution of (8). A positive number ϑ is determined according to the condition a) or b) by (11), or by (12). Suppose for instance a), then the equation (9) has a solution u^* in $D_R \subset L_2$ and $\lim\limits_{n\to\infty} \|u_n - u^*\|_{L_2} = 0$. So that $\varphi_0 = Au^* \in A(D_R)$ is a solution of (8) and

$$\|\varphi_0 - \varphi_n\|_{L_p} = \|Au^* - Au_n\| \leq \|A\|_{L_2\to L_p} \cdot \|u_n - u^*\|_{L_2} \to 0$$

whenever $n \to \infty$. Since $\|A\| = \|A^*\|$ and

$\|A\|_{L_2\to L_p} \leq \|K\|^{\frac{1}{2}}_{L_q\to L_p}$, we have that

$$\|A\|_{L_2\to L_p} \|A^* h(0)\| \leq \|A\|^2_{L_2\to L_p} \|h(0)\| \leq \|K\|_{L_q\to L_p} \|h(0)\|_{L_q}.$$

Hence

$$\|\varphi_0 - \varphi_n\|_{L_p} \leq \lambda\vartheta\alpha^n_\vartheta (1-\alpha_\vartheta)^{-1} \|A\|_{L_2\to L_p} \|A^* h(0)\| \leq$$

$$\leq \lambda\vartheta\alpha^n_\vartheta (1-\alpha_\vartheta)^{-1} \|K\|_{L_q\to L_p} \|h(0)\|_{L_q}.$$

Similar assertions also hold for the case b).

3. Consider Urysohn integral equation

$$(13) \qquad u(s) - \int_G K(s,t,u(t))\,dt = y(s)$$

in a real space $L_2(G)$, where a function $K(s,t,u)$ is defined for $s, t \in G$, $u \in (-\infty, +\infty)$, G is a measurable subset of E_s with $\operatorname{mes} G < \infty$ and $y \in L_2$.

Assume that $K(s,t,u)$ defines an operator

(14) $$A(u)=\int_G K(s,t,u(t))\,dt\ ,$$

which maps L_2 into L_2. Let $Q: L_2 \to L_2$ be a continuous mapping from L_2 into L_2 defined by

(15) $$Q(u)=\int_G Q(s,t)\,u(t)\,dt\ ,$$

where $Q(s,t)$ is determined on $G \times G$. Set $T = I - \lambda Q$ (I denotes the identity mapping of L_2, λ a real number). Suppose that λ is a regular value of Q. Under these conditions, using theorem 1, we shall prove the following

<u>Theorem 5</u>. Let the following conditions be fulfilled:

1^0 for every $u_1, u_2 \in (-\infty, +\infty)$, $s, t \in G$

$$|K(s,t,u_1)-K(s,t,u_2)-\lambda Q(s,t)(u_1-u_2)| \leq \varphi(s,t)|u_1-u_2|,$$

where $\alpha = \left(\int_G\int_G \varphi^2(s,t)\,ds\,dt\right)^{\frac{1}{2}} \leq \dfrac{1}{\|T^{-1}\|}$.

2^0 $$|K(s,t,u)-\lambda Q(s,t)u| \leq \sum_{k=1}^{n} g_k(s,t)|u|^{1-\alpha_k} + h(s,t)$$

$(s,t \in G,\ u \in (-\infty,+\infty))$, where $0<\alpha_k<1$, $(k=1,2,\dots,n)$, $h(s,t) \in L^2_{G\times G}$ and the functions $g_k(s,t)$, $(k=1,2,\dots,n)$ are such that

(16) $$\int_G\left(\int_G |g_k(s,t)|^{\frac{2}{\alpha_k}}\,dt\right)^{\frac{\alpha_k}{2}} ds < \infty\ .$$

Then the equation (13) has at least one solution $u_0 \in L_2$ for every $y \in L_2$.

<u>Proof</u>. Assuming 2^0, then for every $u \in L_2$

$$\|T(u)-F(u)\| = \|\lambda Q(u) - A(u)\| \leq$$
$$\leq C\left(M\sum_{k=1}^{n}\|u\|^{1-\alpha_k} + N\right)^{\frac{1}{2}}\ ,$$

where $M = \max_{k=1,2,\dots,n} \int_G \left(\int_G |g_k(s,t)|^{\frac{2}{\alpha_k}} dt \right)^{\frac{\alpha_k}{2}} ds$,

$N = \int_G \int_G h^2(s,t)\, ds\, dt, \quad F(u) = u - A(u), \quad C = \text{mes}\, G$.

Hence

$$\lim_{\|u\| \to \infty} \frac{\| T(u) - F(u) \|}{\| u \|} = 0 .$$

In view of 1^0 for every $u_1, u_2 \in L_2$

$$\| F(u_1) - F(u_2) - T(u_1 - u_2) \| = \| A(u_1) - A(u_2) - \lambda Q(u_1 - u_2) \| \leqq$$

$$\leqq \alpha \| u_1 - u_2 \|$$

with $\alpha \leqq \frac{1}{\| T^{-1} \|}$. Thus all the assumptions of theorem 1 are satisfied. This completes the proof.

<u>Theorem 6</u>. Let $K(s,t,u)$ be a function satisfying the following conditions:

1^0 For every $u_1, u_2 \in (-\infty, +\infty)$, $(s, t \in G)$ there is

$$| K(s,t,u_1) - K(s,t,u_2) | \leqq \varphi(s,t) | u_1 - u_2 | .$$

2^0 $|K(s,t,u)| \leqq \beta |u| + \sum_{k=1}^{n} g_k(s,t) |u|^{1-\alpha_k} h(s,t)$,

$(s, t \in G, u \in (-\infty, +\infty))$, where $0 < \alpha_k < 1$ $(k = 1, 2, \dots, n)$ $h(s,t) \in L^2_{G \times G}$, β is a number sufficiently small $(0 \leqq \beta \leqq \varepsilon < 1)$ and the functions $g_k(s,t)$ $(k = 1, 2, \dots, n)$ satisfy (16).

If $|\lambda| \leqq \frac{1}{\| \varphi \|_{L^2_{G \times G}}}$ then the equation

$$u(s) - \lambda \int_G K(s,t,u(t))\, dt = y(s)$$

has at least one solution $u_0 \in L_2$ for every $y \in L_2$.

<u>Proof</u>. The proof is similar to the proof of theorem 5.

In next we suppose that A is defined by (14), where $K(s,t,u)$ is a function given on $G \times G \times (-\infty, +\infty)$

and G is a bounded closed subset of E_s .

Lemma 4. Let X be a Banach space, $A : X \to X$ a completely continuous mapping of X into X, $Q : X \to X$ a linear mapping such that

$$\lim_{\|u\| \to \infty} \frac{\| A(u) - \lambda Q(u) \|}{\| u \|} = 0 .$$

If $\lambda \neq 0$ is not a characteristic number of Q , then the equation

(17) $$u - A(u) = y$$

has at least one solution $u_0 \in X$ for every $y \in X$.

Proof. By [6, chapt. IV, lemma 3.1] the operator Q is completely continuous. Since λ is a regular value of Q , $Q_\lambda^{-1} = (I - \lambda Q)^{-1}$ exists, is bounded and everywhere defined. The equation $F(u) = y$ with $F = I - A$ is equivalent to

(18) $$u = R(u) + Q_\lambda^{-1} y ,$$

where $R(u) = Q_\lambda^{-1}(Q_\lambda(u) - F(u)) = Q_\lambda^{-1}(A(u) - \lambda Q(u))$. Furthermore, since Q_λ^{-1} is continuous and $A - \lambda Q$ completely continuous, R is completely continuous. In view of

$$\| R(u) \| \leq \| Q_\lambda^{-1} \| \, \| A(u) - \lambda Q(u) \|$$

we have that $\lim_{\|u\| \to \infty} \frac{\| R(u) \|}{\| u \|} = 0$. Using the theorem of Dubrovskij [9, chapt. II] we see that (18) has at least one solution in X . Thus the equation (17) has at least one solution $u_0 \in X$ for every $y \in X$. This concludes the proof.

Theorem 7. Let one of the following conditions be fulfilled:

1° The operator $A(u)$ defined by (14) is completely continuous in L_2 -space and the function $K(s, t, u)$ is such that

(19) $|K(s,t,u) - \lambda Q(s,t,u)| \leq a + b|u|^{\alpha}$,

$(s, t \in G,\ u \in (-\infty, +\infty))$, where $a, b > 0$, $0 \leq \alpha < 1$, $Q(s,t)$ is a kernel of (15) and $\lambda \neq 0$ is not a characteristic value of Q.

2° $K(s,t,0) = 0, (s,t \in G), K(s,t,u)$ has a bounded derivative $K'_u(s,t,u)$ and $K'_u(s,t,u) \to Q(s,t)$ as $u \to \infty$ uniformly with respect to $s, t \in G$, where $Q(s,t)$ is either identically equal to zero, or defines a linear operator (15) having the property that 1 is not a characteristic value of Q.

Then the equation (13) has at least one solution $u_0 \in L_2$ for every $y \in L_2$.

<u>Proof</u>. The proof of theorem 7 depends on lemma 4. Assuming 1°, it is sufficient to prove that $\lim\limits_{\|u\| \to \infty} \|A(u) - \lambda Q(u)\| \, \|u\|^{-1} = 0$. In fact, using (19)

(20) $\|A(u) - \lambda Q(u)\| \leq (\operatorname{mes} G)^{\frac{1}{2}} [a \operatorname{mes} G + b \int_G |u(t)|^{\alpha} dt]$.

Applying the Hölder's inequality with $p^{-1} = \alpha$, $q^{-1} = 1 - \alpha$ we obtain that

(21) $\int_G |u(t)|^{\alpha} dt \leq (\operatorname{mes} G)^{1-\alpha} (\int_G |u(t)| dt)^{\alpha}$.

According to Cauchy-Schwarz inequality

(22) $(\int_G |u(t)| dt)^{\alpha} \leq (\operatorname{mes} G)^{\frac{\alpha}{2}} \|u\|^{\alpha}$.

By (20),(21) and (22)

$$\lim_{\|u\| \to \infty} \frac{\|A(u) - \lambda Q(u)\|}{\|u\|} \leq (\operatorname{mes} G)^{\frac{1}{2}} \lim_{\|u\| \to \infty} [\frac{a}{\|u\|} + \frac{b (\operatorname{mes} G)^{-\frac{\alpha}{2}}}{\|u\|^{1-\alpha}}] = 0.$$

Assuming 2°, we see that $|K(s,t,u)| \leq M|u|$ for every $s, t \in G$, $u \in (-\infty, +\infty)$, $M = \text{const}$.

According to [6,chapt.I,th.3.2] the mapping $A(u)$ acts from L_2 into L_2 and is completely continuous. Furthermore, A is asymptotic close to a linear mapping Q [cf.6, chapt.V,§ 3]. Thus all the assumptions of lemma 4 are satisfied. This completes the proof.

Remark 3. Some results concerning the solutions of homogeneous Hammerstein integral equations being asymptotic close to linear ones has been established by M.A. Krasnoselskij [6,chapt.III,§ 4,5].

4. Theorem 8. Let $F: X \to X$ be a mapping of a uniformly convex Banach space X into X such that for every u_1, $u_2 \in D_R = \{u \in X : \|u\| \leq R\}$ there is

$$\| PF(u_1) - PF(u_2) - K(u_1 - u_2)\| \leq \alpha \|u_1 - u_2\|,$$

where $P: X \xrightarrow{onto} X$, $K: X \xrightarrow{onto} X$ are linear mappings having the inverses P^{-1}, K^{-1}. Let F be a Fréchet-differentiable at 0, $F(0) = 0$, $a = \|K - PF'(0)\| < 1$ and $\alpha \|K^{-1}\| \leq 1$. Let ε be an arbitrary positive number such that $\varepsilon < 1 - a$.

Then there exists a positive number δ such that for any $y \in X$ with $\|y\| \leq \dfrac{\delta(1-(a+\varepsilon))}{\|P\|}$ the equation $F(u) = y$ has at least one solution in the ball $D_\delta = \{u \in X : \|u\| \leq \delta\}$.

Proof. To prove the theorem 8, use the same arguments as in [10] and the Browder's fixed point theorem [11].

R e f e r e n c e s

[1] J. KOLOMÝ: Application of some existence theorems for the solutions of Hammerstein integral equations. Comment.Math.Univ.Carolinae 7,4(1966), 461-478.

[2] А.И. ПОВОЛОЦКИЙ: Обобщение одной теоремы о разщеплении линейного оператора. Труды Ленингр. Лесотехн. акад., 78 (1957), 27 - 30 .

[3] A. GRANAS: The theory of compact vector fields and some of its applications to topology of functional spaces (I). Rozpr. Matematyczne XXX(1962), 1-93.

[4] S YAMAMURO: A note on the boundedness property of non-linear operators. Yokohama Math. J., Vol. X(1962), 19-23.

[5] М.М. ВАЙНБЕРГ: Вариационные методы исследования нелинейных операторов. Москва 1956.

[6] М.А. КРАСНОСЕЛЬСКИЙ: Топологические методы в теории нелинейных интегральных уравнений. Москва 1956.

[7] M.D. GEORGE: Completely well posed problems for nonlinear differential equations. Proc. Am. Math. Soc., Vol. 15(1964), No 1, 96-100.

[8] W.V. PETRYSHYN: Further remarks on nonlinear P-compact operators in Banach spce. Journ. of Math. Anal. Appl. 16(1966), 243-253.

[9] М.А. КРАСНОСЕЛЬСКИЙ: Некоторые задачи нелинейного анализа. Усп. мат. наук, т. IX (1954), вып. 3, 57-114.

[10] J. KOLOMÝ: Some existence theorems for nonlinear problems. Comment. Math. Univ. Varol., 7, 2(1966), 207-217.

[11] E.F. BROWDER: Nonexpansive nonlinear operators in Banach space. Proc. Nat. Acad. Sci. U.S.A. Vol. 54(1965), 1041-1043.

[12] E.F. BROWDER: Existence of periodic solutions for nonlinear equations of evolutions. Proc. Nat. Acad. Sci. U.S.A. Vol. 53(1965), No 5, 1100-1103.

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