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ON PAIRS OF MATRICES WITH PROPERTY L

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In [1] T.S.Motzkin and O.Taussky have proved the following theorem.

<u>Theorem1</u>. Let A,B be two $n \times n$ matrices satisfying the conditions

1. A and B are hermitian.

2. The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and $\mu_1, \mu_2, \dots, \mu_n$ of A and B respectively can be ordered in such a way that, for every α and β real, $\alpha\lambda_i + \beta\lambda_i$ are all eigenvalues of the matrix $\alpha A + \beta B$ (property L).

Then $AB = BA$.

The aim of this paper is to generalize this theorem, or, more precisely, to prove

<u>Theorem 2</u>. Let A,B be two $n \times n$ matrices, satisfying the following conditions:

1. Both A and B have only real eigenvalues λ_i and μ_i respectively ($i = 1,2,\dots, n$) , and the matrix $\alpha A + \beta B$ can be diagonalized by a similarity transformation (depending in general on α and β) for every real α and β .

2. A, B have the property L .

Then AB = BA and A and B can be diagonalized by the same similarity transformation.

Proof. It is sufficient to construct a matrix T, $\det T \neq 0$, such that $\tilde{A} = T^{-1} A T$ and $\tilde{B} = T^{-1} B T$ are hermitian. Since $\tilde{A}$ and $\tilde{B}$ will also have the property L, they commute by Theorem 1, and so do A and B.
For the construction of such a matrix we consider the matrix $A(\alpha) = \alpha A + B$ and rearrange its eigenvalue $\alpha \lambda_i + \mu_i$ in $k \leq n$ groups in the following manner: two eigenvalues $\alpha \lambda_r + \mu_r$ and $\alpha \lambda_s + \mu_s$ belong to the same group if and only if $\lambda_r = \lambda_s$, $\mu_r = \mu_s$. It will be convenient to consider $A(\alpha)$ for all complex numbers α.
Since the both sides of the equality

$$\det (x E - A(\alpha)) = \prod_{k=1}^{n} (x - \lambda_k \alpha - \mu_k)$$

are polynomials in α we have that $\alpha \lambda_i + \mu_i$ are all eigenvalues of $A(\alpha)$ for all complex α. Moreover, $A(\alpha)$ can be diagonalized by a similarity transformation $T^{-1}(\alpha) A(\alpha) T(\alpha)$ for all complex α. This fact can be seen as follows.
If $l \neq k$, the equality $\alpha \lambda_l + \mu_l = \alpha \lambda_k + \mu_k$ holds either for all complex α (if $\lambda_l = \lambda_k$, $\mu_l = \mu_k$) or for $\alpha = \alpha_{lk} = \frac{\mu_l - \mu_k}{\lambda_k - \lambda_l}$, $\Im m\, \alpha_{lk} = 0$ (for $\lambda_l \neq \lambda_k$), or $\alpha \lambda_l + \mu_l \neq \alpha \lambda_k + \mu_k$ for all complex α (if $\lambda_l = \lambda_k$, $\mu_l \neq \mu_k$). Thus we have that, for every complex α different from α_{lk}, the matrix $A(\alpha)$ has k different eigenvalues $\alpha \lambda_{s_1} + \mu_{s_1}, \alpha \lambda_{s_2} + \mu_{s_2}, \dots, \alpha \lambda_{s_k} + \mu_{s_k}$, each of them having the same multiplicity for all such α which we denote by ρ_j, $j = 1, 2, \dots, k$. Let $N_j(\alpha)$ be the corresponding eigensubspace. Its dimension is ρ_j because

the rank of the matrix $((\alpha\lambda_{s_j}+\mu_{s_j})E - A(\alpha))$ being $\leq n-\rho_j$ for real α, it is $\leq n-\rho_j$ also for complex α (its minors are polynomials in α). Thus $\dim N_j(\alpha) \geq \rho_j$. But it can't be $> \rho_j$ for $\alpha \neq \alpha_{\ell k}$. In each $N_j(\alpha)$, we can choose an orthonormal basis $t_1^j(\alpha), t_2^j(\alpha), \dots, t_{\rho_j}^j(\alpha)$. Moreover, if α_0 is arbitrary complex number different from $\alpha_{\ell k}$, we can choose these $t_r^j(\alpha)$ to be analytic in some neighbourhood of α_0. These bases are not determined uniquely, one of them can be obtained from another one by appropriate unitary transformation. Thus the matrix $T(\alpha)$ diagonalizing $A(\alpha)$ is not determined uniquely (the columns of $T(\alpha)$ are vectors $t_r^j(\alpha)$, $j = 1, 2, \dots, k$, $r = 1, 2, \dots, \rho_j$), but it can be easily seen that the matrix $T(\alpha)\,T^*(\alpha)$ does not depend on the special choice of $t_r^j(\alpha)$ in each point α different from $\alpha_{\ell k}$. Since $t_i^j(\alpha)$ can be chosen analytic in some neighbourhood of each $\alpha \neq \alpha_{\ell k}$, we see that $T(\alpha)T^*(\alpha)$ is analytic for all $\alpha \neq \alpha_{\ell k}$, and it is bounded. Thus it must be a constant regular matrix, say C. Thus we have obtained that there exists, for all $\alpha \neq \alpha_{\ell k}$, a regular uniformly bounded matrix $T(\alpha)$ satisfying, for such α the conditions

(1) $\quad T(\alpha)\,T^*(\alpha) = C$,

(2) $\quad T(\alpha)\,A(\alpha) = \alpha\Lambda + M$,

C being a hermitian regular matrix and

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}, \quad M = \begin{pmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_n \end{pmatrix}.$$

From (1) it follows that $\|T^{-1}(\alpha)\| = \|T^*(\alpha)C^{-1}\| \leq \|T^*(\alpha)\|\|C^{-1}\|$,

and then

(3) $\quad \|T^{-1}(\alpha)\| \leq const$.

Let α_n, $\alpha_n \neq \alpha_{lk}$, $\alpha_n \neq 0$, be real and $\lim\limits_{n\to\infty} \alpha_n = 0$. We can assume that $T(\alpha_n) \to T$ (taking an appropriate subsequence if it is necessary). From (3), T is regular and $T^{-1}(\alpha_n) \to T^{-1}$. From (1) and (2) we obtain

(4) $$TT^* = C ,$$

(5) $$T^{-1}BT = M .$$

Moreover, we have, for all α_n,

$$A(\alpha_n) = T(\alpha_n)(\alpha_n \Lambda + M)T^{-1}(\alpha_n) .$$

Multiplying by T^{-1} and T , we get

$$\alpha_n T^{-1}AT + T^{-1}BT = \alpha_n T^{-1}AT + M =$$

$$= T^{-1}T(\alpha_n)(\alpha_n\Lambda + M)T^{-1}(\alpha_n)T = T^*T^{-1}(\alpha_n)(\alpha_n\Lambda + M)T^{-1}(\alpha_n)T ,$$

where we have used (1),(4),(5). Since the right-hand side and M are hermitian, and $\alpha_n \neq 0$ is real, $T^{-1}AT$ is also hermitian. Thus the matrix with above mentioned properties is constructed. By Theorem 1, $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$ where $\tilde{A} = T^{-1}AT$ and $\tilde{B} = T^{-1}BT = M$ and thus

$$AB = T\tilde{A}T^{-1}T\tilde{B}T^{-1} = T\tilde{B}\tilde{A}T^{-1} = BA .$$

By Theorem 1 in [2] (p.10), A and B can be diagonalized by the same similarity transformation. The proof is complete.

R e f e r e n c e s

[1] T.S.MOTZKIN,O.TAUSSKY: Pairs of matrices with property

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[2] **Д.А.СУПРУНЕНКО; Р.И.ТЫШКЕВИЧ: Перестановочные матрицы, Минск 1966.**

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