Commentationes Mathematicae Universitatis Carolinae

Jiří Kopáček

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*Commentationes Mathematicae Universitatis Carolinae*, Vol. 8 (1967), No. 3, 459--462

Persistent URL: `http://dml.cz/dmlcz/105127`

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8, 3 (1967)

# THE CAUCHY PROBLEM FOR LINEAR HYPERBOLIC SYSTEMS IN $L_p$

Jiří KOPÁČEK, Praha

In [1] P. Brenner has proved that the Cauchy problem for the hyperbolic system

$$(1) \qquad \frac{\partial u}{\partial t} = \sum_{j=1}^{n} A_j \frac{\partial u}{\partial x_j} + Bu \quad ,$$

$$(2) \qquad u(0,x) = u_0(x)$$

is correctly posed in $L_p$, $1 \leq p \leq +\infty$, $p \neq 2$, if and only if $A_j$ commute, provided $A_j$ are hermitian. In this paper we generalize this result to a more general class of hyperbolic systems.

Definition 1. We call the system (1) hyperbolic if the $N \times N$ matrices $A_j$, $B$ satisfy the following conditions:

1. The matrix $A(y) = \sum_{j=1}^{n} y_j A_j$ has, for all $y \in R_n$, only real eigenvalues and can be diagonalized by a similarity transformation $T^{-1}(y) A(y) T(y)$ for all $y \in R_n$.

2. There exist such positive constants $C_1$, $C_2$, $C_3$ that $\|\varphi_t(y)\| = \|e^{t(iA_jy_j + B)}\| \leq C_1 + C_2 |y|^{C_3}$ for all $y \in R_n$, $t \in \langle 0, T \rangle$.

Remark. The condition 2 is fulfilled, e.g., if the matrices $T(y)$, $T^{-1}(y)$ from the condition 1 are bounded on the unit sphere in $R_n$ (hyperbolicity in the sense of Petrovsky) or if B commutes with all $A_j$'s. In the first case, $\| e^{i y_j A_j} \|$ is bounded and from this and the representation $\varphi_t(y) = e^{it y_j A_j} + \int_0^t e^{i(t-u) y_j A_j} \varphi_u(y)\, du$ follows the boundedness of $\| e^{t(i y_j A_j + B)} \|$ by the Gronwall's lemma. In the second case, $\| e^{t i y_j A_j} \| \leq C_1 + C_2 |y|^{C_3}$

with appropriate constants $C_1$, $C_2$, $C_3$ (see e.g. [2], p.93), and we have $e^{t(i y_j A_j + B)} = e^{t i y_j A_j} \cdot e^{tB}$.

Definition 2. We say that the problem (1),(2) is correctly posed in $L_p$, $1 \leq p \leq +\infty$, if, for arbitrary $u_0(x) \in \mathcal{S}$, there exists a solution of (1),(2) in the $L_p$-norm (by which we mean that it satisfies for all $t \in \langle 0, T \rangle$, $\lim_{h \to \infty} \frac{1}{h}(u(t+h) - u(t) - \sum A_j \frac{\partial u}{\partial x_j} - Bu = 0$ in $L_p$ and $u(0,x) = u_0(x)$) continuously depending on $u_0(x)$, i.e., there exists a constant C(T) independent of $u_0(x)$ such that

$$\| u(t, \cdot) \|_{L_p} \leq C(T) \| u_0(x) \|_{L_p}$$

(see [1]). Here $\mathcal{S}$ denotes the space of infinitely differentiable vector-functions in $R_n$ each component $f_j$ of which satisfies the inequalities $\left| |x|^m D^r f_j(x) \right| \leq C^j_{mr} < +\infty$ for all $r = (r_1, r_2, \ldots, r_n)$ and m ( $r_k$, m = 0,1,2,...) and all $x \in R_n$.

Firstly, we note that, for each $u_0 \in \mathcal{S}$, there exists the classical solution $u(t, x) \in C^\infty$ of (1),(2) such that $D^\alpha u(t, x) \in \mathcal{S}$ for all $t \in \langle 0, T \rangle$ and all $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$, $\alpha_k = 0, 1, 2, \ldots$ (which may be obtained by Fourier transformation).

Repeating the proof of Theorem 2 in [1] we obtain the following

<u>Theorem 1</u>. Let the system (1) be hyperbolic in the sense of Definition 1. The Cauchy problem (1),(2) is correctly posed in $L_p$, $1 \le p \le \infty$, if and only if

$$(3) \qquad \sup_{\substack{t \in \langle 0,T\rangle \\ u_0 \in \mathcal{S}}} \|u(t,\cdot)\|_{L_p} / \|u_0(x)\|_{L_p} \le C(T) < +\infty$$

where $u(t,x)$ is the above mentioned solution of (1), (2) corresponding to $u_0(x)$.

It is sufficient to see that (3) is just another form of (*) in [1].

<u>Theorem 2</u>. Let the system (1) be hyperbolic in our sense. If the Cauchy problem (1),(2) is correctly posed in $L_p$, $1 \le p \le +\infty$, $p \ne 2$, then the eigenvalues $\lambda_j(y)$, j = = 1,2, ..., N, of the matrix $A(y) = \sum_{j=1}^{n} y_j A_j$ are linear homogeneous functions of $y_1, y_2, \dots, y_n$, i.e.,

$\lambda_j(y) = \sum_{k=1}^{n} \lambda_k^j y_k$ for all $y \in R_n$, where $\lambda_k^j$ (j = 1,2,..., N, k = 1,2,..., n) are the eigenvalues of $A_k$.

We omit the proof because it is only the repetition of corresponding arguments in [1].

<u>Corollary</u>. If the problem (1),(2) is correctly posed in $L_p$, $1 \le p \le +\infty$, $p \ne 2$, then the matrix $A(y)$ must have multiple eigenvalues for some $y \in R_n$ provided $n > 2$. Thus for strongly hyperbolic system the Cauchy problem is not correctly posed in $L_p$, $1 \le p \le \infty$, if $p \ne 2$ and $n > 2$.

The main result of this paper is the following theorem.

**Theorem 3.** Let the system (1) be hyperbolic in the sense of Definition 1. Then the Cauchy problem (1),(2) is correctly posed in $L_p$, $1 \leq p \leq \infty$, $p \neq 2$, if and only if $A_j$ commute, or if (what is the same) $A_j$ can be diagonalized by the same similarity transformation.

*Proof.* Necessity. By Theorem 2 and Theorem 2 in [3] it follows that if (1),(2) is correctly posed in $L_p$, $1 \leq p \leq \infty$, $p \neq 2$, then $A_j$ commute. By Theorem 1 in [4], p.10, $A_j$ have a common diagonalizing similarity transformation.

Sufficiency. If $A_j$ commute then there exists a regular matrix T such that $T^{-1}A_jT = \Lambda_j$ $(j = 1, 2, \dots, n)$ are diagonal matrices. Then the problem (1),(2) is equivalent to the problem

$$(1') \qquad \frac{\partial v}{\partial t} = \sum_{j=1}^{n} \Lambda_j \frac{\partial v}{\partial x_j} + \tilde{B} v \, ,$$

$$(2') \qquad v(0,x) = T^{-1} u_0(x)$$

with $v = T^{-1}u$, $\tilde{B} = T^{-1} B T$ But the problem (1'), (2') is correctly posed in $L_p$ (e.g. by Theorem 2 in [1]).

## References

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(Received May 18,1967)