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ONE GENERALIZATION OF THE FOURTH HARMONIC POINT

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Preliminary communication x)

By a frame $\mathcal{F}$ in an affine plane $\mathcal{P}$ we shall mean any parallelogram $0 J_x J J_y$. The lines $0J_x$, $0J_y$ are called coordinate axis. $\mathcal{F}$ determines the planar ternary ring $T_{\mathcal{F}}$ ([1],p.16) for which $\mathcal{P}$ can be identified with $T_{\mathcal{F}} \times T_{\mathcal{F}}$ where $0 = (0,0)$, $J_x = (1,0)$, $J = (1,1)$, $J_y = (0,1)$. Then to each point $A \in 0J_x \setminus \{0\}$ there is exactly one point $A'_{\mathcal{F}} \in 0J_x \setminus \{0\}$ such that $A'_{\mathcal{F}} = (a', 0)$ where $a'a = 1$, $A = (a, 0)$.

Condition (1): Be given a fixed frame $\mathcal{F}^* = 0J_x J^* J_y^*$. Then for each $A \in 0J_x \setminus \{0\}$ the point $A'_{\mathcal{F}}$ is independent on the position of the variable frame $\mathcal{F} = 0J_x J J_y$ where J_y runs over $0J_y^*$.

Proposition 1. In an affine plane $\mathcal{P}$ let there be given a fixed frame $\mathcal{F}^* = 0J_x J^* J_y^*$. Then the conclusion of (1) is equivalent to the "left inverse property"

$(2_{\mathcal{F}^*})$ $\quad a(a'b) = b \quad$ for all $a \in T_{\mathcal{F}^*} \setminus \{0\}$, $b \in T_{\mathcal{F}^*}$

where the multiplication is taken with respect to $T_{\mathcal{F}^*}$.

Convention. If the element a' with $a'a = 1$ determined for $a \in T_{\mathcal{F}^*} \setminus \{0\}$ satisfies also the equation $aa' = 1$ then we shall write $a' = a^{-1}$.

Lemma 1. Let T be a Veblen Wedderburn system ([1], p.17) with the left inverse property. Then for

(3) $a(-1) = -a$ for all $a \in T$,

(4) $(a(-1)(-1) = a$ for all $a \in T$,

it holds (3) $\Longleftrightarrow$ (4), and (3) implies

(5) $a(-b) = -b$ for all $a, b \in T$.

Lemma 2. Let a translation affine plane $\mathcal{P}$ satisfy (1). Then (3) holds in $T_{\mathcal{F}*}$ iff in $\mathcal{P}$ there holds

($6_{\mathcal{F}*}$) If $A_1 B_1 C_1$, $A_2 B_2 C_2$ are triangles such that $A_1, A_2 \in \in OJ_y^*$; $B_1, B_2 \in OJ_x$; $C_1, C_2 \in OJ^*$; $A_1 C_1 \parallel A_2 C_2 \parallel OJ_x$; $B_1 C_1 \parallel B_2 C_2 \parallel OJ_y^*$; $A_1 B_1 \parallel J_x J_y^*$ then $A_2 B_2 \parallel J_x J_y''$.

Lemma 3. Let a translation affine plane $\mathcal{P}$ satisfy (1). Then (4) holds in $T_{\mathcal{F}*}$ iff, in $\mathcal{P}$, it holds

($7_{\mathcal{F}*}$) If $A_1 B_1 C_1 D_1$, $A_2 B_2 C_2 D_2$ are parallelograms such that $A_1, C_1, A_2, C_2 \in OJ^*$; $B_1, C_1, B_2 \in ON$ (N the ideal point of the line $J_x J_y^*$); $C_1 D_1 \parallel C_2 D_2 \parallel OJ_x$; $A_1 D_1 \parallel A_2 D_2 \parallel OJ_y^*$ then $B_2 \in ON$.

Proposition 2. Let $\mathcal{P}$ be a translation affine plane satisfying (1) and ($6_{\mathcal{F}*}$). Then ($6_{\mathcal{F}}$) is valid for all frames $\mathcal{F} = OJ_x J J_y$ with $J_y \in OJ_y^*$.

Lemma 4. Let $\mathcal{P}$ be an affine plane with a fixed frame $\mathcal{F}^* = OJ_x J^* J_y^*$. Then the "right inverse property"

($8_{\mathcal{F}*}$) $(a b')b = a$ for all $a \in T_{\mathcal{F}*}$, $b \in T_{\mathcal{F}*} \setminus \{0\}$.

is satisfied in $T_{\mathcal{F}*}$ iff, in $\mathcal{P}$, there holds

($9_{\mathcal{F}*}$) If $A_1 B_1 C_1 D_1$, $A_2 B_2 C_2 D_2$ are parallelograms such that $A_1 B_1 \parallel C_1 D_1 \parallel A_2 B_2 \parallel C_2 D_2 \parallel OJ_x$; $A_1 D_1 \parallel B_1 C_1 \parallel \parallel A_2 B_2 \parallel C_2 D_2 \parallel OJ_x$; $A_1 D_1 \parallel B_1 C_1 \parallel A_2 D_2 \parallel B_2 C_2 \parallel OJ_y^*$; $B_2 \in OB_1$; $A_1 C_1 = A_2 C_2 = OJ^*$ then $D_2 \in OD_1$

Proposition 3. Let $\mathcal{P}$ be an affine plane satisfying (1) and ($9_{\mathcal{F}^*}$). Then ($9_{\mathcal{F}}$) holds for all frames $\mathcal{F} = 0 J_x J J_y$, $J_y \in 0 J_y^*$ iff the "general right inverse property" is valid in $T_{\mathcal{F}^*}$:

($10_{\mathcal{F}^*}$) $((ac)(c^{-1}b)) \cdot c = a(bc)$ for all $a, b \in T_{\mathcal{F}^*}$ and $c \in T_{\mathcal{F}^*} \setminus \{0\}$.

Remark. If $T_{\mathcal{F}^*}$ possesses associative multiplication then ($10_{\mathcal{F}^*}$) is fulfilled. Moreover, if $T_{\mathcal{F}^*}$ is an alternative field, ($10_{\mathcal{F}^*}$) is satisfied. Further, the associativity of multiplication in $T_{\mathcal{F}^*}$ is equivalent to

($11_{\mathcal{F}^*}$) $(ac)(c^{-1}b) = ab$ for all $a, b \in T_{\mathcal{F}^*}$; $c \in T_{\mathcal{F}^*} \setminus \{0\}$.

Lemma 4'. Let $\mathcal{P}$ be an affine plane with a fixed frame $\mathcal{F}^* = 0 J_x J^* J_y^*$. Then, in $T_{\mathcal{F}^*}$, there holds

($8'_{\mathcal{F}^*}$) $a'(ab) = b$ for all $a \in T_{\mathcal{F}^*} \setminus \{0\}$; $b \in T_{\mathcal{F}^*}$

iff $\mathcal{P}$ satisfies

($9'_{\mathcal{F}^*}$) If $A_1 B_1 C_1 D_1$, $A_2 B_2 C_2 D_2$ are parallelograms such that $A_1 B_1 \parallel C_1 D_1 \parallel A_2 B_2 \parallel C_2 D_2 \parallel 0 J_x$; $A_1 D_1 \parallel B_1 C_1 \parallel A_2 D_2 \parallel B_2 C_2 \parallel 0 J_y^*$; $B_1 C_1 = A_2 D_2$; $A_1 \in 0 A_2$; $B_1 \in 0 B_2$; $C_1 \in 0 C_2$ then $D_1 \in 0 D_2$.

Proposition 3'. Let $\mathcal{P}$ be an affine plane with a fixed frame $\mathcal{F}^* = 0 J_x J^* J_y^*$ and let ($8_{\mathcal{F}^*}$), ($8'_{\mathcal{F}^*}$) be satisfied. Then ($8'_{\mathcal{F}^*}$) holds for all frames $\mathcal{F} = 0 J_x J J_y$ with $J_y \in 0 J_y^*$.

Definition 1. Let $\mathcal{P}$ be a translation affine plane satisfying (1). Let $T_{\mathcal{F}^*}$ satisfy the condition $1 + 1 \neq 0$. If A,B,C are pairwise distinct points on the coordinate axis $0 J_x$ such that $C \neq M_{AB}$ (the "middle point" of A,B) the triple (A,B,C) will be called an <u>admissible triple</u>. To each admissible triple (A,B,C) we associate the point $H^{\mathcal{F}*}_{ABC}$ in the following manner: Write $A = (a, 0)$, $B = (b, 0)$, $C = (c, 0)$ with respect to $T_{\mathcal{F}*}$ and construct the points $SB \cap J^*Y = B_1$, $SC \cap J^*Y = C_1$ (Y the ideal point of OJ^*_y, $S = (a, 1)$), further the point D_1 such that $B_1 = M_{C_1 D_1}$ and finally the point $H^{\mathcal{F}*}_{ABC} = SD_1 \cap OJ_x$.

<u>Proposition 4.</u> From the assumptions of Definition 1 it follows $H^{\mathcal{F}}_{ABC} = H^{\mathcal{F}*}_{ABC}$ for all frames $\mathcal{F} = OJ_x J J_y$ with $J_y \in OJ^*_y$ and for all admissible triples (A,B,C) .

<u>Lemma 6.</u> Let $\mathcal{P}$ be a translation affine plane satisfying (1),($6_{\mathcal{F}*}$),($9_{\mathcal{F}*}$) and $1 + 1 \neq 0$ in $T_{\mathcal{F}*}$. Then for $A = (1, 0)$, $B = (-1, 0)$, $C = (c, 0) \neq (0, 0)$ it follows $H^{\mathcal{F}*}_{ABC} = (c^{-1}, 0)$.

<u>Definition 2.</u> Let $\mathcal{P}$ be a translation affine plane satisfying the assumptions of Lemma 6. By a <u>von Staudt projectivity</u> on OJ_x we shall mean a bijection σ of OJ_x onto itself preserving at both sides all admissible triples and all points $H^{\mathcal{F}*}_{ABC}$ (where (A,B,C) runs over all admissible triples).

<u>Proposition 5.</u> Let $\mathcal{P}$ be a translation affine plane satisfying the assumption of Lemma 6. If σ is a von Staudt projectivity of OJ_x with fixed points O, J_x then the mapping $\sigma_0 : T_{\mathcal{F}*} \longrightarrow T_{\mathcal{F}*}$ defined by $A^{\sigma} = (a^{\sigma_0}, 0)$ for all $A = (a, 0) \in OJ_x$ satisfies the conditions

(i_{σ_0}) $(a+b)^{\sigma_0} = a^{\sigma_0} + b^{\sigma_0}$ for all $a, b \in T_{\mathcal{F}*}$,

(ii_{σ_0}) $(a^{-1})^{\sigma_0} = (a^{\sigma_0})^{-1}$ for all $a \in T_{\mathcal{F}*} \setminus \{0\}$.

Conversely, if $\wp : T_{\mathcal{F}*} \longrightarrow T_{\mathcal{F}*}$ is a bijection with fixed elements 0,1 and if $(i_{\wp})$,$(ii_{\wp})$ are fulfilled then the mapping $\wp^{\bullet} : O J_x \longrightarrow O J_x$ defined by $A^{\wp^{\bullet}} = (a^{\wp}, 0)$ for all $A = (a, 0) \in O J_x$ is a von Staudt projectivity of $O J_x$.

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R e f e r e n c e s

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