Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 8 (1967), No. 3, 503--514

Persistent URL: http://dml.cz/dmlcz/105130

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8,3 (1967)

A NOTE ON THE CONTINUITY PROPERTIES OF NONLINEAR OPERATORS

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1. Introduction. M.M. Vajnberg [1] has proved that every strongly continuous mapping defined on a closed ball D_R $(\|x\| \leq R)$ of a reflexive Banach space X into X is completely compact (i.e., compact and uniformly continuous). Some necessary and sufficient conditions for the strong continuity of F have been proposed by E.S. Citlanadze [2] and E.H. Rothe [3]. These results have been extended and generalized for instance, by M.M. Vajnberg [1,§ 7],[4], and M.I. Kadec [5]. T. Kato [6] has shown that every hemicontinuous locally bounded monotone operator F from Banach space X to its dual X^* is always demicontinous. This result was generalized for vaguely continuous mapping by F. E. Browder [7]. A two-way connection between the range and demicontinuity of nonlinear operators has been established by W.V. Petryshyn [8,th.2,corr.5],cf.also [9,th.5,6] . Some properties of vaguely continuous operators were also discussed in [10].

The purpose of this note is to give some conditions for the strong and weak continuity of nonlinear operators and to derive some basic properties of weak continuous operators acting in Banach spaces (or in linear normed spaces).

2. **Notations and definitions.** Let X, Y be Banach spaces (or linear normed spaces), X^*, Y^* their adjoint(dual) spaces as the set of all bounded conjugate-linear functionals on X, Y, respectively. The pairing between $e^* \in X^*$ (or Y^*) and $x \in X$ is denoted by (x, e^*). We shall use the symbols " $\longrightarrow$ ", " $\xrightarrow{w}$ " to denote the strong convergence in X, Y (or in X^*, Y^*, or in the set of real numbers) and weak convergence in X, Y (or in X^*, Y^*), respectively.

Let F be a mapping with domain $D = D(F) \subset X$ and values in Y. Then:

(1) F is said to be strongly continuous if $x_n \xrightarrow{w} x$ in D implies $F(x_n) \rightarrow F(x)$.

(2) F is said to be **weakly** continuous if $x_n \xrightarrow{w} x$ in D implies $F(x_n) \xrightarrow{w} F(x)$.

(3) F is said to be demicontinuous [11] if $x_n \longrightarrow x$ in D implies $F(x_n) \xrightarrow{w} F(x)$.

(4) F is said to be uniformly demicontinuous on D if for any given constant $\varepsilon > 0$ and $e^* \in Y^*$ there exists a positive number δ such that for every x_1, $x_2 \in D$ with $\| x_1 - x_2 \| < \delta$ there is $|(F(x_1) - F(x_2), e^*)| < \varepsilon$.

(5) F is said to be vaguely continuous [10, 7] if $x \in D$, $v \in X$ and $x + tv \in D$ for $0 < t < t_0$ for some $t_0 > 0$ imply that there exists a sequence $\{t_n\}$ with $t_n > 0$ for all n, $t_n \rightarrow 0$ whenever $n \rightarrow +\infty$ such that $F(x + t_n v) \xrightarrow{w} F(x)$.

(6) F is said to be compact (weakly compact) on D if for every bounded subset $M \subset D$, $F(M)$ is compact

(weakly compact) in Y .

(7) F is said to be completely demicompact on a bounded set $M \subset D$ if for any couples (x'_n, x''_n), $x'_n, x''_n \in M$ with $\|x'_n - x''_n\| \to 0$ as $n \to \infty$ there exist the subsequences (x'_{n_k}, x''_{n_k}) such that $F(x'_{n_k}) \xrightarrow{w} y_0$, $F(x''_{n_k}) \xrightarrow{w} y_0$ and $y_0 \in Y$.

(8) F is said to be locally weakly sequentially bounded if $x_n \in D$, $x \in D$, $x_n \xrightarrow{w} x$ imply that $\{F(x_n)\}$ is bounded in Y .

(9) F is said to be weakly closed in $D_R = \{x \in X : \|x\| \leqq R\}$ if $x_n \xrightarrow{w} x$ in D_R and $F(x_n) \xrightarrow{w} y$ imply $F(x) = y$.

Now suppose that F is a mapping with domain $D(F) = D \subset X$ and values in X^* .

Then:

(10) F is said to be monotone [11, 12] if $Re\,(F(u) - F(v), u - v) \geqq 0$ for all $u, v \in D$.

(11) F is said to be D-maximal monotone if for $u_0 \in D$, $w_0 \in X^*$ the inequality $Re\,(w_0 - F(u), u_0 - u) \geqq 0$ for all $u \in D$ implies that $w_0 = F(u_0)$.

(12) D is said to be quasi-dense [6] if for each $u \in D$ there exists a dense subset M_u of X such that for each $v \in M_u$, $u + tv \in D$ for sufficiently small $t > 0$.

3. Recall that there is known only one theorem [1, th. 7.1] , [5] giving necessary and sufficient condition for the strong continuity of non-potential and non-smooth

operators. Now we prove

Theorem 1. Suppose that X, Y are linear normed spaces. Let one of the following conditions be fulfilled:

a) $F: D_R \longrightarrow Y$ is compact and weakly closed operator on D_R, where $D_R = \{x \in X : \|x\| \leqq R\} \subset X$.

b) $F: D \longrightarrow X^*$, $(D \subset X)$ is D-maximal monotone and compact on D.

c) $F: D \longrightarrow X^*$ is monotone, vaguely continuous and compact on D, where D is quasi-dense in X.

Then F is strongly continuous in D_R, D respectively.

Proof. a) Suppose that $x_0, x_n \in D_R$, $x_n \xrightarrow{w} x_0$. Since $F(D_R)$ is compact, there exists a subsequence such that $F(x_{n_k}) \longrightarrow y_0$, $y_0 \in Y$. Hence $F(x_{n_k}) \xrightarrow{w} y_0$. Since $x_{n_k} \xrightarrow{w} x_0$ and F is weakly closed, $F(x_0) = y_0$. We shall prove that $F(x_n) \xrightarrow{w} F(x_0)$. Suppose, on the contrary, there exist $\varepsilon_0 > 0$, $e_0^* \in Y^*$ and an increasing sequence $n_1, n_2, \ldots$ of integers such that

(1) $\quad |(F(x_{n_j}) - F(x_0), e_0^*)| \geqq \varepsilon_0$.

Since $\{F(x_{n_j})\} \in F(D_R)$, there exists a subsequence $x_{n_{j_k}}$ such that $F(x_{n_{j_k}}) \longrightarrow x$. Hence $F(x_{n_{j_k}}) \xrightarrow{w} x$. Because $x_{n_{j_k}} \xrightarrow{w} x_0$ and F is weakly closed in D_R, $F(x_0) = x$. Hence $F(x_{n_{j_k}}) \xrightarrow{w} F(x_0)$, which contradicts (1). But $\{F(x_n)\} \in$ $\in F(D_R)$ and $F(D_R)$ is compact. Since weak convergence is equivalent to the strong one on compact set, we have that $F(x_n) \longrightarrow F(x_0)$. b) Let $\{x_n\}$ be a sequence in D with $x_n \xrightarrow{w} x_0$, $x_0 \in D$. Then

$\| x_n \| \leqq c, \quad c > 0$ and $\{ F(x_n) \}$ is compact. We have to show that $F(x_n) \longrightarrow F(x_0)$. Suppose that $\{F(x_n)\}$ does not converge to $F(x_0)$. Then there exists a subsequence $\{ x_{n_k} \}$ such that $F(x_{n_k}) \not\longrightarrow F(x_0)$. Passing to a subsequence $F(x_{n_{k_j}})$, we have that $F(x_{n_{k_j}}) \longrightarrow y_0$. Let u be any element of D. Then $F(x_{n_{k_j}}) - F(u) \rightarrow y_0 - F(u)$ and $Re\,(F(x_{n_{k_j}}) - F(u), x_{n_{k_j}} - u) \geqq 0$ by the monotonicity. Since $x_{n_{k_j}} \xrightarrow{w} x_0$, $Re\,(y_0 - F(u), x_0 - u) \geqq 0$ for every $u \in D$. Since F is D-maximal monotone, $y_0 = F(x_0)$, which is a contradiction. Hence $F(x_n) \longrightarrow F(x_0)$. The assertion c) is a corollary of Browder's theorem [7] and b). This concludes the proof.

Corollary 1. Suppose that X is a reflexive Banach space, Y a linear normal space. Let one of the following conditions be fulfilled:

a) $F: D_R \longrightarrow Y$ is compact and weakly closed operator on $D_R \subset X$.

b) $F: D_R \longrightarrow X^*$, $(D_R \subset X)$ is D_R-maximal monotone and compact on D_R.

Then F is completely compact and hence uniformly continuous on D_R.

Theorem 2. Suppose that X, Y are linear normed spaces. Let one of the following conditions be fulfilled:

a) $F: D_R \longrightarrow Y$ is weakly compact and weakly closed on $D_R \subset X$.

b) $F: D_R \rightarrow X^*$, $D_R \subset X$ (or $F: D_R \rightarrow Y$, where Y is a reflexive Banach space) is locally weakly sequentially bounded and weakly closed on D_R.

Then F is weakly continuous on D_R .

Theorem 3. Let X be a reflexive Banach space, Y a linear normed space, $F: D_R \longrightarrow Y$ a weakly continuous mapping of a closed ball D_R $(\|x\| \leqq R) \subset X$ into Y . Then F is uniformly demicontinuous, weakly compact and bounded on D_R .

Proof. Suppose the contrary; then there exist $\varepsilon_0 > 0$, $z_0^* \in Y^*$ with the following property: for every n $(n = 1, 2, \ldots)$ there exist x_n', $x_n'' \in D_R$ such that $\|x_n' - x_n''\| < \frac{1}{n}$ and $|(F(x_n') - F(x_n''), z_0^*)| \geqq \varepsilon_0$.

Since D_R is weakly compact, there exists a subsequence $\{x_{n_k}'\}$ such that $x_{n_k}' \xrightarrow{w} x_0$. Because D_R is weakly closed, $x_0 \in D_R$. As $n \to \infty$ $x_n' - x_n'' \xrightarrow{w} 0$. Passing to a subsequence $\{x_{n_k}''\}$ we have that $x_{n_k}'' \xrightarrow{w} x_0$. Since F is weakly continuous on D_R, $F(x_{n_k}') \xrightarrow{w} F(x_0)$, $F(x_{n_k}'') \xrightarrow{w} F(x_0)$. Let z^* be any element of Y^*. Ther

$$|(F(x_{n_k}') - F(x_{n_k}''), z^*)| \leqq |(F(x_{n_k}') - F(x_0), z^*)| + \\ + |(F(x_0) - F(x_{n_k}''), z^*)| .$$

Hence $|(F(x_{n_k}') - F(x_{n_k}''), z^*)| \to 0$ whenever $n \to \infty$, which is a contradiction. Thus F is uniformly demicontinuous on D_R . Let $\{x_n\}$ be any sequence of D_R . Since D_R is weakly compact, there exists a subsequence x_{n_k} such that $x_{n_k} \xrightarrow{w} x_0$ and $x_0 \in D_R$. Since F is weakly continuous, $F(x_{n_k}) \xrightarrow{w} F(x_0)$. Therefore F is weakly compact on D_R and hence F is weakly bounded on D_R . But weak boundedness is equivalent to boundedness. This completes the proof.

Remark 1. If F is a compact closed mapping of a closed ball $D_R \subset X$ into Y, where X, Y are linear normed spaces, then F is continuous on D_R. Recall that a continuous mapping defined on $D_R \subset X$ is not uniformly continuous on D_R in general even if we impose on F the condition of boundedness. Let us remark that Theorem 3 is valid if we replace D_R by an arbitrary convex closed bounded set $M \subset X$.

Corollary 2. Suppose that X is a reflexive Banach space, Y a linear normed space. Assume that one of the following two conditions is satisfied:

a) $F : D_R \longrightarrow Y$ is weakly compact and weakly closed on $D_R \subset X$.

b) $F : D_R \longrightarrow X^*$, $D_R \subset X$ (or $F: D_R \to Y$, where Y is a reflexive Banach space) is locally weakly sequentially bounded and weakly closed on D_R.

Then F is uniformly demicontinuous, weakly compact and bounded on D_R.

Theorem 4. Let X, Y be linear normed spaces, F a mapping of X into Y. Then F is completely demicompact on a bounded subset $M \subset X$ if and only if F is uniformly demicontinuous and weakly compact on M.

Proof. (The proof is similar to that of M.M. Vajnberg's theorem [1,chap.I].) Suppose that F is completely demicompact on M. Taking (x_n, x_n), $x_n \in M$, there exists a subsequence $\{x_{n_k}\}$ such that $\{F(x_{n_k})\}$ weakly converges in Y. Hence F is weakly compact. Suppose that F is not uniformly demicontinuous on M. Then there exist

$\varepsilon_0 > 0$, $z_0^* \in Y^*$ with the following property: for every n $(n = 1, 2, \dots)$ there exist $x_n', x_n'' \in M$ such that $\| x_n' - x_n'' \| < \frac{1}{n}$ and $|(F(x_n') - F(x_n''), z_0^*)| \geq \varepsilon_0$. But this is a contradiction. Conversely, assume that F is weakly compact and uniformly demicontinuous on M. Let (x_n', x_n'') be any sequence of couples of M such that $\| x_n' - x_n'' \| \to 0$ as $n \to \infty$. From weak compactness of F it follows the existence of the subsequence (x_{n_k}', x_{n_k}'') such that $F(x_{n_k}') \xrightarrow{w} y_1$, $F(x_{n_k}'') \xrightarrow{w} y_2$. Assume $y_1 \neq y_2$ and set $\| y_1 - y_2 \| = 2\varepsilon$. According to Hahn-Banach theorem there exists $z_1^* \in Y^*$ $\| z_1^* \| = 1$ such that $|(y_1 - y_2, z_1^*)| = \| y_1 - y_2 \| = 2\varepsilon$. Hence there exists a subsequence (x_{m_k}', x_{m_k}'') such that $x_{m_k}' - x_{m_k}'' \to 0$ as $k \to \infty$ and $|(F(x_{m_k}') - F(x_{m_k}''), z_1^*)| \geq \varepsilon$ $(k = 1, 2, \dots)$. But this contradicts the uniform demicontinuity of F. This completes the proof.

Corollary 3. Let X be a reflexive Banach space, Y a linear normed space, $F : D_R \to Y$ a weakly continuous mapping of a closed ball D_R $(\| x \| \leq R) \subset X$ into Y. Then F is completely demicompact on D_R.

Remark 2. If either a) or b) of Corollary 2 is satisfied, then F is completely demicompact on D_R.

Suppose that X is a reflexive separable Banach space. M.I. Kadec has shown that we can introduce an equivalent norm in X under which X is strictly normed. Assume that X is provided by this norm. Let G be a closed linear subspace of X. Define the operator P of metric projection on G by

$$\| x - Px \| = \min_{y \in G} \| x - y \| .$$

Since X is reflexive and strictly normed (and hence strictly convex), P exists and is single-valued. An element $x \in X$ is called orthogonal to G if $Px = 0$. According to [5], taking a complete linearly independent system of the elements $e_1^*, e_2^*, \ldots, e_n^*, \ldots$ of X^*, we form a decreasing sequence of subspaces $X \supset X_1 \supset X_2 \supset \ldots$, where $X_n = \{x \in X : e_k^*(x) = 0, \ k = 1, 2, \ldots, n\}$. Denote by X^n the set of all elements of X orthogonal to X_n. For egery $x \in X$ there is a unique decomposition $x = P^n x + P_n x$, where P^n, P_n are operators of metric projections on X^n, X_n, respectively.

Theorem 5. Let X be a separable reflexive Banach space, F a demicontinuous mapping of X into X. Then F is weakly continuous on a closed ball $D_R(\|x\| \leq R) \subset X$ if and only if for any given $\varepsilon > 0$ and $e^* \in Y^*$ there exists an integer $n_0(\varepsilon, e^*)$ such that for every $n \geq n_0$ and $x \in D_R$ there is

$$(2) \qquad |(F(P^n x) - F(x), e^*)| < \varepsilon .$$

Proof. Suppose F is weakly continuous on D_R and (2) does not hold. Then there exist $\varepsilon_0 > 0$, $e_0^* \in X^*$ and the subsequence $x_{n_k} \in D_R$, $x_{n_k} \xrightarrow{w} x_0$, $x_0 \in D_R$ such that

$$(3) \qquad |(F(P^{n_k} x_{n_k}) - F(x_{n_k}), e_0^*)| \geq \varepsilon_0 .$$

According to [5, lemma 2], $P^{n_k} x_{n_k} \xrightarrow{w} x_0$. Hence $F(P^{n_k} x_{n_k}) \xrightarrow{w} F(x_0)$ and (3) contradicts the weak continuity of F. Conversely, assume that $x_m \in D_R$, $x_0 \in D_R$, $x_m \xrightarrow{w} x_0$. Let $\varepsilon > 0$ be any positive number, $e^* \in Y^*$ and suppose that there exists an integer

$n_0(\varepsilon, e^*)$ such that for every $n \geq n_0$, $x \in D_R$ there is

$$|(F(P^n x) - F(x), e^*)| < \frac{\varepsilon}{4} .$$

Then

$$|(F(x_m) - F(x_0), e^*)| \leq |(F(x_m) - F(P^n x_m), e^*)| +$$

$$+ |(F(P^n x_m) - F(P^n x_0), e^*)| + |(F(P^n x_0) - F(x_0), e^*)| <$$

$$< \frac{\varepsilon}{2} + |(F(P^n x_m) - F(P^n x_0), e^*)|$$

for every m, $n \geq n_0$. According to [5] $P^n x_m \to$ $\to P^n x_0$ as $m \to \infty$. Since F is demicontinuous, $F(P^n x_m) \xrightarrow{w} F(P^n x_0)$. Thus $F(x_m) \xrightarrow{w} F(x_0)$ and this concludes the proof.

<u>Theorem 6</u>. Let X be a reflexive Banach space, Y a linear normed space, F a weakly continuous mapping of a convex closed bounded subset $M \subset X$ into Y. Then there exists $x_0 \in M$ such that $\| F(x_0) \| =$ $= \inf_{x \in M} \| F(x) \|$.

<u>Proof</u>. According to Theorem 3 F is bounded on M. Set $d = \inf_{x \in M} \| F(x) \|$. Then there exists a sequence $\{x_n\} \in M$ such that $\lim_{n \to \infty} \| F(x_n) \| = d$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \xrightarrow{w} x_0$. Then $x_0 \in M$ and $F(x_{n_k}) \xrightarrow{w}$ $\xrightarrow{w} F(x_0)$ by weak continuity of F. Hence $\| F(x_0) \| \leq$ $\leq \underline{\lim}_{k \to \infty} \| F(x_{n_k}) \| = \lim_{k \to \infty} \| F(x_{n_k}) \| = d$. On the other hand $d \leq \| F(x_0) \|$. Hence $d = \| F(x_0) \|$ and this completes the proof.

Remark 3. Some of these theorems will be applied in forthcoming paper which deals with the weak differentiability of mappings in functional spaces.

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(Received April 10, 1967)