Commentationes Mathematicae Universitatis Carolinae

Břetislav Novák

Mean value theorems in the theory of lattice points with weight

Commentationes Mathematicae Universitatis Carolinae, Vol. 8 (1967), No. 4, 711--733

Persistent URL: http://dml.cz/dmlcz/105144

Commentationes Mathematicae Universitatis Carolinae
8,4 (1967)

MEAN VALUE THEOREMS IN THE THEORY OF LATTICE POINTS WITH WEIGHT

Břetislav NOVÁK, Praha

§ 1. Introduction. Let k be a natural number, $k \geq 2$, and let

$$Q(u_j) = \sum_{l,j=1}^{k} a_{lj} u_l u_j$$

be a positive definite quadratic form with integer coefficients and the determinant D. Let further $M_1, M_2, \dots, M_k$ be natural numbers and $b_1, b_2, \dots, b_k$ integers. For arbitrary real numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ and $x > 0$ let

$$A(x) = A(x; \alpha_j) = \sum e^{2\pi i \sum_{j=1}^{k} \alpha_j u_j} ,$$

where the summation is over all systems $u_1, u_2, \dots, u_k$ of real numbers satisfying

$$u_j \equiv b_j \pmod{M_j}$$

$(j = 1, 2, \dots, k)$ and

$$Q(u_j) \leq x .$$

Let us put as usually

$$V(x) = V(x; \alpha_j) = \frac{M x^{\frac{k}{2}} e^{2\pi i \sum_{j=1}^{k} \alpha_j b_j}}{\Gamma(\frac{k}{2} + 1)} \delta$$

$\left(M = \frac{\pi^{k/2}}{\sqrt{D} \prod_{j=1}^{k} M_j} \right.$; $\delta = 1$ if all numbers $\alpha_1 M_1, \alpha_2 M_2, \dots, \alpha_k M_k$ are integers, $\delta = 0$ otherwise) and let us consider the "lattice rest"

$$(1) \qquad P(x) = P(x; \alpha_j) = A(x) - V(x).$$

As is known (see [5] pp.11-84), we have

$$P(x) = O(x^{\frac{k}{2} - \frac{k}{k+1}})$$

and (if $A(x) \not\equiv 0$ - we shall exclude from our considerations the case where $A(x) = 0$ identically)

$$P(x) = \Omega(x^{\frac{k-1}{4}}) .$$

In the papers [6] - [11] there were proved the following results:

I. Let $k > 4$.

a) There always holds

$$P(x) = O(x^{\frac{k}{2}-1}) .$$

b) If $\alpha_1, \alpha_2, \ldots, \alpha_k$ are rational numbers we have either

$$P(x) = \Omega(x^{\frac{k}{2}-1})$$

or

$$P(x) = O(x^{\frac{k}{4}-\frac{1}{10}}) .$$

c) If at least one of the numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ is irrational, then

$$P(x) = o(x^{\frac{k}{2}-1}) .$$

d) If $\varphi(x)$ is a positive non-increasing function, $\varphi(x) = o(1)$, there exists a system $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$$P(x) = o(x^{\frac{k}{2}-1}) \quad \text{and} \quad P(x) = \Omega(x^{\frac{k}{2}-1}\varphi(x))$$

hold.

e) For almost all systems $\alpha_1, \alpha_2, \dots, \alpha_k$ (in the sense of the Lebesgue measure in the k-dimensional Euclidean space E_k) there is

$$P(x) = O(x^{\frac{k}{4}} \lg^{3k} x)$$

(see [6], Theorems 3,4,5 and [10], p. 67).

II. Let $k > 5$, $\alpha_1 = \alpha_2 = \dots = \alpha_k = \alpha$, and let $\gamma = \gamma(\alpha)$ be the supremum of all numbers $\beta > 0$ for which the inequality

$$\min_{p \text{ integer}} |\alpha k - p| \leq \frac{c}{k^\beta}$$

is satisfied for infinitely many natural k's, c being a positive constant depending at most on α and β. Let us put

$$f = \left(\frac{k}{4} - \frac{1}{2}\right) \frac{2\gamma + 1}{\gamma + 1}$$

(for $\gamma = +\infty$ put $f = \frac{k}{2} - 1$). Then

$$P(x) = O(x^{f+\varepsilon})$$

for every $\varepsilon > 0$. If $b_1 = b_2 = \dots = b_k = 0$, then we have, for every $\varepsilon > 0$,

$$P(x) = \Omega(x^{f-\varepsilon})$$

(see [7], Theorem 4).

From the results presented above there follow corresponding O-estimates of the function

$$T(x) = \sqrt{M(x)/x},$$

where

$$M(x) = \int_0^x |P(y)|^2 dy .$$

The direct investigation of the function $M(x)$ provides often results which are even sharper:

III. It is always

$$\liminf_{x \to +\infty} \frac{M(x)}{x^{r/2+1/2}} > 0$$

and thus

$$M(x) = \Omega(x^{\frac{r}{2}+\frac{1}{2}}) .$$

Further,

$$M(x) = O(x^{r-1})$$

for $r \geq 4$,

$$M(x) = O(x^2 \lg x)$$

for $r = 3$ and

$$M(x) = O(x^{3/2})$$

for $r = 2$.

(See [9], Theorem 3.)

These results cannot be improved as it may be seen from the following assertions:

IV. a) Let the numbers $\alpha_1, \alpha_2, \ldots, \alpha_r$ be rational. Then

$$M(x) = H_r x^{r-1} + o(x^{r-1})$$

for $r \geq 4$,

$$M(x) = H_3 x^2 \lg x + O(x^2 \lg^{1/2} x)$$

for $r = 3$, where H_r are nonnegative constants depending only on Q, M_j, b_j and α_j $(j = 1, 2, \ldots, r)$.[1)]

1) We have $H_r > 0$ if, e.g., $b_1 = b_2 = \ldots = b_r = 0$ (see [7], Lemma 9 and [9], Theorem 1).

If we have for some form Q and suitable numbers M_j, b_j and α_j $(j = 1, 2, \ldots, r)$, $H_r = 0$, then even [2)]

$$M(x) = O(x^{\frac{r}{2}+\frac{1}{2}}) .$$

b) For almost all systems $\alpha_1, \alpha_2, \ldots, \alpha_r$ (again in the sense of Lebesgue measure in E_r) there is

$$M(x) = O(x^{\frac{r}{2}+\frac{1}{2}} \lg^{3r+2} x)$$

(See [9], Theorems 1,2 and [8], Theorem 1.)

The main aim of the presented paper is to complete the results on the O-estimations of function $M(x)$. Our examinations will be based on the following Theorem, which shall be proved using Jarník's method (see [1] -[3]):

Main Theorem. Let $\overline{Q}$ be the form conjugated to Q, and, for a natural number k, let

$$R_k = \min \overline{Q}\left(\frac{m_j}{M_j} - \alpha_j k\right),$$

the minimum being taken over all systems $m_1, m_2, \ldots, m_r$ of integers. Then

$$M(x) = O\left(x^{\frac{r}{2}} \sum_{1 \leq k \leq \sqrt{x}} \min^{\frac{r}{2}-\frac{1}{2}}\left(\frac{x}{k^2}, \frac{1}{R_k}\right) \frac{k}{\sqrt{x}}\right)$$

---

2) Let us remark that in this case Walfisz ([11]) has shown with help of the theory of modular forms that for $r \geq 4$, even $M(x) = K_r x^{\frac{r}{2}+\frac{1}{2}} + O(x^{\frac{r}{2}} \lg^2 x)$, K_r being a positive constant depending only on Q, M_j, b_j and α_j $(j = 1, 2, \ldots, r)$ (see [11] and [10], Lemma 11).

(for $A = 0$, $B \geq 0$, put $\min(B, \frac{1}{A}) = B$).

§ 2. Notations and auxiliary Theorems. In the whole paper we shall preserve the following notations and agreements:

The letter c means (eventually also various) positive constants, which depend on Q, M_j, b_j and α_j ($j = 1, 2, \ldots, n$). $c(\varepsilon)$, $c(\beta_j)$, respectively, etc. are positive constants (various) depending moreover on ε, $\beta_1, \beta_2, \ldots, \beta_n$, respectively, etc. The symbols O, o and Ω have the usual meaning, i.e., they refer to the limiting process $x \to +\infty$ and the constants involved are of the "type" c. We express the validity of the relation $|A| \leqq c B$ shortly by $A << B$. n, k, k' and k'' mean natural numbers, m_1, $m_2, \ldots, m_n$, h, h', h'', p integers. If h and k (h' and k' etc.) are to appear simultaneously then always $(h,k) = 1$ ($(h',k') = 1$ etc.). For a real t let $\langle t \rangle$ be the distance of t to the nearest integer, i.e.,

$$\langle t \rangle = \min_p |t - p| .$$

Further, let us put

$$P_k = \max_{j=1,2,\ldots,n} \langle \alpha_j M_j k \rangle .$$

It is easy to show (see [6], Remark 2) that

$$P_k^2 << R_k << P_k^2 .$$

In the whole work it will be assumed that the number x is sufficiently large, i.e., $x > c$. Let us put

$$M(y) = M_1(y) ,$$

and let

$$M_2(x) = \int_0^x M_1(y)\,dy .$$

For a complex number s, $\operatorname{Re} s > 0$, let

$$\theta(s) = \theta(s;\alpha_j) = \sum e^{-sQ(m_j M_j + b_j) + 2\pi i \sum_{j=1}^{k} \alpha_j (m_j M_j + b_j)} ,$$

where the summation is over all systems $m_1, m_2, \ldots, m_k$.
As known, the function $\theta(s)$ is a holomorphic function in the half plane $\operatorname{Re} s > 0$. By an integral we always mean the (absolute convergent) Lebesgue integral; for real a we put

$$\int_{(a)} f(s)\,ds = i \int_{-\infty}^{\infty} f(a+it)\,dt$$

and (for $s = \frac{1}{x} + it$, $-\infty \leq a \leq b \leq +\infty$)

$$\int_a^b f(s)\,dt = \int_a^b f\left(\frac{1}{x} + it\right) dt ,$$

if the integrals on the right hand sides exist.

Let us remind some known properties of the Farey's fractions corresponding to $\sqrt{x}$, i.e. the fractions of the form h/k, where $k \leq \sqrt{x}$ (see [5] pp.249-250):
If $h'/k' < h/k < h''/k''$ are three succeeding fractions of this form (i.e. between h'/k' and h''/k'' lies just one Farey's fraction corresponding to $\sqrt{x}$ - that is $\frac{h}{k}$) then necessarily $hk' - h'k = 1$, $h''k - hk'' = 1$, $k + k' > \sqrt{x}$, $k + k'' > \sqrt{x}$.
If we put thus, for $k \leq \sqrt{x}$,

$$\mathcal{L}_{h,k} = \left\langle 2\pi \frac{h+h'}{k+k'},\ 2\pi \frac{h+h''}{k+k''} \right) ,$$

then, for $t \in \mathcal{L}_{h,k}$, the relation

$$(2) \qquad \left| t - \frac{2\pi h}{k} \right| \leqq \frac{2\pi}{k\sqrt{x}}$$

holds. The intervals $\mathcal{L}_{h,k}$ are, of course, disjunctive and they cover the entire real axis. If we put

$$w = \frac{2\pi}{[\sqrt{x}\,] + 1}$$

(for real t, $[\,t\,]$ is the integral part of the number t) then clearly

$$\mathcal{L}_{0,1} = \langle -w, w) .$$

At the end of this paragraph let us present several auxiliary assertions.

<u>Lemma 1</u>. For $a > 0$ and $b > 0$, we have

$$M_2(x) = -\frac{1}{4\pi^2} \int_{(a)} \int_{(b)} \frac{F(s)\,G(s')}{ss'(s+s')^2}\, e^{x(s+s')}\, ds\, ds' + O(x) ,$$

where

$$F(s) = \Theta(s) - \frac{M e^{2\pi i \sum_{j=1}^{k} \alpha_j b_j}}{s^{k/2}}\,\sigma,\quad G(s) = \overline{F(\bar{s}')} .$$

The <u>proof</u> can be carried out almost in the same way as in the papers [1] -[3].

<u>Lemma 2</u>. Let $s = \frac{1}{x} + it$. If $t << w$ then

$$(3) \qquad \frac{F(s)}{s} << x^{\frac{k}{4}+\frac{1}{2}} .$$

If $\left| t - \frac{2\pi h}{k} \right| << \frac{1}{k\sqrt{x}}$ (this being accomplished, according to (2) for $t \in \mathcal{L}_{h,k}$) and $h \neq 0$, then

$$(4) \qquad F(s) << \frac{x^{\frac{r}{2}}}{k^{\frac{r}{2}}} \; \frac{e^{-\frac{c R_k x}{k^2(1+x^2|t-\frac{2\pi h}{k}|^2)}}}{(1+x^2|t-\frac{2\pi h}{k}|^2)^{\frac{r}{4}}} .$$

Analogous assertions hold for the function $G(s')$.

Proof. See [8], Lemma 3 and [9], Lemma 7. Let us remark that, if $\sigma = 1$, then necessarily $R_k = 0$ for all k's.

Lemma 3. Using the notation of IV.a), § 1, we have, for $r \geq 4$,

$$H_r = \frac{M^2 x^{r-1}}{4\pi^2 (r-1)\Gamma^2(\frac{r}{2})} \sum_{k \equiv 0 (\mathrm{mod}\, H)} \sum_{h \neq 0} \frac{|S_{h,k}|^2}{k^{2r-2} h^2} ,$$

where

$$S_{h,k} = \sum_{a_1, a_2, \dots, a_r = 1}^{k} e^{-2\pi i \frac{h}{k} Q(a_j M_j + b_j) + 2\pi i \sum_{j=1}^{r} \alpha_j (a_j M_j + b_j)}$$

and H is the least common denominator of the numbers $\alpha_1 M_1, \alpha_2 M_2, \dots, \alpha_r M_r$. If $(H, 2D \prod_{j=1}^{r} M_j^2) = 1$, there is $|S_{h,k}| = k^{\frac{r}{2}}$ for $k \equiv 0 (\mathrm{mod}\, H)$.

Proof. See [9], Theorem 1 and [6], Lemma 2 and Definition 2.

Lemma 4. Let $r \geq 6$, $\alpha_1 = \alpha_2 = \dots = \alpha_r = \alpha$ and let the inequality

$$\langle \alpha k \rangle >> \frac{1}{k^{\beta}}$$

be satisfied for all k's (and thus $\beta \geq 1$). Then

$$S_k(x;\alpha,M_1,M_2,\dots,M_k)=x^{\frac{k}{4}-\frac{1}{2}}\sum_{k\le\sqrt{x}}\min{}^{\frac{k}{4}-\frac{1}{2}}\left(\frac{x}{k^2},\frac{1}{P_k^2}\right)<<x^{\left(\frac{k}{4}-\frac{1}{2}\right)\frac{2\beta+1}{\beta+1}}g(x),$$

where $g(x)=\lg x$ for $k=6$ and $\beta=1$, $g(x)=$ $=1$ in all other cases.

<u>Proof</u>. See [7], the proof of Theorem 1 (relations (36), (37),(41),(44),(49)-(51) and b) of this proof).

§ 3. <u>Proof of the Main Theorem</u>. We shall follow the considerations of the Paper [8]. Let us always write

$s=\frac{1}{x}+it$, $s'=\frac{1}{x}+it'$, t and t' being real numbers.

From Lemma 1 (for $a=b=\frac{1}{x}$) we have, taking regard to the obvious relation

$$e^{x(s+s')}<<1,$$

(5) $$M_2(x)<<\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{F(s)F(\bar{s}')}{ss'(s+s')^2}\right|dt\,dt'+O(x).$$

Because of the symmetry of the integrand we can write

(6) $$M_2(x)<<T_1+T_2+T_3+O(x),$$

where

$$T_1=\int_{-2w}^{2w}\int_{-2w}^{2w}\dots\,dt\,dt',$$

$$T_2=\int_{-w}^{w}\int_{2w}^{\infty}\dots\,dt\,dt'+\int_{-w}^{w}\int_{-\infty}^{-2w}\dots\,dt\,dt',$$

$$T_3 = \int_w^\infty \int_w^\infty \dots dt\, dt' + \int_w^\infty \int_{-\infty}^{-w} \dots dt\, dt'$$

(the integrands which are not presented are the same as those in (5)). According to (3), there is

$$(7) \quad T_1 << x^{\frac{k}{2}+2} \int_0^{2w} \left(\int_0^t \frac{dt'}{(1+x(t-t'))^2} \right) dt << x^{\frac{k}{2}+\frac{3}{2}} \quad .$$

To estimate T_2 and T_3, let us first of all consider the following assertions: Let $\beta = c$, $\beta \geq \frac{1}{2}$. Then

$$(8) \int_0^{\frac{c}{k\sqrt{x}}} \frac{e^{-\frac{cR_k x}{k^2(1+x^2u^2)}}}{(1+x^2u^2)^\beta} du << \int_0^{\frac{c}{k\sqrt{x}}} \frac{du}{(1+x^2u^2)^\beta} << \begin{cases} \frac{1}{x} & \text{for } \beta > \frac{1}{2} \\ \frac{lg\, x}{x} & \text{for } \beta = \frac{1}{2} \end{cases}$$

If however $R_k \neq 0$, we can write

$$\int_0^{\frac{c}{k\sqrt{x}}} \frac{e^{-\frac{cR_k x}{k^2(1+x^2u^2)}}}{(1+x^2u^2)^\beta} du << \left(\frac{k^2}{R_k x}\right)^\beta \int_0^{\frac{c}{k\sqrt{x}}} \left(\frac{cR_k x}{k^2(1+x^2u^2)}\right)^\beta e^{-\frac{cR_k x}{k^2(1+x^2u^2)}} du \quad .$$

The last integral, for $\beta > \frac{1}{2}$, can be estimated by means of the expression

$$\int_0^{\sqrt{\frac{R_k}{k^2x}}} c\, du + \left(\frac{cR_k x}{k^2}\right)^\beta \int_{\sqrt{\frac{R_k}{k^2x}}}^\infty \frac{du}{(x^2u^2)^\beta} << \sqrt{\frac{R_k}{k^2x}}$$

(for $\xi \geq 0$ there is $\xi^c e^{-c\xi} << 1$). For $\beta > \frac{1}{2}$, $\beta = c$, we thus obtain, according to (8),

$$(9)\quad \int_0^{\frac{c}{k\sqrt{x}}} \frac{e^{-\frac{cR_k x}{k^2(1+x^2u^2)}}}{(1+x^2u^2)^{\beta}}\,du << \min\left(\frac{1}{x}, \frac{k^{2\beta-1}}{R_k^{\beta-\frac{1}{2}}x^{\beta+\frac{1}{2}}}\right).$$

First of all, let us estimate T_2. Let us remark that, according to (2) for $t \in \mathcal{L}_{h,k}$, $h > 0, t > 2w, |t'| \leq w$, we have

$$|s+s'| >> \frac{h}{k} \qquad \text{and} \qquad |s| >> \frac{h}{k}.$$

If we now use (3) and (4) (we decompose the integration path into intervals $\mathcal{L}_{h,k}$ and in each of them we use the corresponding estimate (4)) we obtain, according to (9) (for $\beta = \frac{\kappa}{4}, \beta > \frac{1}{2}$, i.e., for $\kappa > 2$) or according to (8) (for $\beta = \frac{\kappa}{4} = \frac{1}{2}$, i.e., for $\kappa = 2$), successively (making use of (2))

$$T_2 << x^{\frac{\kappa}{4}+\frac{1}{2}} \int_0^{w} \sum_{k\leq\sqrt{x}} \sum_{h=1}^{\infty} \left(\frac{k}{h}\right)^3 \frac{x^{\frac{\kappa}{2}}}{k^{\kappa/2}} \int_0^{\frac{c}{k\sqrt{x}}} \frac{e^{-\frac{cR_k x}{k^2(1+x^2u^2)}}}{(1+x^2u^2)^{\kappa/4}}\,du <<$$

$$<< x^{\frac{3\kappa}{4}} \sum_{k\leq\sqrt{x}} \frac{1}{k^{\kappa/2-3}} \min\left(\frac{1}{x}, \frac{k^{\frac{\kappa}{2}-1}}{R_k^{\kappa/4-1/2}x^{\kappa/4+1/2}}\right) \lg x.$$

An easy rearranging provides

$$(10)\qquad T_2 << x^{\frac{\kappa}{2}+1} \sum_{k\leq\sqrt{x}} \min{}^{\frac{\kappa}{4}-\frac{1}{2}}\left(\frac{x}{k^2}, \frac{1}{R_k}\right) \frac{k}{\sqrt{x}}.$$

Let us pass over to the estimation of T_3. Obviously,

$$\int_w^\infty \int_w^\infty \left| \frac{F(s)\,F(\bar{s}')}{s s'(s+s')^2} \right| dt\,dt' << \int_w^\infty \int_w^\infty \frac{|F(s)|^2 + |F(\bar{s}')|^2}{|s s'(s+s')^2|}\, dt\,dt' \quad ,$$

and an analogous inequality is obtained also for the second integral appearing in T_3 . From the symmetry of the integrands there follows that

$$(11) \qquad T_3 << \int_w^\infty \int_w^\infty \frac{|F(s)|^2 + |F(\bar{s})|^2}{t t'(\frac{1}{x} + |t-t'|)^2}\, dt'\,dt \quad .$$

Analogously as in [8] (relations (29)-(33)), we find easily that, for $t \geq w$,

$$\int_w^\infty \frac{dt'}{t'(\frac{1}{x} + |t-t'|)^2} << \frac{x}{t} \quad ,$$

and thus substituting into (11) we obtain

$$T_3 << x \int_w^\infty \frac{|F(s)|^2 + |F(\bar{s})|^2}{t^2}\, dt \quad .$$

We again decompose the integration path into intervals

$\mathcal{L}_{h,k}$ $(h > 0,\ k \leq \sqrt{x})$ and in each of them we use the corresponding estimate (4). According to (9) (for $\beta = \frac{\kappa}{2} > \frac{1}{2}$), we successively obtain (for $t \in \mathcal{L}_{h,k}$, $h > 0$, we have, according to (2), $t >> \frac{h}{k}$)

$$T_3 << x^{\kappa+1} \sum_{k \leq \sqrt{x}} \sum_{h=1}^{\infty} \frac{k^2}{h^2} \frac{1}{k^\kappa} \int_0^{\frac{c}{h\sqrt{x}}} \frac{e^{-\frac{cR_k x}{h^2(1+x^2u^2)}}}{(1+x^2u^2)^{\kappa/2}}\, du <<$$

$$(12) \qquad << x^{\kappa+1} \sum_{k \leq \sqrt{x}} \frac{1}{k^{\kappa-2}} \min\left(\frac{1}{x}, \frac{k^{\kappa-1}}{R^{\frac{\kappa}{2}-\frac{1}{2}}\, x^{\frac{\kappa}{2}+\frac{1}{2}}}\right) <<$$

$$<< x^{\frac{k}{2}+1} \sum_{k \leq \sqrt{x}} \min^{\frac{k}{2}-\frac{1}{2}} \left(\frac{x}{k^2}, \frac{1}{R_k} \right) \frac{k}{\sqrt{x}} \quad .$$

Let us now denote

$$F(x) = x^{\frac{k}{2}+1} \sum_{k \leq \sqrt{x}} \min^{\frac{k}{2}-\frac{1}{2}} \left(\frac{x}{k^2}, \frac{1}{R_k} \right) \frac{k}{\sqrt{x}} \quad .$$

Obviously ($R_k << 1$)

$$(13) \qquad F(x) >> x^{\frac{k}{2}+1} \sum_{k \leq \sqrt{x}} \frac{k}{\sqrt{x}} >> x^{\frac{k}{2}+\frac{3}{2}} \quad .$$

According to (6),(7),(10) and (12) we can write ($F(x) >> x$ by (13))

$$(14) \qquad M_2(x) << F(x) \quad .$$

The function $M(x)$ being non-negative and non-decreasing, we have

$$M(x) \leq \frac{1}{x} \int_x^{4x} M(y)\,dy = \frac{1}{x} (M_2(4x) - M_2(x)) \ ,$$

and, according to (14),

$$(15) \qquad M(x) << \frac{1}{x} (F(4x) + F(x)) \quad .$$

Now we have

$$F(4x) << x^{\frac{k}{2}+1} \sum_{k \leq 2\sqrt{x}} \min^{\frac{k}{2}-\frac{1}{2}} \left(\frac{4x}{k^2}, \frac{1}{R_k} \right) \frac{k}{2\sqrt{x}} <<$$

$$<< x^{\frac{k}{2}+1} \sum_{k \leq 2\sqrt{x}} \min^{\frac{k}{2}-\frac{1}{2}} \left(\frac{x}{k^2}, \frac{1}{R_k} \right) \frac{k}{\sqrt{x}} =$$

$$= F(x) + x^{\frac{k}{2}+1} \sum_{\sqrt{x} < k \leq 2\sqrt{x}} \min{}^{\frac{k}{2}-\frac{1}{2}}\left(\frac{x}{k^2}, \frac{1}{R_k}\right) \frac{k}{\sqrt{x}} <<$$

$$<< F(x) + x^{\frac{k}{2}+1} \sum_{\sqrt{x} < k \leq 2\sqrt{x}} \frac{x^{\frac{k}{2}-1}}{k^{k-2}} << F(x) + x^{\frac{k}{2}+\frac{3}{2}}$$

and thus, according to (13),

$$F(4x) << F(x) .$$

Finally, we obtain from (15)

$$(16) \quad M(x) << \frac{1}{x} F(x) = x^{\frac{k}{2}} \sum_{k \leq \sqrt{x}} \min{}^{\frac{k}{2}-\frac{1}{2}}\left(\frac{x}{k^2}, \frac{1}{R_k}\right) \frac{k}{\sqrt{x}} ,$$

this proving the Main Theorem.

§ 4. __Consequences of the Main Theorem.__ First of all, let us present two "exceeding" consequences of the relation (16). It always holds

$$M(x) << x^{\frac{k}{2}} \sum_{k \leq \sqrt{x}} \frac{x^{\frac{k}{2}-1}}{k^{k-2}} << \begin{cases} x^{3/2} & \text{for } k = 2, \\ x^2 \lg x & \text{for } k = 3, \\ x^{k-1} & \text{for } k \geq 4, \end{cases}$$

and thus the relation (16) yields immediately the O-estimates presented in III,§ 1. On the other hand,

$$M(x) << x^{\frac{k}{2}} \sum_{k \leq \sqrt{x}} \frac{1}{R_k^{k/2-1/2}} \frac{k}{\sqrt{x}} << x^{\frac{k}{2}} \sum_{k \leq \sqrt{x}} \frac{1}{R_k^{k/2-1/2}}$$

if at least one of the numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ is irrational (and thus $R_k \neq 0$ for all k's). This relation was the starting point for the O-estimates IV.b),§ 1 in the paper [8].

Let us now rearrange the relation (16) in the following way: because of

$$\min^{\frac{k}{2}-\frac{1}{2}}\left(\frac{x}{k^2}, \frac{1}{R_k}\right)\frac{k}{\sqrt{x}} \leq \min^{\frac{k}{2}-1}\left(\frac{x}{k^2}, \frac{1}{R_k}\right)$$

(if $R_k = 0$ or $R_k \neq 0$ and $x/k^2 \leq \frac{1}{R_k}$, the equality takes place; if $R_k \neq 0$, $x/k^2 > \frac{1}{R_k}$, i.e., for $\frac{k}{\sqrt{x}} < \sqrt{R_k}$, we have $\frac{k}{\sqrt{x}}\frac{1}{R_k^{k/2-1/2}} < \frac{1}{R_k^{k/2-1}}$ on the left hand side, and the inequality takes place), we can write

$$(17) \qquad M(x) \ll x^{\frac{k}{2}} \sum_{k \leq \sqrt{x}} \min^{\frac{k}{2}-1}\left(\frac{x}{k^2}, \frac{1}{R_k}\right) .$$

From the assertion I.c), § 1 there follows, for $k > 4$: If at least one of the numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ is irrational then

$$(18) \qquad M(x) = o(x^{k-1}) .$$

For $k = 4$, we cannot derive this result from the results of the paper [6]. Therefore we shall use the relation (17).

<u>Theorem 1</u>. Let $k \geq 4$ and let at least one of the numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ be irrational. Then (18) holds.

<u>Proof</u> (analogously to [6], Theorem 3). According to the assumptions, there is $R_k \neq 0$ for all k's and $k-2 \geq 2$. If we produce, for every $x > c$, a natural number $\psi(x)$ such that

$$\sum_{k \leq \psi(x)} \frac{1}{R_k^{k/2-1}} \leq \frac{x^{\frac{k}{2}-1}}{lg\, x} < \sum_{k \leq \psi(x)+1} \frac{1}{R_k^{k/2-1}} ,$$

then $\psi(x)$ is non-decreasing function, $\lim_{x \to +\infty} \psi(x) = + \infty$. But according to (17) we have

$$M(x) << x^{k/2} \Big(\sum_{k \leq \psi(x)} \frac{1}{R_k^{k/2-1}} + x^{\frac{k}{2}-1} \sum_{k > \psi(x)} \frac{1}{k^{k-2}} \Big) <<$$

$$<< x^{k-1} \Big(\frac{1}{lg\, x} + \frac{1}{\psi(x)} \Big) = o(x^{k-1}) ,$$

and the Theorem is thereby proved.

The estimate (18) cannot be improved generally. Using the known method of categories [3] an assertion analogous to I.d), § 1 can be stated:

<u>Theorem 2</u>. Let $k \geq 4$ and let $\varphi(x)$ be a non-increasing positive function, $\varphi(x) = o(1)$. Then there exists a system $\alpha_1, \alpha_2, \dots, \alpha_k$ such that (18) takes place and

$$M(x) = \Omega(x^{k-1} \varphi(x)) \tag{19}$$

holds.

<u>Proof</u>. Let $\mathfrak{M}$ be a set of all points $(u_1, u_2, \dots, u_k) \in E_k$ such that $0 \leq u_j \leq \frac{1}{M_j}$ $(j = 1, 2, \dots, k)$, $\mathfrak{N}$ let be a set of all points from $\mathfrak{M}$ having rational

----

3) The first one who used this method for Ω-estimates in the theory of lattice points was Jarník in the paper [4].

coordinates. For a natural n, let $\mathfrak{M}_n$ be a set of all points $(\beta_1, \beta_2, \ldots, \beta_k) \in \mathfrak{M}^\circ$ ($= \operatorname{Int} \mathfrak{M}$) such that, for a suitable $x = x(n, \beta_j) > n$, there is

$$\frac{M(x, \beta_j)}{x^{k-1}\varphi(x)} > n .$$

From the continuity of the function $M(x; \beta_j)$ for a steady x on the set $\mathfrak{M}^\circ$ (let us remark that, for $(\beta_1, \beta_2, \ldots, \beta_k) \in \mathfrak{M}^\circ$, there is $A(x; \beta_j) = P(x; \beta_j)$, and the function $A(x; \beta_j)$ is, for a steady x, continuous in the entire space E_k) there follows that every set $\mathfrak{M}_n$ is open.

Let $\mathfrak{R}$ be the set of all points $(\beta_1, \beta_2, \ldots, \beta_k) \in \mathfrak{M}^\circ$ such that, for the least common denominator H of the numbers $\beta_1 M_1, \beta_2 M_2, \ldots, \beta_k M_k$, we have $(H, 2D\prod_{j=1}^{k} M_j^2) = 1$. According to the Lemma 3 and the assertion IV.a), § 1, we obtain that, for every $(\beta_1, \beta_2, \ldots, \beta_k) \in \mathfrak{R}$, there is

$$M(x; \beta_j) \geq c(\beta_j)\, x^{k-1}$$

for $x > c(\beta_j)$. Thus, choosing a natural n we have, for every $(\beta_1, \beta_2, \ldots, \beta_k) \in \mathfrak{R}$,

$$\frac{M(x; \beta_j)}{x^{k-1}\varphi(x)} \geq \frac{c(\beta_j)}{\varphi(x)} > n$$

for all sufficiently large $x > c(\beta_j)$, and it immediately follows that $\mathfrak{R} \subset \bigcap_{n=1}^{\infty} \mathfrak{M}_n$. Since the set $\mathfrak{R}$ is obviously dense in $\mathfrak{M}$, all the sets $\mathfrak{M}_n$ are dense in $\mathfrak{M}$, too.

Conclusively, the sets $\mathfrak{M} - \mathfrak{M}_n$ are nowhere dense in $\mathfrak{M}$, and thus the set

$$\mathfrak{N} \cup (\mathfrak{M} - \mathfrak{M}^0) \cup \bigcup_{n=1}^{\infty} (\mathfrak{M} - \mathfrak{M}_n)$$

is of the first category in $\mathfrak{M}$, i.e., ($\mathfrak{M}$ is a complete space) there exists a point

$$(\alpha_1, \alpha_2, \dots, \alpha_n) \in (\mathfrak{M}^0 - \mathfrak{N}) \cap \bigcap_{n=1}^{\infty} \mathfrak{M}_n \quad .$$

The relation (18) thus holds. Since $(\alpha_1, \alpha_2, \dots, \alpha_k) \in$ $\in \bigcap_{n=1}^{\infty} \mathfrak{M}_n$, the inequality

$$\frac{M(x)}{x^{k-1}\varphi(x)} > n$$

is satisfied for every n for a suitable $x = x(\alpha_j, n) > n$, and (19) holds, too.

Let further $\alpha_1 = \alpha_2 = \dots = \alpha_k = \alpha$ be an irrational number. From II, § 1, it follows (in the notation introduced there), for $k \geq 6$, the estimate

$$\limsup_{x \to +\infty} \frac{\lg M(x)}{\lg x} \leq 2f + 1 \quad .$$

For $k = 4, 5$, we obtain, using the estimate from the paper [7], weaker results:

$$\limsup_{x \to +\infty} \frac{\lg M(x)}{\lg x} \leq \max\left(\frac{k}{2}, 2f\right) + 1$$

for $k = 5$ and

$$\limsup_{x \to +\infty} \frac{\lg M(x)}{\lg x} \leq 3$$

for $k = 4$. Considering (17) we can prove the following generalization:

Theorem 3. Let $k \geq 4$, $\alpha_1 = \alpha_2 = \ldots = \alpha_k = \alpha$. Let

$$(20) \qquad \langle \alpha h \rangle >> \frac{1}{h^\beta}$$

for all h's. Then

$$M(x) << x^{(\frac{k}{2}-1)\frac{2\beta+1}{\beta+1}+1} g(x),$$

where $g(x) = lg\, x$ for $k = 4$ and $\beta = 1$, $g(x) = 1$ simultaneously in other cases.

Proof. According to (17) and § 2 ($R_h >> P_h^2$) we can write

$$M(x) << x^{\frac{k}{2}} \sum_{h \leq \sqrt{x}} min^{\frac{k}{2}-1}\left(\frac{x}{h^2}, \frac{1}{P_h^2}\right),$$

i.e.,

$$M(x) << x\, S_{2k-2}(x, \alpha, M_1, M_2, \ldots, M_k, M_1, M_1, \ldots, M_1)$$

and the assertion follows from Lemma 4.

Connecting the Ω-estimate III, § 1 and Theorem 3, we obtain the following result:

Theorem 4. Let $k = 4$, $\alpha_1 = \alpha_2 = \ldots = \alpha_k = \alpha$. Let (20) with the value $\beta = 1$ hold for all h's (i.e., if $\{a_0, a_1, a_2, \ldots\}$ is the continued fraction expressing the number α, then $a_n << 1$). Then

$$0 < \lim_{x \to +\infty} inf \frac{M(x)}{x^{5/2}}, \lim_{x \to +\infty} sup \frac{M(x)}{x^{5/2} lg x} < +\infty .$$

Remark. a) If at least one of the numbers α_1,

$\alpha_2, \dots, \alpha_n$ is irrational then it follows from (17) that

$$M(x) << x^{\frac{k}{2}} \sum_{k \leq \sqrt{x}} \frac{1}{R_k^{k/2-1}} \quad .$$

Jsing this estimate in the paper [8] we could slightly improve the O -estimate IV $b)$, § 1.

b) An assertion analogous to Theorem 2 can be stated for $k = 3$: If $\varphi(x)$ is a positive and non-increasing function, $\varphi(x) = o(1)$, there exists a triplet of numbers $\alpha_1, \alpha_2, \alpha_3$ such that at least one of them is irrational and moreover

$$M(x) = \Omega(x^2 \varphi(x) \lg x)$$

(and obviously $M(x) = O(x^2 \lg x)$). The proof is to be carried out analogously, we have only to mention that the constant H_3 in IV.a), § 1 is non-zero if the least common denominator of the numbers $\alpha_1 M_1, \alpha_2 M_2, \alpha_3 M_3$ is relatively prime to $2 D M_1^2 M_2^2 M_3^2$. The validity of this assertion follows from Theorem 1 of the paper [9] and Lemma 2 of the paper [6].

c) If $\alpha_1 = \alpha_2 = \dots = \alpha_n = \alpha$ and if f is defined in the same way as in II, § 1 we obtain from Theorem 3 (and III, § 1 in the case of a rational α) the estimate

$$M(x) = O(x^{2f+1+\varepsilon})$$

(for an arbitrary $\varepsilon > 0$, the constants in the O -estimate are of the type $c(\varepsilon)$).

d) The proof of Theorem 3 could be carried out directly, analogously to the proof of Theorem 1 in the paper [7].

It is anyhow interesting to compare (17) with this result (see [6], Theorem 2): Let $\kappa > 4$, then

$$P(x) = O(x^{\frac{\kappa}{4}-\frac{1}{2}} \sum_{k \le \sqrt{x}} \min^{\frac{\kappa}{4}-\frac{1}{2}}(\frac{x}{k^2}, \frac{1}{R_k}) \lg^2 k .$$

References

[1] V. JARNÍK: Über die Mittelwertsätze der Gitterpunktlehre, Math. Zeitschr. 33 (1931), 62-84.

[2] V. JARNÍK: Über die Mittelwertsätze der Gitterpunktlehre, Zweite Abhandlung, Math. Zeitschr. 33 (1931), 85-97.

[3] V. JARNÍK: Über die Mittelwertsätze der Gitterpunktlehre, 5. Abhandlung, Čas. pěst. mat. 69 (1940), 148-174.

[4] V. JARNÍK: Eine Bemerkung zur Gitterpunktlehre, Čas. pěst. mat. 69 (1940), 57-60.

[5] E. LANDAU: Ausgewählte Abhandlungen zur Gitterpunktlehre, Berlin 1962.

[6] B. NOVÁK: Verallgemeinerung eines Peterssonschen Satzes und Gitterpunkte mit Gewichten, Acta Arithmetica XIII, 4 (to appear).

[7] B. NOVÁK: On lattice points with weight in high-dimensional ellipsoids, Acta Arithmetica (to appear).

[8] B. NOVÁK: Über Gitterpunkte mit Gewichten in mehrdimensionalen Ellipsoiden: Mittelwertsätze, Czech. Math. J. 17, 4 (1967) (to appear).

[9] B. NOVÁK: Mittelwertsätze der Gitterpunktlehre:rationale Gewichte(to appear).

[10] А.З. ВАЛФИШ: Абсциссы сходимости некоторых рядов Дирихле, Труды Тбилисского мат.инст.XXII(1956), 33-75.

[11] A. WALFISZ: Über die Koeffizientensummen einiger Modulformen,Math.Ann.108(1933),75-90.

(Received September 8, 1967)