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Svatopluk Fučík

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SOLVING OF NONLINEAR OPERATORS' EQUATIONS IN BANACH SPACE

Svatopluk FUČÍK, Praha

1. Introduction. This paper deals with the generalization of the following well-known theorem: Let X be a Banach space and $A : X \to X$ be a bounded linear operator with the norm $\|A\| < 1$. Then $(I + A)X = X$. Some surjectivity theorems are obtained for nonlinear mappings under similar conditions as the condition $\|A\| < 1$ in linear case.

Section 2 solves the following question. If an operator H possesses the fixed point property which conditions we have to supply on H or $I + H$ that $(I + H)X = X$.

Section 3 deals with sufficient conditions under which an operator H possesses the fixed point property. The proofs are not given. Some lemmas are true under weaker assumptions (see Remark 3).

Substituing the hypotheses concerning fixed point property by sufficient conditions from Section 3 we obtain several surjectivity theorems containing as particular case some known results.

2. Main theorems. Let X be a real Banach space with the norm $\|\cdot\|$, θ its zero element; X^* denotes the

adjoint (dual) space of all bounded linear functionals on X. The pairing between $x^* \in X^*$ and $x \in X$ is denoted by (x, x^*). Let S_R (or K_R) denote the set of all x such that $\|x\| = R$ (or $\|x\| \leq R$).

Definition 1: ([6],[7],[8],[9])

a) A gauge function is a real valued continuous function μ defined in the interval $\langle 0, \infty)$ such that

$$\mu(0) = 0,$$

$$\lim_{t \to \infty} \mu(t) = \infty,$$

μ is strictly increasing.

b) The duality mapping in X with a gauge function μ is a mapping J from X into the set 2^{X^*} of all subsets of X^* such that

$$Jx = \begin{cases} \{\theta^*\} & , x = \theta \\ \{x^*; x^* \in X^*, (x, x^*) = \|x\| \cdot \|x^*\|, \|x^*\| = \mu(\|x\|)\}, & x \neq 0. \end{cases}$$

Remark 1: ([6],[7],[8],[9])

a) The set Jx is non-empty.

b) Let X be a Banach space with a strictly convex dual space X^*. Let J be the duality mapping in X with a gauge function μ. Then the set Jx consists of precisely one point.

Definition 2: Let h be a mapping with domain X and values in X $(h : X \to X)$.

a) h is said to be surjective if for every $y_0 \in X$ there exists $x_0 \in X$ such that $hx_0 = y_0$ (i.e. $hX = X$).

b) h is said to be coercive if

$$\lim_{\substack{\|x\| \to \infty \\ y^* \in Jx}} \frac{(hx, y^*)}{\|x\|} = +\infty .$$

Theorem 1: Let X be a real Banach space and J be a duality mapping with a gauge function μ. Let $\limsup_{t \to \infty} \frac{\mu(t)}{t} < +\infty$. Suppose that $h: X \to X$ is the mapping such that for every $x \in X$ is

$$hx = x + Hx$$

and that the following two hypotheses are fulfilled:

I. For every $y_0 \in X$ and $R > 0$ so that the inequality $(y_0 - Hx, y^*) \le \mu(\|x\|)\|x\|$ holds for each $x \in S_R$ and $y^* \in Jx$, there exists $x_{y_0,R} \in X$ such that $Hx_{y_0,R} = y_0 - x_{y_0,R}$.

II. h is coercive.

Then h is surjective.

Proof: Let $y_0 \in X$. For $y^* \in Jx$ there is

$$\frac{(y_0 - Hx, y^*)}{\|x\|} = \mu(\|x\|) + \frac{(y_0, y^*)}{\|x\|} - \frac{(hx, y^*)}{\|x\|} \le$$

$$\le \mu(\|x\|) + \|y_0\| \frac{\mu(\|x\|)}{\|x\|} - \frac{(hx, y^*)}{\|x\|} .$$

By assumption $\limsup_{t \to \infty} \frac{\mu(t)}{t} < +\infty$ and the hypothesis II we obtain the existence of $R_0 > 0$ such that for every $\|x\| \ge R_0$ and $y^* \in Jx$ there is $\|y_0\| \frac{\mu(\|x\|)}{\|x\|} - \frac{(hx, y^*)}{\|x\|} \le 0$. Then $\frac{(y_0 - Hx, y^*)}{\|x\|} \le \mu(\|x\|)$ for every $x \in S_{R_0}$ and $y^* \in Jx$, i.e.

$(y_0 - Hx, y^*) \le \mu(\|x\|)\cdot\|x\|$ for every $x \in S_{R_0}$ and $y^* \in Jx$. By the hypothesis I there exists $x_{y_0,R_0} \in X$ such that $Hx_{y_0,R_0} = y_0 - x_{y_0,R_0}$, i.e. $hx_{y_0,R_0} = y_0$.

<u>Definition 3</u>: ([12]) Let X be a real Banach space and $H : X \to X$ be a mapping. H is said to be quasi-bounded if there exist $K \ge 0$, $\rho_0 \ge 0$ such that $\frac{\|Hx\|}{\|x\|} \le K$ for every $x \in X$, $\|x\| \ge \rho_0$. By the quasi-norm for the quasi-bounded operator H we mean $\|H\| = \inf_{\rho_0 \le \rho < \infty} \{ \sup_{\|x\| \ge \rho} \frac{\|Hx\|}{\|x\|} \}$.

<u>Theorem 2</u>: Let X be a real Banach space, $h: X \to X$ be a mapping such that for every $x \in X$ is $hx = x + Hx$ and $0 \le K < 1$. The following hypotheses are fulfilled:

III. For every $y_0 \in X$ and $R > 0$ with the property $(y_0 - H)(S_R) \subset K_R$ there exists $x_{y_0,R} \in X$ such that $Hx_{y_0,R} = y_0 - x_{y_0,R}$.

IV. H is the quasi-bounded operator with the constant K.

Then h is surjective operator.

<u>Proof</u>: Let $y_0 \in X$, $\varepsilon > 0$ be such that $K + \varepsilon < 1$, $\rho_1 = \frac{\|y_0\|}{\varepsilon}$ and $\rho_2 = \rho_0 + \rho_1$ (ρ_0 is from definition 3). For every $x \in X$, $\|x\| \ge \rho_2$ we

have $\frac{\|y_0\|}{\|x\|} \leq \frac{\|y_0\|}{\rho_1} \leq \varepsilon$. For $x \in S_{\rho_2}$ we obtain from the triangle inequality and the hypothesis IV

$$\frac{\|y_0 - Hx\|}{\|x\|} \leq \frac{\|y_0\|}{\|x\|} + \frac{\|Hx\|}{\|x\|} \leq \varepsilon + K < 1 \text{, i.e.}$$

$\|y_0 - Hx\| \leq \|x\|$ and by hypothesis III there exists $x_{y_0,\rho_2} \in X$ such that $Hx_{y_0,\rho_2} = y_0 - x_{y_0,\rho_2}$, i.e. $h\, x_{y_0,\rho_2} = y_0$.

__Example:__ If $H = -I$ (I denotes the identity mapping) we see that if $K = 1$ the theorem 2 is not valid.

__Remark 2__: If J is the duality mapping with gauge function μ and $\limsup_{t \to \infty} \frac{\mu(t)}{t} < +\infty$ then the theorem 2 is a consequence of the theorem 1, for

$$((y_0 - Hx), Jx) \leq \|y_0 - Hx\| \, \|Jx\| \leq \|x\| \mu(\|x\|)$$

and

$$\frac{(hx, Jx)}{\|x\|} \geq \mu(\|x\|) - \frac{\|Hx\| \mu(\|x\|)}{\|x\|} \geq (1-K)\mu(\|x\|).$$

3. __Sufficient conditions for the hypotheses.__

__Definition 4__: Let X be a real Banach space and $F: X \to X$ be a mapping.

a) F is said to be strongly continuous if $x_n \rightharpoonup x_0$ (weak convergence) implies $Fx_n \to Fx_0$ (strong convergence).

b) F is said to be weakly continuous if $x_n \rightharpoonup x_0$ implies $Fx_n \rightharpoonup Fx_0$.

c) F is said to be completely continuous if for each bounded subset $M \subset X$, $F(M)$ is a compact and continuous on X .

d) F is said to be a nonexpansive mapping on X if for every $x, y \in X$ there is $\| Fx - Fy \| \leq \| x - y \|$.

e) F is said to be a contractive mapping on X if there exists q $(0 \leq q < 1)$ such that for every $x, y \in X$ we have $\| Fx - Fy \| \leq q \| x - y \|$.

f) F is said to be a monotone on Hilbert space X if $(Fx - Fy, x - y) \geq 0$ for every $x, y \in X$.

g) F is said to be demicontinuous if $x_n \rightarrow x_0$ implies $Fx_n \rightharpoonup Fx_0$.

Lemma 1: ([2],[10],[17]). Let H be a completely continuous operator on X . Then the hypothesis III is valid.

Lemma 2: ([2],[10]) Let H be a completely continuous operator on a Hilbert space X . Then the hypotheses I and II are fulfilled.

Lemma 3: (Banach contractive mapping principle) Let H be a contractive mapping on X . Then the hypotheses I and III are valid.

Lemma 4: ([2],[7],[8],[9],[18]) Let H be a weakly continuous mapping on a Hilbert space X . Then the hypotheses I and III are valid.

Definition 5: A Banach space X is said to be uniformly convex if given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\| x - y \| \geq \varepsilon$ for $\| x \| \leq 1$ and $\| y \| \leq 1$ implies $\| \frac{x+y}{2} \| \leq 1 - \delta(\varepsilon)$.

Lemma 5: ([5]) Let X be a uniformly convex Banach space and H be a nonexpansive mapping on X . Then the hypothesis III is valid.

Lemma 6: ([9]) Let A be a monotone demicontinuous mapping on a Hilbert space X and $H = I - A$. Then the hypotheses I and III are valid.

Lemma 7: ([11],[13],[15]) Let X be a Hilbert space, $A : X \to X$ be completely continuous and $B : X \to X$ be a contractive mapping. Let $H = A + B$. Then the hypotheses I and III are valid.

Lemma 8: ([11],[13],[15]) Let X be a real Hilbert space, $A : X \to X$ be strongly continuous and $B : X \to X$ be a nonexpansive mapping. Let $H = A + B$. Then the hypotheses I and III are valid.

Remark 3: We can show that G -operators ([6],[7],[8],[9]) and P -compact operators ([19]) have the fixed point property. We write the consequences of this for Hilbert space only. Lemma 5 is true for Banach spaces in which every ball has uniform structure ([5]).

Remark 4: The sufficient conditions for the validity of the hypothesis IV are given in [12],[14],[16],[19].

The mapping $H : X \to X$ is said to be assymptotically differentiable if there exists a bounded linear operator $H': X \to X$ such that $\lim\limits_{\|x\| \to \infty} \frac{\|Hx - H'x\|}{\|x\|} = 0$.

If H is assymptotically differentiable then H is quasi-bounded and $\|H\| \leq \|H'\|$.

4. Consequences of the Theorem 1 and 2 and from Section 3.

Theorem 3: ([12],[16]) Let X be a real Banach space and $h : X \to X$ be a mapping such that for every $x \in X$ is $hx = x + Hx$, where H is completely continuous. Let the hypothesis IV with $0 \leq K < 1$ be fulfilled. Then h is a surjective operator.

Theorem 4: Let X be a real Hilbert space, $h: X \to X$ be a mapping such that for every $x \in X$ is $hx = x + Hx$, where H is completely continuous and h is coercive. Then h is a surjective mapping.

Theorem 5: Let X be a real Hilbert space, $h: X \to X$ be a mapping such that for every $x \in X$ is $hx = x + Hx$, where H is weakly continuous. If h is a coercive mapping then h is a surjective one.

Theorem 6: Let X be a uniformly convex Banach space and H be a nonexpansive mapping on X satisfying the hypothesis IV with $0 \leq K < 1$. Then $hx = x + Hx$ is surjective.

Theorem 7: ([4]) Let X be a real Hilbert space, $h: X \to X$ monotone, demicontinuous and coercive. Then h

is surjective. (This is a consequence of Lemma 6.)

Theorem 8: Let X be a Hilbert space, $A: X \to X$ be completely continuous, $B: X \to X$ be contractive and $hx = x + Ax + Bx$ be coercive. Then h is a surjective mapping.

Remark 5: This theorem is a generalization of Nashed-Wong [16] in Hilbert space case.

Theorem 9: Let X be à Hilbert space, $A: X \to X$ be strongly continuous, $B: X \to X$ be a nonexpansive mapping and $hx = x + Ax + Bx$ coercive. Then h is surjective.

Theorem 10: ([14]) Let X be a Hilbert space and $F: X \to X$ be a mapping such that for every $x \in X$ it has the Gâteaux derivative $F'(x)$. Let $PF'(x)$ be a normal mapping for every $x \in X$ (A is normal if $AA^* = A^*A$, where A^* denotes the mapping adjoint to A) such that $(PF'(x)h, h) \geq 0$ for every $x \in X, h \in X$ where P is a linear mapping of X onto X having an inverse P^{-1}, $\|P\| \leq (\sup_{x \in X} \|F'x\|)^{-1}$. If the quasi-norm $\|I - PF\| < 1$, then the equation $Fx = y^*$ has at least one solution for every $y^* \in X$.

Proof: The equation $Fx = y^*$ is equivalent to $x - Gx = z^*$, where $z^* = Py^*$; $Gx = x - PFx$ is a nonexpansive mapping and by means of Theorem 6 we conclude this proof.

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