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A CONTINUOUS GEOMETRY AS A MATHEMATICAL MODEL FOR QUANTUM MECHANICS

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Introduction. The usual assertions of quantum mechanics are that observables are self-adjoint operators in Hilbert space, states are vectors in this space, and the expectation value of an observable A in the state Ψ is $(A\Psi, \Psi)$. The "pure" states are associated with unit vectors (i.e. "points") in the Hilbert space. G. Mackey [1] has produced a set of axioms for the standard model for quantum mechanics which gives a mathematical theory in accordance with the above assertions. The spectral theory for operators on a Hilbert space plays a vital role in this analysis.

This theory is developed from the basic assumption that underlying every physical system is an orthocomplemented lattice, namely the lattice of experimentally verifiable propositions about the system. This lattice is called the "logic" of the system. In the case of classical mechanics the logic is a Boolean algebra, the distributivity corresponding to the fact that all observables are simultaneously measurable. However, in the case of quantum mechanics the distributivity is lost, since observables exist which are not simultaneously measurable. In the standard model, where we assume the logic to be isomorphic to the lattice of closed subspaces of an infinite-dimensional, separable Hilbert space, the lattice is not even modular. Modularity would imply that the lattice was a continuous geometry [2] and all the desirable properties of Hilbert space, in particular the spectral theory for operators on it, existence of "points", and the well-established theory of Hilbert spaces, are lost. Consequently, a model for quantum mechanics with any physical relevance based on a continuous geometry has never been produced. However, a spectral theory for elements in a continuous ring has recently been developed [3] and we attempt to replace the Hilbert space of the standard model by a particular kind of continuous ring. Continuous rings arise in connection with the continuous geometries of von Neumann (see [4]-[9]), and if R is a continuous ring it is associated with a continuous geometry, namely the lattice of its principal left ideals [9]. If R replaces the Hilbert space in the model, then the logic of the system is the continuous geometry associated with R.

1. Continuous geometries and regular rings.

A continuous geometry is a complete, complemented, modular, irreducible lattice satisfying two continuity conditions and which admits infinite chains.

1.1. Definition. An associative ring R with a unit is regular if $axa = a$ is solvable in R for all $a \in R$.

If L is a continuous geometry with homogeneous basis of order ≥ 4 then there exists a regular ring R, called the co-ordinate ring of L, such that L and the lattice of all principal left ideals in R are isomorphic. It is possible to define a unique dimension function "dim" on the lattice, which in turn makes it possible to define a unique rank function r on the co-ordinate ring, by setting $r(a) = dim((a_e))$ for $a \in$ $\in R$. The range of r is the closed interval $[0, 1]$, and the ring is irreducible and also complete in the rank metric $\delta(a, b) = r(a-b)$. A complete, regular, irreducible rank ring is called a continuous ring and it can be proved that the lattice of principal left ideals of such a ring is a continuous geometry. There is therefore a one to one correspondence between continuous geometries and continuous rings.

We will need the following properties of regular rings (proved in [10]).

1.2. Theorem. For an associative ring R with a unit the following are equivalent.

i) R is regular.

ii) Every principal left ideal is generated by a unique idempotent.

iii) The principal left ideals of R form a complemented sublattice of all the left ideals of R.

iv) Every principal left ideal of R has a complement in the lattice of all left ideals.

1.3. Theorem. i) If R is the union of a finite set of principal left ideals $\{(a_i)_\ell : i = 1, 2, \ldots, n\}$ then there exists a unique collection of idempotents $e_1, e_2, \ldots, e_n$ such that $(a_i)_\ell = (e_i)_\ell$, $e_1 + \ldots\ldots + e_n = 1$ and $e_i e_j = 0$ for $i \neq j$.

ii) $e^2 = e$ implies $(e)_\ell^r = (1-e)_r$ and $(e)_r^\ell = (1-e)_\ell$.

iii) $(a)_\ell^{r\ell} = (a)_\ell$ and $(a)_r^{\ell r} = (a)_r$ for all $a \in R$.

iv) The mapping $(a)_\ell \longrightarrow (a)_\ell^r$ is a lattice anti-isomorphism of the lattice of principal left ideals of R onto the lattice of principal right ideals.

v) If $(e)_\ell$ is a right ideal and $e^2 = e$, then e is central.

vi) The center Z of a regular ring is also regular.

vii) R is irreducible if and only if the center is a field.

2. Eigenvalues in continuous rings.

R will denote a continuous ring, Z its center and r the rank function on R.

The results of this section are proved in [3].

2.1. Definition. Let $a \in R$. Then an element $b \in R$ is said to be an eigenvalue of a if and only if $b \in Z$ and $r(a-b) < 1$.

We can prove

2.2. Lemma. There are only a countable number of irreducible, monic, a-singular (i.e. $\kappa[p(a)] < 1$) polynomials $p(x)$ with coefficients in Z.

Hence we immediately get

2.3. Theorem. Let $a \in R$. Then a has only a countable number of eigenvalues.

2.4. Definition. $x \in R$ is a principal vector of degree k corresponding to the eigenvalue b of a, if $x(a-b)^k = 0$ but $x(a-b)^{k-1} \neq 0$.

Let $M(a)$ denote the space spanned by all the principal vectors associated with eigenvalues of a.

2.5. Definition. If $b \in Z$ is an eigenvalue of $a \in R$ then the multiplicity of b is $dim\,[(a-b)_r^\ell]$.

Using these definitions it is possible to build up a theory of eigenvalues in continuous rings which parallels that for linear transformations on a finite-dimensional vector space.

2.6. Definition. A regular ring is said to be $*$-regular if there exists an involutory anti-automorphism $a \longrightarrow a^*$ of the ring onto itself, such that $aa^* = 0$ if and only if $a = 0$.

If R is $*$-regular an element $a \in R$ for which $a = a^*$ is called self-conjugate. Self-conjugate idempotents are called projections.

The most basic result concerning $*$-regular rings is that every left ideal $(a)_\ell$ is generated by a uniquely defined projection $\pi_\ell a$, called a left projection. The term "lattice of projections" will denote the lattice of principal left ideals of the $*$-regular ring R , where $e \vee f$ is the projection defined by $(e)_\ell \vee (f)_\ell$, $e \wedge f$ the projection defined by $(e)_\ell \wedge (f)_\ell$, and $e \leq f$ means that $(e)_\ell \leq$ $\leq (f)_\ell$.

We say that a $*$-regular ring is <u>complete</u> if the lattice of its projections is complete. A result of Kaplansky [2] gives the following

2.7. <u>Theorem</u>. The projections of a complete irreducible $*$-regular ring form a continuous geometry.

From now on we assume R to be a continuous $*$-regular ring. We define an inner product on R by setting $(u, v) = uv^*$ for all $u, v \in R$. $a \in R$ is <u>hermitian</u> if $a = a^*$, <u>unitary</u> if $(ua, va) = (u, v)$ for all $u, v \in R$, and <u>normal</u> if $aa^* = a^*a$.

Using these definitions we can obtain a theory of normal, hermitian and unitary elements parallelling the classical theory in linear transformations. In particular we get the following results.

2.8. <u>Theorem</u>. Let R be a continuous $*$-regular ring. Then to every normal element $a \in R$ with $M(a) = R$ there correspond elements $b_1, b_2, \ldots\ldots \in Z$ and projections $e_1, e_2, \ldots$ such that i) the b_i are pairwise distinct, ii) the e_i are pairwise orthogonal and non-zero, iii) $\sum_i e_i = 1$, iv) $a = \sum_i b_i e_i$.

The representation $a = \sum_i b_i e_i$ is the spectral form of a , and is unique.

2.9. Theorem. If $\sum_i b_i e_i$ is the spectral form of $a \in R$, then a necessary and sufficient condition that an element $b \in R$ commute with a is that it commute with each e_i .

If a is a normal element with spectral form $\sum_i b_i e_i$ and f is a Z -valued function defined at least at the points $b_i \in Z$, then we define $f(a)$ by $f(a) = \sum_i f(b_i) . e_i$.

2.10. Theorem. Two normal elements a and b of a continuous $*$-regular ring R with $M(a) = M(b) = R$, are commutative if and only if there exist two Z -valued functions f and g defined on Z , the center of R ,and there also exists a normal element $c \in R$ with $M(c) = R$ such that $a = f(c)$ and $b = g(c)$.

2.11. Theorem. If a is normal with $M(a) = R$, and b commutes with a , then b commutes with a^* .

3. A model for quantum mechanics.

Let $\mathcal{L}$ be a partially ordered set with an involutory anti-automorphism, i.e. a mapping $a \to a'$ of $\mathcal{L}$ onto itself such that $a'' = a$ and $a_1 \leq a_2$ if and only if $a_2' \leq a_1'$. We say that a_1 and a_2 are disjoint, and write $a_1 \perp a_2$ if $a_1 \leq a_2'$.

$a \to a'$ is said to be an orthocomplementation if i) there exists a least upper bound for any countable set $\{a_1, a_2, \dots\}$ of elements disjoint in pairs, written $a_1 \vee a_2 \vee \cdots$

ii) $a \vee a' = b \vee b'$ for all $a, b \in \mathcal{L}$ (call this quantity i).

iii) $a \leq b$ implies $b = a \vee (b' \vee a)'$.

3.1. Definition. Say $m : \mathcal{L} \to [0, 1]$ is a probability measure on an orthocomplemented modular lattice $\mathcal{L}$ if $m(1) = 1$, $m(1') = 0$, and if $a_i \perp a_j$ for $i \neq j$ then $m(a_1 \vee a_2 \vee \dots) = \sum_i m(a_i)$.

From now on we suppose $\mathcal{L}$ to be isomorphic to the lattice of projections of a continuous $*$-regular ring R . $\mathcal{L}$ is orthocomplemented ([10], p.124, prop.89).

We have at least one probability measure on $\mathcal{L}$, namely the rank function r on R . If e is a projection and r_e is defined by $r_e(f) = r(e \wedge f)/r(e)$ for all projections f , then r_e is a probability measure on $\mathcal{L}$ ([11], p.195).

3.2. Definition. If for some family $\mathcal{F}$ of probability measures on $\mathcal{L}$ $m(a_1) \leq m(a_2)$ for all $m \in \mathcal{F}$ implies $a_1 \leq a_2$ then $\mathcal{F}$ is said to be a full family.

3.3. Lemma. The set $\mathcal{S} = \{r_e : e$ a projection$\}$ is a full family of probability measures on $\mathcal{L}$.

Proof. Obviously $r_e(f) = 1$ if and only if $e \leq f$. Let $r_e(f_1) \leq r_e(f_2)$ for all projections e . In particular, $r_{f_1}(f_1) \leq r_{f_1}(f_2)$, i.e. $r_{f_1}(f_2) = 1$.

Therefore $f_1 \leq f_2$ and $\mathcal{S}$ is full.

Let $\mathcal{B}$ denote the set of countable subsets of Z, the center of R, together with Z itself.

3.4. <u>Definition</u>. A function $L : E \to L_E$ of $\mathcal{B}$ to $\mathcal{L}$ is an $\mathcal{L}$-<u>valued measure</u> if

i) $E \cap F = \Phi$ implies $L_E \perp L_F$.

ii) $L_{E_1 \vee E_2 \vee \dots} = L_{E_1} \vee L_{E_2} \vee \dots$ whenever $E_i \cap E_j = \Phi$ for $i \neq j$.

iii) $L_\Phi = 1'$, $L_Z = 1$.

Now let Θ denote the set of all $\mathcal{L}$-valued measures on $\mathcal{B}$, and let $m \in \mathcal{S}$. Define $p(L, m, E) = m(L_E)$ for each triple $L \in \Theta$, $m \in \mathcal{S}$, $E \in \mathcal{B}$.

We will now show that Θ, $\mathcal{S}$, p possess certain properties (labelled properties 1-6), these properties being modifications of the axioms for the standard quantum mechanical model, as proposed by Mackey. We also show that we can obtain a comparable theory of quantum statics to the standard theory. Physically, Θ will be the "observables" and $\mathcal{S}$ the "states".

3.5. <u>Property 1</u>. i) $p(L, m, \Phi) = m(L_\Phi) = m(1') = 0$, for all $L \in \Theta$ and $m \in \mathcal{S}$, ii) $p(L, m, Z) = m(L_Z) = m(1) = 1$, for all $L \in \Theta$ and $m \in \mathcal{S}$.

iii) $p(L, m, \bigcup_i E) = m(L_{\bigcup_i E_i}) = m(\bigcup_i L_{E_i}) = \sum_i m(L_{E_i}) = \sum_i p(L, m, E_i)$, for all $L \in \Theta$ and $m \in \mathcal{S}$, if $E_i \cap E_j = \Phi$ for $i \neq j$.

This says that $p(L, m,)$ is a probability measure, and physically we understand this to mean that if the system is in state m , the probability that the observable L have a value in E is $p(L,m,E)$.

3.6. Property 2. i) Let $p(L_1,m,E) = p(L_2,m,E)$ for all $m \in \mathcal{S}$, $E \in \mathcal{B}$. Then $m(L_{1E}) = m(L_{2E})$ for all $m \in \mathcal{S}$ and $E \in \mathcal{B}$. But $\mathcal{S}$ is a full family and therefore $L_{1E} = L_{2E}$ for all $E \in \mathcal{B}$. Therefore $L_1 = L_2$.

ii) Let $p(L,m_1,E) = p(L,m_2,E)$ for all $L \in \Theta$, $E \in \mathcal{B}$, i.e. $m_1(L_E) = m_2(L_E)$ for all $L \in \Theta$.

3.7. Lemma. Every element of $\mathcal{L}$ is of the form L_E.

Proof. If $a \in \mathcal{L}$, define $L : \mathcal{B} \to \mathcal{L}$ by

$L : E \to a$ if and only if $1 \in E$, $0 \notin E$.

$L : E \to a'$ if and only if $1 \notin E$, $0 \in E$.

$L : E \to 1$ if and only if $1 \in E$, $0 \in E$.

$L : E \to 1'$ if and only if $1 \notin E$, $0 \notin E$.

Then L is an $\mathcal{L}$ -valued measure and $L_{\{1\}} = a$.

Hence, using the lemma in ii) above, we get $m_1(a) = m_2(a)$ for all $a \in \mathcal{L}$ and therefore $m_1 = m_2$.

This is interpreted physically by saying that two different observables have a different probability distribution in some state, and two different states yield a different probability distribution for some observable.

3.8. Property 3. Let f be any function from Z to Z such that $f^{-1}(E) \in \mathcal{B}$ for any $E \in \mathcal{B}$, and suppose L is any $\mathcal{L}$ -valued measure on $\mathcal{B}$ and

$m \in \mathcal{S}$. We wish to show that there exists an $\mathcal{L}$-valued measure L^* on $\mathcal{B}$ such that, for any $m \in \mathcal{S}$, $E \in \mathcal{B}$, $p(L, m, f^{-1}(E)) = p(L^*, m, E)$.

For $E \in \mathcal{B}$ write $L_E^{f^{-1}} = L_{f^{-1}(E)}$. Then $L^{f^{-1}}$ defined by $L^{f^{-1}}: E \longrightarrow L_E^{f^{-1}}$ is an $\mathcal{L}$-valued measure on $\mathcal{B}$ as is easily proved. Putting $L^* = L^{f^{-1}}$ we get $p(L^{f^{-1}}, m, E) = m(L_E^{f^{-1}}) = m(L_{f^{-1}(E)}) = p(L, m, f^{-1}(E))$ for each $m \in \mathcal{S}$, as required.

This property enables us to consider L^2, L^3, $L + L^2$, $1 - L$ etc. as observables when L is an observable.

We no longer express states as convex linear combinations of other states, and we do not single out a special class of states as "pure" (which is what happens in the standard model). Our physical interpretation of states is that each state is associated with a certain class of configurations of the system in that particular state. If the system is in the state defined by the projection e, then it is in any other state, defined by the projection f say, with a certain probability, namely $r_e(f)$.

3.9. <u>Definition.</u> $L \in \Theta$ is a <u>question</u> if $p(L, m, \{0,1\}) = 1$ for all $m \in \mathcal{S}$, i.e. if $m(L_{\{0,1\}}) = 1$ for all $m \in \mathcal{S}$.

Suppose L is a question. Then $m(L_{\{0,1\}}) = 1 = m(1)$ for all $m \in \mathcal{S}$. Since $\mathcal{S}$ is full this implies that $L_{\{0,1\}} = 1$. Similarly it can be shown that $L_E = 1$ if $\{0,1\} \subseteq E$, and $L_E = 1'$ if $E \cap \{0,1\} = \phi$.

$L_{\{1\}} \vee L_{\{0\}} = L_{\{0,1\}} = 1$, and as L is an $\mathcal{L}$-valued measure we get $L_{\{1\}} \perp L_{\{0\}}$ i.e. if $L_{\{1\}} = a$, then $L_{\{0\}} \leq a'$. From the definition of orthocomplementation this means $a' = L_{\{0\}} \vee (a \vee (L_{\{0\}}))' = L_{\{0\}} \vee 1' = L_{\{0\}}$ (since $L'_{\{0\}} \leq 1$ and so $L_{\{0\}} \geq 1'$).

Define the function $1 - L : \mathcal{B} \to \mathcal{L}$ for $L \in \Theta$, sending $E \in \mathcal{B}$ to $(1-L)_E \in \mathcal{L}$, by $(1-L)_E = L_{f^{-1}(E)}$, where $f(x) = 1 - x$.
From the discussion on Property 3 we deduce that this is an $\mathcal{L}$-valued measure on $\mathcal{B}$. Then $p(1-L, m, E) = m((1-L)_E) = m(L_{f^{-1}(E)}) = p(L, m, f^{-1}(E))$. Therefore $p(1-L, m, \{0,1\}) = p(L, m, \{0,1\})$, and we deduce that $1 - L$ is a question if and only if L is.

If L_1, L_2 are questions we write $L_1 \leq L_2$ if and only if $m(L_{1\{1\}}) \leq m(L_{2\{1\}})$ for all $m \in \mathcal{S}$. It is easily proved that this defines a partial ordering on the questions.

We say that questions L_1, L_2 are <u>disjoint</u> and write $L_1 \perp L_2$ if and only if $L_1 \leq 1 - L_2$ (or equivalently, if and only if $m(L_{1\{1\}}) + m(L_{2\{1\}}) \leq 1$, for all $m \in \mathcal{S}$).

We say that a question L is the <u>sum</u>, written $L = L_1 + L_2 + \ldots$, of disjoint questions $L_1, L_2, \ldots$, if $m(L_{\{1\}}) = m(L_{1\{1\}}) + m(L_{2\{1\}}) + \ldots$ for all $m \in \mathcal{S}$.

Such a sum, if it exists, is unique.

3.10. Property 4. If $L_1, L_2, \dots$ are questions, and $L_i \perp L_j$; for $i \neq j$, then there exists a question L such that $L = L_1 + L_2 + \dots$.

Proof. $L_i \perp L_j$; for $i \neq j$ if and only if $L_i \leq 1 - L_j$; for $i \neq j$, and by definition of $\leq$ this means $m(L_{i\{1\}}) \leq 1 - m(L_{j\{1\}})$ for all $m \in \mathcal{S}$ and for $i \neq j$. Now $L'_{j\{1\}} \vee L_{j\{1\}} = 1$, $j = 1, 2, \dots$ and so, by the definition of probability measure on $\mathcal{L}$ we get $m(L'_{j\{1\}}) = 1 - m(L_{j\{1\}})$, $j = 1, 2, \dots$ and therefore $m(L_{i\{1\}}) \leq m(L'_{j\{1\}})$ for all $m \in \mathcal{S}$, $i, j = 1, 2, \dots, i \neq j$. Since $\mathcal{S}$ is full we deduce that $L_{i\{1\}} \leq L'_{j\{1\}}$ for $i \neq j$, i.e. $L_{i\{1\}} \perp L_{j\{1\}}$ for $i \neq j$.

Since $\mathcal{L}$ is complete, $b = \bigvee_i L_{i\{1\}}$ exists. Let us define on the sets $E \in \mathcal{B}$ a function L with range in $\mathcal{L}$ by

$L: E \to b$ if and only if $1 \in E$, $0 \notin E$.

$L: E \to b'$ if and only if $1 \notin E$, $0 \in E$.

$L: E \to 1$ if and only if $1 \in E$, $0 \in E$.

$L: E \to 1'$ if and only if $1 \notin E$, $0 \notin E$.

It is easily seen that L defined above is a question. Using this L, and the definition of probability measure in $\mathcal{L}$, we get $m(L_{\{1\}}) = m(\bigvee_i L_{i\{1\}}) = \sum_i m(L_{i\{1\}})$, i.e. L is the sum of the disjoint questions $L_1, L_2, \dots$.

Example. If $E \in \mathcal{B}$ and Q_E is the characteristic function of E, then the observable $Q_E(L) = Q_E^L$,

for any given $L \in \Theta$, is a question, since, by definition, $p(Q_E^L, m, F) = p(L,m,Q_E^{-1}(F))$ for all $m \in \mathcal{S}$ and $F \in \mathcal{B}$; and therefore $p(Q_E^L, m, \{0,1\}) =$
$= p(L,m,Q_E^{-1}(\{0,1\})) = p(L,m,Z) = 1$.

3.11. Lemma. If $L \in \Theta$ is fixed, then the set $\{Q_E^L ; E \in \mathcal{B}\}$ is a family of questions determining L uniquely.

Proof. Suppose that $Q_E^L = Q_E^{L'}$ for all $E \in \mathcal{B}$. Then for all $m \in \mathcal{S}$ and $F \in \mathcal{B}$ we have $p(Q_E^L,m,F) = p(Q_E^{L'},m,F)$. Therefore $p(L,m,Q_E^{-1}(F)) =$
$= p(L',m,Q_E^{-1}(F))$ for all $m \in \mathcal{S}$, E, $F \in \mathcal{B}$. In particular, $p(L,m,Q_E^{-1}(\{1\})) = p(L',m,Q_E^{-1}(\{1\}))$ for all $m \in \mathcal{S}$ and $E \in \mathcal{B}$. This implies that $p(L,m,E) = p(L',m,E)$ for all m and E , and hence that $L = L'$.

3.12. Definition. Let $\mathcal{Q}$ denote the set of all questions. Then a function $q : E \rightarrow q_E$ from $\mathcal{B}$ to $\mathcal{Q}$ which satisfies i) $E \cap F = \phi$ implies $q_E \perp q_F$, ii) $E_i \cap E_j = \phi$ for $i \neq j$ implies $q_{\cup E_i} = \sum_i q_{E_i}$, iii) $q_\phi = 0$, $q_Z = 1$; is called a question-valued measure.

The function $Q_{()}^L : E \rightarrow Q_E^L$ is obviously a question-valued measure, and in lemma 3.11 we showed that each observable is uniquely determined by $Q_{()}^L$, i.e. there exists a one to one correspondence between the observables and certain question-valued measures.

3.13. Property 5. Property 5 says that we can eliminate the word "certain" above. That is, if we are given a function $q : E \to q_E$ with q a question-valued measure on $\mathcal{B}$ to $\mathcal{Q}$, then there exists $L \in \Theta$ such that $q_E = Q_E^L$ for all $E \in \mathcal{B}$.

From Property 2 i) we need only show that there exists an $L \in \Theta$ such that for all $F \in \mathcal{B}$, $m \in \mathcal{S}$ and for the given q, $p(q_E, m, F) = p(Q_E^L, m, F)$. As q_E, Q_E^L are questions it is sufficient to show equality for $F = \{1\}$, i.e. we need only show $p(q_E, m, \{1\}) = p(Q_E^L, m, \{1\}) = p(L, m, Q_E^{-1}\{1\}) = p(L, m, E)$ for some $L \in \Theta$. We take $L = q_{\{1\}}$ and show that this is the required L.

If $1 \in E$ our requirement can be written as $p(q_{\{1\}}, m, \{1\}) + p(q_{E-\{1\}}, m, \{1\}) = p(q_{\{1\}}, m, \{1\}) + p(q_{\{1\}}, m, E-\{1\})$ or as $p(q_{E-\{1\}}, m, \{1\}) = p(q_{\{1\}}, m, E-\{1\})$. If $1 \notin E$ we have to prove $p(q_E, m, \{1\}) = p(q_{\{1\}}, m, E)$. Hence it is sufficient to prove the following

3.14. <u>Lemma</u>. If q is a question-valued measure then $p(q_E, m, F) = p(q_F, m, E)$ for all $m \in \mathcal{S}$, and for all disjoint $E, F \in \mathcal{B}$.

<u>Proof</u>. $p(q_E, m, E \vee F) + p(q_{E \vee F}, m, F) = p(q_E, m, E) +$

$+ p(q_E, m, F) + p(q_E, m, F) + p(q_F, m, F) = p(q_{E \vee F}, m, E \vee F)$. Subtracting $p(q_{E \vee F}, m, F)$ from the first and last terms above gives $p(q_E, m, E \vee F) = p(q_{E \vee F}, m, E)$, and subtracting $p(q_E, m, E)$ from both sides of this gives $p(q_E, m, F) =$ $= p(q_F, m, E)$.

3.15. <u>Theorem</u>. The mapping $L \rightarrow L_{\{1\}}$ is a one to one, order-preserving mapping of the questions of the system onto such that if $a \rightarrow L$, then $a' \rightarrow 1 - L$.

<u>Proof</u>. We have seen in our discussion of questions that if L is a question and $L_{\{1\}} = a$, then $(1 - L)_{\{1\}} = a'$, and so the mapping $L \rightarrow L_{\{1\}}$ satisfies the last requirement of the theorem.

Let L_1, L_2 be questions with $L_1 \leq L_2$, i.e. $m(L_{1\{1\}}) \leq m(L_{2\{1\}})$ for all $m \in \mathcal{S}$. Since $\mathcal{S}$ is full this means that $L_{1\{1\}} \leq L_{2\{1\}}$ in $\mathcal{L}$ so we have order-preservation.

Let $a \in \mathcal{L}$, and define the function $L : \mathcal{B} \rightarrow \rightarrow \mathcal{L}$ by

$$L : E \rightarrow a \text{ if and only if } 1 \in E,\ 0 \notin E.$$
$$L : E \rightarrow a' \text{ if and only if } 1 \notin E,\ 0 \in E.$$
$$L : E \rightarrow 1 \text{ if and only if } 1 \in E,\ 0 \in E.$$
$$L : E \rightarrow 1' \text{ if and only if } 1 \notin E,\ 0 \notin E.$$

We have already seen that such a function is a question, and $L_{\{1\}} = a$. Therefore the mapping $L \rightarrow L_{\{1\}}$ is onto.

The mapping is also one to one, for if $L_{1\{1\}} = L_{2\{1\}}$ then $m(L_{1\{1\}}) = m(L_{2\{1\}})$ for all $m \in \mathcal{S}$, or $p(L_1, m, \{1\}) = p(L_2, m, \{1\})$ for all $m \in \mathcal{S}$. Since L_1, L_2 are questions we deduce that $p(L_1, m, E) = p(L_2, m, E)$ for all $m \in \mathcal{S}$ and all $E \in \mathcal{B}$. From Property 2 i) we then get $L_1 = L_2$.

3.16. <u>Definition</u>. Let $L_1, L_2 \in \mathcal{Q}$. We say that L_1 and L_2 are <u>simultaneously answerable</u> if there exists an observable A and $E_1, E_2 \in \mathcal{B}$ such that $L_1 = Q^A_{E_1}$, $L_2 = Q^A_{E_2}$. Observables A and B are <u>simultaneously observable</u> if Q^A_E, Q^B_F are simultaneously answerable for all pairs $E, F \in \mathcal{B}$.

Suppose that $A, B \in \mathcal{O}$ and that there exists a $C \in \mathcal{O}$ and functions f and g from Z to Z, such that $A = f(C)$ and $B = g(C)$. Then it is easy to show that $Q^A_E = Q^{f(C)}_E = Q^C_{f^{-1}(E)}$ and $Q^B_F = Q^{g(C)}_F = Q^C_{g^{-1}(F)}$, and hence that A and B are simultaneously observable.

We know that each observable defines and is defined by a question-valued measure on $\mathcal{B}$ (lemma 3.11). Since the questions are isomorphic to the lattice of projections on the ring, the observables must be isomorphic to the projection-valued measures on $\mathcal{B}$.

3.17. <u>Lemma</u>. The projection-valued measures on $\mathcal{B}$ are in one to one correspondence with those elements of the ring which have a spectral form.

<u>Proof</u>. Let m be a projection-valued measure on $\mathcal{B}$. If λ, μ are arbitrary members of Z then $m(\lambda) = e_\lambda$ and $m(\mu) = e_\mu$, say, where e_λ, e_μ are projections. Since m is a measure we deduce that if $\lambda \neq \mu$ then $e_\lambda \perp e_\mu$. Although Z may be uncountable only a countable number of non-zero idempotents defined in this way can exist ([12],p.119). Let these idempotents be $e_1, e_2, \ldots$ with $e_i = m(\lambda_i)$.

Form $a = \sum_i \lambda_i e_i$.

A second projection-valued measure yield a different projection to that given by the first measure for some $\lambda \in Z$ and hence yields a different spectral form.

Now we see that each spectral form arises from a projection-valued measure in the following manner: given a spectral form $a = \sum_i \lambda_i e_i$ where $\lambda_i \in Z$, $i = 1, 2, \ldots$. define m_a on $\mathcal{B}$ as follows. If $E \in \mathcal{B}$, $m_a(E) = \sum e_{i_j}$, where $\lambda_{i_j} \in E \cap \{\lambda_1, \lambda_2, \ldots\}$. Then m_a is certainly a projection-valued measure on $\mathcal{B}$, $m_a(\lambda_i) = e_i$, $i = 1, 2, \ldots$, $m_a(\tau) = 0$ for $\tau \notin$ $\notin \{\lambda_1, \lambda_2, \ldots\}$, and $m_a \rightarrow \sum_i \lambda_i e_i$ under the mapping mentioned in the first part of the proof.

We therefore have a one to one correspondence between the projection-valued measures and the ring elements with a spectral form. Hence the observables are in one to one correspondence with the ring elemets with a spectral form.

Let $a \in R$ have a spectral form, $\kappa_e \in \mathcal{S}$, and $E \in \mathcal{B}$. To determine the probability that in the state defined by e , a measurement of the observable defined by a will yield a value in E , we first of all take the projection-valued measure p^a associated with by the spectral theorem. p^a_E is then the projection associated with the question "does the value of the observable defined by a lie in E ?" The required probability is $\kappa_e(p_E^a)$.

Let e be a projection and a principal vector of degree 1 of a, corresponding to the eigenvalue λ. Then $e \leq p_{\{\lambda\}}^a$ and so we get $\kappa_e(p_{\{\lambda\}}^a) = 1$.

Conversely, suppose $\kappa_e(p_{\{\lambda\}}^a) = 1$. Then $e \leq p_{\{\lambda\}}^a$ and so $ea = \lambda e$, i.e. e is a principal vector of a of degree 1, corresponding to the eigenvalue λ.

If a and b have spectral forms then a and b commute if and only if $p_E^a \ p_F^b = p_F^b \ p_E^a$ for all E, $F \in \mathcal{B}$ (theorem 2.9). Also, if a and b commute, we know that there exists an element $c \in R$ with spectral form, and functions f and g such that $a = f(c)$ and $b = g(c)$ (theorem 2.10). Thus a and b are simultaneously observable.

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