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SOME PROPERTIES OF SET FUNCTORS

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In the present paper functors from the category S of all sets into itself are studied. Special attention is paid to the preservation of various configurations in S. There are investigated the functors preserving intersections, proimages, difference kernels, products and subdirect products. The paper has six parts. The first one brings the basic conventions, notations, familiar definitions and facts used in the sequel. The second part contains two propositions concerning the preservation of intersection of finite families and proimages and it brings several examples. In the third part there are defined the small functors. These are exactly the functors expressible as direct limits of the small diagrams composed from covariant hom-functors. For the small functors their first and second charachters are defined. The first character of a small functor F specifies, roughly speaking, the number of hom-functors we have to take in order to express F as a factor functor of their disjoint union. The second character determinates the supremum of their dimensions. Further we bring several lemmas, issuing in the theorem 3,1 describing all the functors preserving the difference kernels and intersections. In the fourth and fifth sections there are investigated the functors preserving products or subdirect products, respectively. We bring a number of lemmas on these. The fourth part results in the theorem 4,1 describing all the functors preserving products, the fifth one in the theorem 5,1 describing all the functors preserving subdirect products. The sixth section examines the possibility of the embedding of a given functor into a hom-functor. There are again brought various examples. The results of the present paper can be generalized to functors of a category with suitable properties to the category S .

1.

We recall some definitions and give some conventions.

I. Conventions from the set theory.

Every ordinal number is the set of all smaller ordinal numbers; in particular, $0 = \emptyset$, $1 = \{\emptyset\}$, $2 = \{0,1\}$, cardinal numbers are those ordinal numbers α such that if β is a smaller ordinal number, then there is no one-to-one mapping of β onto α .

The ordered couple of elements x_1, x_2 is denoted by $\langle x_1, x_2 \rangle$. If X is a set, then by e_X the identical mapping of X onto itself is denoted, by $\vartheta_X : 0 \to X$ the empty mapping is denoted. A mapping $f: X \to Y$ is called inclusion if $f(x) = x$ for all $x \in X$. As usual, every mapping onto a set is called a surjection, every one-to-one mapping is called an injection.

II. All functors throughout this paper will be covariant functors from the category S of all sets (the empty set including!) and all their mappings into itself. Often we consider functors only up to the natural equivalence $\simeq$.

III. Let F, G be functors;

F is a <u>subfunctor</u> of G if there exists a monotransformation $\mu : F \to G$;

F is a <u>factorfunctor</u> of G if there exists an epitransformation $\nu : G \to F$.

If $\nu : G \to F$ is a natural transformation, then by $\nu(G)$ the subfunctor H of F with $H(X) = \nu_X(G(X))$ for every set X is denoted.

IV. Let us list some of the used functors (the notation from [1] is kept):

I denotes the identical functor;

C_M denotes the constant functor to a set M;

Q_M denotes the covariant hom-functor from a set M, i.e. $Q_M(X) = \mathcal{H}om(M, X)$.

Thus $Q_0 \simeq C_1$. The functor C_0 is called <u>trivial</u>, the others are <u>non-trivial</u>.

V. For every non-trivial functor F denote by F^* its domain-range-restriction to the category of all non-void sets and all their mappings (such a restriction exists since $F \neq C_0$ implies $F(X) \neq \emptyset$ for every non-void set X).

VI. In the sequel we consider the disjoint union V of functors over a set (for the definition see for example also [1]). But we are not quite correct in the computing with them. If $F = \bigvee_{\iota \in J} Q_{M_\iota}$, we often suppose $Q_{M_\iota}(X) \subset F(X)$. This simplifies the denotation.

VII. Recall (cf.[2]) that if $f : X \to Y$ is an injection (or a surjection) and $X \neq \emptyset$, then $F(f)$ is also an injection (or a surjection, respectively). For, choose $\kappa : Y \to X$ with $\kappa \circ f = e_X$ (or $f \circ \kappa = e_Y$); then $F(\kappa) \circ F(f) = e_{F(X)}$ (or $F(f) \circ F(\kappa) = e_{F(Y)}$, respectively).

VIII. For every functor F and every mapping $f : X \to Y$ denote by $F(X)_f$ the set of all $[F(f)](x)$ with $x \in F(X)$. If f is an inclusion, we shall write also $F(X)_Y$ instead of $F(X)_f$.

IX. Recall that a functor F is said to be separating if, whenever $A, B \subset X$, $A \cap B = \emptyset$, then $F(A)_X \cap F(B)_X = \emptyset$.

Every functor F can be expressed as $F \cong F_d \vee F_s$, where F_s is separating and F_d has no non-trivial separating subfunctor (cf. Statement 4,3 from [3]).

2.

We recall that a functor F preserves intersections (or preserves intersections of finite collections or preserves non-void intersections of finite collections) if, whenever X is a set and $\{Y_\alpha ; \alpha \in A\}$ a collection of its subsets and

$Y = \bigcap_{\alpha \in A} Y_\alpha$ (or moreover A is finite or moreover A is finite and $Y \neq \emptyset$, respectively), then $F(Y)_X = \bigcap_{\alpha \in A} F(Y_\alpha)_X$;

preserves proimages (or preserves proimages of non-void sets, respectively), if, whenever $f: X \to Y$ is a surjection, $A \subset Y$ (or moreover $A \neq \emptyset$, respectively), $B = f^{-1}(A)$, then $F(B)_X = [F(f)]^{-1}(F(A)_Y)$;

preserves difference kernels (or preserves non-void difference kernels) if, whenever $f, g: X \to Y$ are mappings, A is their difference kernel, i.e. $A = \{x \in X;\ f(x) = g(x)\}$ (or moreover $A \neq \emptyset$, respectively), then $F(A)_X$ is the difference kernel of $F(f)$ and $F(g)$;

preserves difference kernels of stars if, whenever $\mathcal{S}$ is a star (i.e. $\mathcal{S} = \{\langle f_\iota, g_\iota \rangle;\ \iota \in \mathcal{J}\}$ where $\mathcal{J} \neq \emptyset$, $f_\iota, g_\iota : X \to Y_\iota$ are mappings), A is the difference kernel of $\mathcal{S}$ (i.e. $A = \{x \in X;\ f_\iota(x) = g_\iota(x)$ for all $\iota \in \mathcal{J}\}$), then $F(A)_X$ is the difference kernel of the star $F\mathcal{S}$ (i.e. $F\mathcal{S} = \{\langle F(f_\iota), F(g_\iota)\rangle;\ \iota \in \mathcal{J}\}$).

Note 2.1: It is known and easy to see that F preserves difference kernels of stars if and only if it preserves difference kernels and intersections.

Proposition 2.1: Every functor preserves non-void intersections of finite collections.

Proof: Let X be a set, A, B its subsets, $A \cap B \neq \emptyset$. Then evidently $F(A \cap B)_A \subset F(A)$,

$F(A \cap B)_B \subset F(B)$, consequently $F(A \cap B)_X \subset \subset F(A)_X \cap F(B)_X$. Let $x \in F(A)_X \cap F(B)_X$; we have to prove $x \in F(A \cap B)_X$. Denote by $i_A : A \to X$, $i_B : B \to X$, $p : A \cap B \to A$ the inclusions. Denote by a or b the elements of $F(A)$ or $F(B)$ with $[F(i_A)](a) = x$, $[F(i_B)](b) = x$. Choose $c \in A \cap B$ and define the mapping $k : X \to A$ such that $k(x) = x$ for $x \in A$, $k(x) = c$ for $x \in \in X - A$. Then $k \circ i_A = e_A$ and there exist some $t : B \to A \cap B$ such that $k \circ i_B = p \circ t$. Put $d = [F(t)](b) \in F(A \cap B)$. Then $a = [F(k \circ i_A)](a) = = [F(k)](x) = [F(k \circ i_B)](b) = [F(p \circ t)](b) = [F(p)](d)$, consequently $a \in F(A \cap B)_p$, which implies $x \in \in F(A \cap B)_X$.

Corollary 2.1: Every separating functor preserves intersections of finite collections.

Convention 2.1: Let P, M be sets, $p : P \to M$ a mapping. Denote by $C_{P,p,M}$ the functor F defined as follows:

if $P = \emptyset$, $M = \emptyset$, then $F = C_0$;

if $M \neq \emptyset$, then $F^* = C^*_M$, $F(\emptyset) = P$ and if $X \neq \emptyset$, then $F(v_X) = p$.

If $P = \emptyset$, we shall write $C_{0,M}$ instead of $C_{P,p,M}$.

Examples 2.1:

a) If $M \neq \emptyset$, then the functor $C_{0,M}$ does not preserve intersections of finite collections.

b) Now we describe the functor F with the following properties: F is separating, preserves intersections of finite collections but it does not preserve intersec-sections. Let N be the set of all natural numbers. Let $\nu : Q_N \longrightarrow F$ be the epitransformation such that the equality $\nu_X(x) = \nu_X(y)$ with $x = \{x_n\}_{n=1}^{\infty} \in Q_N(X)$, $y = \{y_n\}_{n=1}^{\infty} \in Q_N(X)$ holds if and only if there exists $m \in N$ such that $x_n = y_n$ for all $n \geq m$. It is easy to see that F has the required properties.

Proposition 2.2: If a functor F preserves non-void difference kernels, then it preserves proimages of non-void sets.

Proof: Let $f : X \rightarrow Y$ be a surjection, $A \subset Y$, $A \neq \emptyset$, $B = f^{-1}(A)$. Denote by $i_B : B \rightarrow X$, $i_A : A \rightarrow Y$ the inclusions. Then there exists a mapping $g : B \rightarrow A$ with $i_A \circ g = f \circ i_B$, which implies $F(B)_X \subset [F(f)]^{-1}(F(A)_Y)$. Conversely, let $z \in [F(f)]^{-1}(F(A)_Y)$. We have to prove $z \in F(B)_X$. Let Z be the set received from $Y^1 \vee Y^2$ (where Y^1 and Y^2 are copies of the set Y) by identification of every point of A^1 with the corresponding point of A^2; let $i_1 : Y \rightarrow Z$ or $i_2 : Y \rightarrow Z$ be the embeddings of Y onto Y^1 or Y^2, respectively. Then A is the difference kernel of i_1 and i_2 and B is the difference kernel of $i_1 \circ f$ and $i_2 \circ f$. We have $[F(f)(z) \in F(A)_Y$, which implies $[F(i_1 \circ f)](z) = [F(i_2 \circ f)](z)$. Consequently $z \in F(B)_X$.

Examples 2.2:

a) The question, whether a preservation of all difference kernels implies a preservation of proimages of all sets, remains open. Nevertheless, it is evidently true for all functors F with $F(\emptyset) = \emptyset$.

b) Example of a separating functor not preserving proimages: Let $\nu : Q_3 \longrightarrow F$ be the epitransformation such that

$$\nu_X(\langle x_1, x_2, x_3\rangle) = \nu_X(\langle y_1, y_2, y_3\rangle)$$

if and only if $x_1 = x_2 = y_1 = y_2$.
Then F has the required properties.

c) The converse of Proposition 2,2 does not hold. The functor which we receive from Q_2 by identification of every couple (x, y) with the couple (y, x) preserves proimages. It does not preserve difference kernels.

3.

Definition 3.1: Let F be a functor. If A, X are sets, $A \subset F(X)$, denote by $F_{\langle A, X\rangle}$ the following subfunctor G of F: for every set Y $G(Y)$ is the set of all $y \in F(Y)$ such that $y = [F(f)](a)$ for some $a \in A$, $f : X \longrightarrow Y$; if $g : M \longrightarrow P$ is a mapping, then $G(g) : G(M) \longrightarrow G(P)$ is the domain-range-restriction of $F(g)$.

<u>Definition 3.2</u>: Let F be a functor. Every couple $\langle A, X \rangle$ with $A \subset F(X)$ is said to be a <u>reaching couple</u> of F if either $F = C_0$ or $A \neq \emptyset$ and $F^* = (F_{\langle A,X \rangle})^*$.

If there exists a reaching couple of a functor F, then F is said to be <u>small</u>.

<u>Definition 3.3</u>: Let F be a small functor. The smallest couple $(\mathfrak{m}, \mathfrak{n})$ of cardinal numbers (in the lexico-graphic well order of the class of all couples of cardinal numbers) such that there exists a reaching couple $\langle A, X \rangle$ of F with $\operatorname{card} A = \mathfrak{m}, \operatorname{card} X = \mathfrak{n}$, will be called the <u>character</u> of F. Then $\mathfrak{m}$ will be called the <u>first character</u> of F and denoted by ${}^1\chi_F$; $\mathfrak{n}$ will be called the <u>second character</u> of F and denoted by ${}^2\chi_F$.

<u>Proposition 3.1:</u> Let $\langle A, X \rangle$ be a reaching couple of a functor F. Then ${}^1\chi_F \leq \operatorname{card} A$, ${}^2\chi_F \leq \operatorname{card} X$.

<u>Proof</u>: The first inequality is the immediate consequence of the definition, the proof of the second one is easy.

<u>Proposition 3.2:</u>

a) A functor F is small if and only if either $F = C_0$ or F^* is a factorfunctor of a $(\bigvee_{\iota \in J} Q_{M_\iota})^*$.

b) ${}^1\chi_{C_0} = 0 = {}^2\chi_{C_0}$; if F^* is a factorfunctor of $(\bigvee_{\iota \in J} Q_{M_\iota})^*$, then ${}^1\chi_F \leq \operatorname{card} J$, ${}^2\chi_F \leq \sup_{\iota \in J} \operatorname{card} M_\iota$.

c) Let F be small; if ${}^{1}\chi_F = 0$ then $F = C_0$; if $\mathcal{J}, M_\iota$ are sets, $\mathrm{card}\ \mathcal{J} \geqq {}^{1}\chi_F > 0$ and $\mathrm{card}\ M_\iota \geqq$ $\geqq {}^{2}\chi_F$ then F^* is a factorfunctor of $(\bigvee_{\iota \in \mathcal{J}} Q_{M_\iota})^*$.

Proof is easy.

Note 3.1: A functor F is small if and only if the image of the category $\mathcal{S}$ has a generator, or, if and only if either $F = C_0$ or F^* is a direct limit of a diagram (over a small category) of functors Q^*.

Convention 3.1: If F is a functor, X is a set, $Y \subset X$, we put $\overline{Y}^X = F(Y)_X - \bigcup_{Z \subsetneqq Y} F(Z)_X$.

Proposition 3.3: Let F be a functor preserving difference kernels. Let $\mu : Q_M \longrightarrow F$ be a natural transformation with $\mu_M(e_M) \in \overline{M}^M$. Then μ is a monotransformation.

Proof: Let $\mu_Y(\rho) = \mu_Y(\sigma)$ for some $\rho, \sigma \in$ $\in Q_M(Y)$. Then $\rho = [Q_M(\rho)](e_M)$, $\sigma = [Q_M(\sigma)](e_M)$. Consequently, if we put $x = \mu_M(e_M)$, then $[F(\rho)](x) =$ $= [F(\sigma)](x)$; thus x is an element of the difference kernel of $F(\rho)$ and $F(\sigma)$. Denote by D the difference kernel of ρ and σ. Then $x \in F(D)_M$ which implies, together with $x \in \overline{M}^M$, $M = D$.

Definition 3.4: A functor F is said to be regular if every monotransformation $\mu^* : C_1^* \longrightarrow F^*$ can be extended on a monotransformation $\mu : C_1 \longrightarrow F$.

Note 3.2: A functor $C_{P, \pi, M}$ is regular if and only if π is a surjection.

Lemma 3.1: Let $\mu : G \rightarrow F$ be a monotransformation, $f : X \rightarrow Y$ be a mapping. If either $X \neq \emptyset$ or G is regular, then there is no $x \in F(X)$ with

$$(*) \quad [F(f)](x) \in \mu_Y(G(Y)) - [F(f)](\mu_X(G(X))) .$$

Proof: I. First suppose $X = \emptyset$ and G regular. If for some $x \in F(X)$ the assertion $(*)$ holds then necessarily $Y \neq \emptyset$. Put $y = [F(f)](x)$. There exists a monotransformation $\nu : C_1^* \rightarrow G^*$ such that $\mu_Y \circ \nu_Y(1) = y$. If $\tilde{\nu} : C_1 \rightarrow G$ is an extension of ν, $z = \mu_\emptyset \circ \tilde{\nu}_\emptyset(1)$, then $y = [F(f)](z) \in [F(f)](\mu_X(G(X)))$ which is a contradiction.

II. Now let $X \neq \emptyset$. Let $f = g \circ h$, where $h : X \rightarrow Z$ is a surjection, $g : Z \rightarrow Y$ is an injection. If for some $x \in F(X)$ the assertion $(*)$ holds, then $z = [F(h)](x) \in F(Z) - \mu_Z(G(Z))$. Then necessarily $[F(g)](z) \in \mu_Y(G(Y)) - [F(g)](\mu_Z(G(Z)))$. Choose some $\kappa : Y \rightarrow Z$ with $\kappa \circ g = e_Z$. Then $z =$

$$= [F(\kappa \circ g)](z) \in [F(\kappa)](\mu_Y(G(Y))) =$$

$$= \mu_Z([G(\kappa)](G(Y))) \subset \mu_Z(G(Z))$$

which is a contradiction.

Lemma 3.2: Let F preserve intersections. Then F is regular.

Proof is easy.

Lemma 3.3: A regular subfunctor of a functor preserving difference kernels and intersections preserves difference kernels and intersections.

Proof is easy. Use Lemma 3,1.

Convention 3,2: Let F be a functor preserving intersections. For every set X and every $x \in F(X)$ put

$$^{x}X = \bigcap_{\substack{Y \subset X \\ x \in F(Y)_X}} Y \quad .$$

Then evidently $x \in \overline{^{x}X}^{X}$, $F(X) = \bigcup_{Y \subset X} \overline{Y}^{X}$.

Lemma 3,4: A functor preserving difference kernels and intersections is small.

Proof: Let F preserve difference kernels and intersections. Suppose that F is not small. Then $F \neq C_0$. Choose a set X such that $card\ X \geq card\ F(2)$. Choose a set $Y \neq \emptyset$ and $y \in F(Y)$ such that there are no $f: X \to Y$ and $x \in F(X)$ with $[F(f)](x) = y$. Put $M = {}^{y}Y$. Then $card\ M > card\ X$. Denote by $i_M: M \to Y$ the inclusion and by m the element of $F(M)$ with $[F(i_M)](m) = y$. Then $m \in \overline{M}^{M}$; consequently the natural transformation $\mu: Q_M \to F$ with $\mu_M(e_M) = m$ is a monotransformation. Thus $card\ Q_M(2) \leqq card\ F(2)$ which is a contradiction because $card\ F(2) < card\ 2^X \leqq card\ Q_M(2)$.

Lemma 3,5: Let $\nu: \bigvee_{\iota \in J} Q_{M_\iota} \longrightarrow F$ be an epitransformation such that all domain-restrictions $\nu_\iota: Q_{M_\iota} \to F$ of ν are monotransformations. If $\varphi \in Q_{M_\iota}(X)$, $\varphi' \in Q_{M_{\iota'}}(X)$, $\nu_X(\varphi) = \nu_X(\varphi')$, then $\varphi(M_\iota) = \varphi'(M_{\iota'})$.

Proof: If there exists some $y \in \varphi'(M_{\iota'}) - \varphi(M_\iota)$, choose a mapping $\sigma : X \to X$ such that $\sigma(x) = x$ whenever $x \in \varphi(M_\iota)$, $\sigma(y) \neq y$. Then $\nu_X(\varphi') =$ $= \nu_X(\varphi) = \nu_X(\sigma \cdot \varphi) = \nu_X(\sigma \cdot \varphi')$; this contradicts the fact that $\nu_{\iota'}$ is a monotransformation.

Theorem 3.1: Let F be a functor. The following statements are equivalent:

(i) F preserves difference kernels and intersections;

(ii) F is small, regular and either $F = C_0$ or for every X, $x \in F(X)$ the functor $F_{\langle\{x\}, X\rangle}$ is naturally equivalent to $C_{0,1}$ or to some Q_M;

(iii) there exists an epitransformation $\nu : \bigvee_{\iota \in \mathcal{J}} Q_{M_\iota} \to F$ such that all domain-restrictions $\nu_\iota : Q_{M_\iota} \to F$ of ν are monotransformations.

Proof: (i) $\Longrightarrow$ (ii): F is small and regular as it follows from pervious lemmas. Let $F \neq C_0$, X be a set, $x \in F(X)$. If $X = \emptyset$, then $F_{\langle\{x\}, X\rangle} \simeq Q_\emptyset \simeq C_1$. If $X \neq \emptyset$, ${}^xX = \emptyset$, then $F_{\langle\{x\}, X\rangle} \simeq C_{0,1}$. Let $X \neq \emptyset$, $M = {}^xX \neq \emptyset$. Put $G = F_{\langle\{x\}, X\rangle}$. Let $i : M \to X$ be the inclusion. Denote by a the element of $F(M)$ with $[F(i)](a) = x$. Then $a \in \overline{M}^M$, $\langle\{a\}, M\rangle$ is a reaching couple of G. Let H be the functor with $H^* = G^*$, $H(\emptyset)$ is the set of all $z \in F(\emptyset)$ with $[F(\vartheta_1^{\emptyset})](z) \in G(1)$. Since H is a regular subfunctor of F, it preserves difference kernels and intersections. Hence the natural transformation $\nu : Q_M \to$ $\to H$ with $\nu_M(e_M) = a$ is a monotransformation. The restriction $\nu^* : Q_M^* \longrightarrow H^*$ is an epitransformation.

Since $M \neq \emptyset$, there exists no monotransformation $\mu :$ $: C_1^* \longrightarrow H^*$. Consequently, $H(\emptyset) = \emptyset$ which implies $G = H \simeq Q_M$.

(ii) $\Longrightarrow$ (iii): If $F = C_0$, put $\mathcal{J} = \emptyset$. Let $F \neq C_0$. Let $\langle A, X \rangle$ be a reaching couple of F, $A \neq \emptyset$. Denote by B the set of all $a \in A$ such that $F_{\langle \{a\}, X \rangle}$ is naturally equivalent to some Q_{M_a}, $M_a \neq \emptyset$. If $a \in A - B$, then either $F_{\langle \{a\}, X \rangle} \simeq C_1$ or $F_{\langle \{a\}, X \rangle} \simeq C_{0,1}$. Since F is regular, there exists some $a' \in F(\emptyset)$ with $[F(\nu_X^{\emptyset})](a') = a$. For every $x \in F(\emptyset)$ denote by $\mu^x : C_1 \longrightarrow F$ the monotransformation with $\mu^x_{\emptyset}(1) = x$. The definition of the epitransformation

$$\nu : (\bigvee_{a \in B} Q_{M_a}) \vee \bigvee_{x \in F(\emptyset)} (C_1)_x \longrightarrow F$$

with the required properties is evident.

(iii) $\Longrightarrow$ (i): If $\nu : \bigvee_{\iota \in \mathcal{J}} Q_{M_\iota} \longrightarrow F$ is an epitransformation such that all ν_ι are monotransformations, then it is easy to prove that F preserves difference kernels. We prove that F preserves intersections. Let $\{X_\alpha ; \alpha \in A\}$ be a non-void collection of subsets of X, $Y = \bigcap_{\alpha \in A} Y_\alpha$,

$x \in \bigcap_{\alpha \in A} F(Y_\alpha)_X$. Choose $y \in Q_{M_\iota}(X)$ with $\nu_X(y) = x$. For every $\alpha \in A$ there exists $y_\alpha \in Q_{M_{\iota_\alpha}}(Y_\alpha)_X$ such that $\nu_X(y_\alpha) = x$. Lemma 3,5 implies $y(M_\iota) = y_\alpha(M_{\iota_\alpha})$. Since $y_\alpha(M_{\iota_\alpha}) \subset Y_\alpha$ there is $y \in Q_{M_\iota}(Y_\alpha)_X$. Thus $y \in \bigcap_{\alpha \in A} Q_{M_\iota}(Y_\alpha)_X = Q_{M_\iota}(Y)_X$, hence $x \in F(Y)_X$.

Note 3.3: The following proposition needed later can be proved analogously to some parts of the proof of the Theorem:

Let F be a functor which preserves difference kernels and intersections, $F(\emptyset) = \emptyset$; let $\langle \mathfrak{m}, \mathfrak{n} \rangle$ be the character of F . Then there exists an epitransformation $\nu : \bigvee_{\iota \in \mathcal{J}} Q_{M_\iota} \longrightarrow F$ such that

a) $M_\iota \neq \emptyset$ for all $\iota \in \mathcal{J}$;

b) all ν_ι are monotransformations;

c) $\mathrm{card}\, \mathcal{J} = \mathfrak{m}$, $\sup_{\iota \in \mathcal{J}} \mathrm{card}\, M_\iota = \mathfrak{n}$.

4.

Definition 4.1: Let $\mathfrak{X} = \{X_\alpha ; \alpha \in A\}$ be a non-void collection of sets. A couple $\langle X ; \{\xi_\alpha ; \alpha \in A\} \rangle$, where X is a set, $\xi_\alpha : X \longrightarrow X_\alpha$ are mappings, is said to be a subdirect product of $\mathfrak{X}$ if,

a) whenever $x, y \in X$, $x \neq y$, there exists $\alpha \in A$ such that $\xi_\alpha(x) \neq \xi_\alpha(y)$;

b) if all X_α are non-void sets, then all ξ_α are surjections.

If moreover for every $\alpha \in A$ and every $x_\alpha \in X_\alpha, \alpha \in A$ there exists some $x \in X$ with $\xi_\alpha(x) = x_\alpha$ for all $\alpha \in A$ then $\langle X ; \{\xi_\alpha ; \alpha \in A\} \rangle$ is called, as usual, product of $\mathfrak{X}$.

The definition of functors preserving subdirect products or products, respectively, is evident.

Lemma 4.1: Let a functor F preserve subdirect products. Then F is small. If F preserves products then ${}^{1}\chi_F \leqq 1$.

Proof: Let $F \neq C_0$.

a) Let F preserve subdirect products. For every $M \subset F(2)$ choose, whenever it is possible, a set X_M and a point $x_M \in F(X_M)$ such that for every $m \in M$ there exists $f: X_M \longrightarrow 2$ with $[F(f)](x_M) = m$. Put $X = \bigvee_M X_M$; let $i_M: X_M \longrightarrow X$ be the embedding; put $a_M = [F(i_M)](x_M)$. Let A be the set of all a_M. We prove that $\langle A, X \rangle$ is a reaching couple of F. Let $Y \neq \emptyset$, $y \in F(Y)$. Denote by $\mathfrak{G}$ the set of all mappings $g: Y \rightarrow 2$. For every $g \in \mathfrak{G}$ put $2_g = 2$. Let $\langle P; \{\pi_g ; g \in \mathfrak{G}\} \rangle$ be the product of the collection $\{2_g ; g \in \mathfrak{G}\}$; let $\psi: Y \rightarrow P$ be the mapping with $\pi_g \circ \psi = g$. Put $p = [F(\psi)](y)$. Let M_0 be the set of all $[F(\pi_g)](p)$. For every $g \in \mathfrak{G}$ choose some $f_g: X \rightarrow 2$ with $[F(f_g)](a_{M_0}) = [F(\pi_g)](p)$. Let $\varphi: X \rightarrow P$ be the mapping with $\pi_g \circ \varphi = f_g$ for all $g \in \mathfrak{G}$. Since $\langle F(P); \{F(\pi_g); g \in \mathfrak{G}\} \rangle$ is a subdirect product, then necessarily $[F(\varphi)](a_{M_0}) = p$. Now it is sufficient to choose $\tau: P \rightarrow Y$ with $\tau \circ \psi = e_Y$. Then $[F(\tau \circ \varphi)](a_{M_0}) = y$.

b) Let F preserve products. For every $x \in F(2)$ put $2_x = 2$. Let $\langle P; \{\pi_x ; x \in F(2)\} \rangle$ be the product of the collection $\{2_x ; x \in F(2)\}$. Denote by a the point of $F(P)$ such that $[F(\pi_x)](a) = x$ for all $x \in F(2)$. The couple $\langle \{a\}, P \rangle$ is a reaching couple of F, the proof is analogous to a).

Lemma 4.2: Let a functor F preserve subdirect products, $F \neq C_0$ and let F have no non-trivial separating subfunctor. Let $\nu : Q_M^* \longrightarrow F^*$ be an epitransformation. Then $F^* \simeq C_1^*$.

Proof: If $R \neq \emptyset$, $z, z' \in Q_M(R)$, then the fact $\nu_R(z) = \nu_R(z')$ will be written by $z \sim z'$. Let $\lambda_0 : M \longrightarrow 2$ or $\lambda_1 : M \longrightarrow 2$ be the constant mapping on 0 or 1, respectively. Since F has no non-trivial separating subfunctor, $\lambda_0 \sim \lambda_1$. We prove $\lambda \sim \lambda_0$ for an arbitrary $\lambda : M \longrightarrow 2$. Put $M_0 = \lambda^{-1}(0)$, $M_1 = \lambda^{-1}(1)$. Let $X = \{a, b, c, d\}$ be a four-point set, $\varphi, \psi : X \longrightarrow 2$ be the mappings with $\varphi(a) = \varphi(b) = 0$, $\varphi(c) = \varphi(d) = 1$, $\psi(a) = \psi(c) = 0$, $\psi(b) = \psi(d) = 1$. Then $\langle F(X); \{F(\varphi), F(\psi)\} \rangle$ is a subdirect product. Let $\rho, \rho' : M \longrightarrow X$ be the mappings with $\rho(x) = a$, $\rho'(x) = b$ whenever $x \in M_0$, $\rho(x) = c$, $\rho'(x) = d$ whenever $x \in M_1$. Then $\rho, \rho' \in Q_M(X)$ and

$$[Q_M(\varphi)](\rho) = \varphi \circ \rho = \lambda = \varphi \circ \rho' = [Q_M(\varphi)](\rho'),$$

$$[Q_M(\psi)](\rho) = \psi \circ \rho = \lambda_0 \sim \lambda_1 = \psi \circ \rho' = [Q_M(\psi)](\rho'),$$

consequently $\rho \sim \rho'$. Let $\sigma : X \longrightarrow X$ be the mapping with $\sigma(a) = a$, $\sigma(c) = c$, $\sigma(b) = b = \sigma(d)$. Then necessarily $\rho = \sigma \circ \rho \sim \sigma \circ \rho'$. Hence $\lambda = \varphi \circ \rho \sim$

$\sim \varphi \circ \sigma \circ \varphi' = \lambda_0$. Thus $card\, F(2) = 1$. The rest of the proof is evident.

<u>Lemma 4.3</u>: Let F preserve subdirect products and let it have no non-trivial separating subfunctor. Then $F \simeq C_{P,p,M}$ where either $P = \emptyset$ or p is a surjection.

<u>Proof</u>: Let $F \neq C_0$. F is small, consequently there exists an epitransformation $\nu : (\bigvee_{\iota \in J} Q_{M_\iota})^* \rightarrow F^*$. Then $\nu_\iota(Q^*_{M_\iota}) \simeq C^*_1$ for every $\iota \in J$; thus $F^* \simeq$ $\simeq C^*_M$ for some M .

<u>Lemma 4.4</u>:

a) Let F preserve products; then either $F = C_0$ or $F \simeq C_{0,1}$ or $F \simeq C_1$ or F is separating.

b) Let F preserve subdirect products; then either $F \simeq C_{P,p,M}$ where p is a surjection, or $F \simeq C_{0,M} \vee G$ where G is separating and preserves subdirect products.

<u>Proof</u>: Express F as $F = F_d \vee F_s$ where F_s is separating and F_d has no non-trivial separating subfunctor and use the previous Lemmas.

<u>Lemma 4.5</u>: Let F be a separating functor which preserves subdirect products. Then F preserves difference kernels and intersections.

<u>Proof</u>: I. First prove that F preserves difference kernels. Let $\varphi, \sigma : X \rightarrow Y$ be mappings, A be their difference kernel, $i : A \rightarrow X$ the inclusion. Since $\varphi \circ i = \sigma \circ i$ then $F(A)_X \subset \{x \in F(X) ;$ $[F(\varphi)](x) = [F(\sigma)](x)\}$. Let $\langle X \times Y ; \{\pi_X , \pi_Y\}\rangle$

be the product of the collection $\{X, Y\}$. Let $i_\varphi : X \to X \times Y$, $i_\sigma : X \to X \times Y$ be the mappings with $i_\varphi(x) = \langle x, \varphi(x) \rangle$, $i_\sigma(x) = \langle x, \sigma(x) \rangle$ for all $x \in X$.

The following equalities hold evidently:

$$(*) \qquad \begin{array}{ll} \pi_X \circ i_\varphi = e_X & \pi_Y \circ i_\varphi = \varphi \\ \pi_X \circ i_\sigma = e_X & \pi_Y \circ i_\sigma = \sigma \end{array} \Big\}$$

If $[F(\varphi)](x) = [F(\sigma)](x)$ for some $x \in F(X)$, then necessarily $x \in F(A)_X$. For, if we put $x_\varphi = [F(i_\varphi)](x)$, x_σ $[F(i_\sigma)](x)$, the assertion $(*)$ implies $[F(\pi_X)](x_\varphi) = [F(\pi_X)](x_\sigma)$, $[F(\pi_Y)](x_\varphi) = [F(\pi_Y)](x_\sigma)$, consequently $x_\varphi = x_\sigma \in F(X)_{i_\varphi} \cap F(X)_{i_\sigma}$. Now use Corollary 2,1. Thus $x_\varphi \in F(A)_{i_\varphi \circ i}$, consequently $x \in F(A)_i$.

II. Now we prove that F preserves difference kernels of stars. Let $\mathcal{S} = \{\langle \varphi_\iota, \sigma_\iota \rangle ; \iota \in \mathcal{J}\}$ be a star with $\varphi_\iota, \sigma_\iota : X \to Y_\iota$. Denote by $\langle Y; \{\pi_\iota ; \iota \in \mathcal{J}\} \rangle$ the product of the collection $\{Y_\iota ; \iota \in \mathcal{J}\}$. Let $\varphi, \sigma : X \to Y$ be the mappings with $\pi_\iota \circ \varphi = \varphi_\iota$, $\pi_\iota \circ \sigma = \sigma_\iota$. Denote by

A .... the difference kernel of $\mathcal{S}$,

B .... the difference kernel of $F\mathcal{S}$,

C .... the difference kernel of the mappings φ and σ,

D .... the difference kernel of the mappings $F(\varphi)$ and $F(\sigma)$.

Then evidently A = C, I. implies $F(C)_X = D$, $B = D$ is satisfied for subdirect-product-preserving functors. Thus $F(A)_X = B$.

Now use Note 2,1.

Theorem 4.1: The following properties of a functor F are equivalent:

(i) F preserves products;

(ii) $F = C_0$ or $F \simeq C_{0,1}$ or $F \simeq C_1$ or F preserves difference kernels and intersections and ${}^1\chi_F = 1$.

(iii) $F = C_0$ or $F \simeq C_{0,1}$ or $F \simeq Q_M$ for some set M.

Proof: (i) $\Longrightarrow$ (ii) follows from Lemma 4,4, Lemma 4,5 and Lemma 4,1.

(ii) $\Longrightarrow$ (iii) follows from Proposition 3,3.

(iii) $\Longrightarrow$ (i) is evident.

5.

Definition 5.1: A natural transformation $\nu : Q_{M_1} \vee \vee Q_{M_2} \to F$ is said to be elementary if

a) $M_1 \neq \emptyset \neq M_2$;

b) the domain restrictions $\nu_1 : Q_{M_1} \to F$, $\nu_2 : Q_{M_2} \to F$ of ν are monotransformations;

c) there exist a set P and surjections $m_1 : M_1 \to P$, $m_2 : M_2 \to P$ such that the following assertion is satisfied: the equality $\nu_Y(\varphi_1) = \nu_Y(\varphi_2)$, with $\varphi_1 \in \in Q_{M_1}(Y), \varphi_2 \in Q_{M_2}(Y)$, holds if and only if there exists a mapping $p : P \to Y$ with $\varphi_1 = p \circ m_1$, $\varphi_2 = p \circ m_2$.

Note 5.1: It is easy to see: Let $\nu : Q_{M_1} \vee Q_{M_2} \to F$ be an elementary epitransformation. Then there exists a non-void set P and monotransformations $\lambda_1 : Q_P \to Q_{M_1}$,

$\lambda_2 : Q_P \to Q_{M_2}$ such that F is the direct limit of the diagram $\langle \{Q_P, Q_{M_1}, Q_{M_2}\}; \{\lambda_1, \lambda_2\}\rangle$.

Note 5,2: We recall that if $m_1 : M_1 \to P$, $m_2 : M_2 \to P$ are surjections and

$$\begin{tikzcd} & M_1 \arrow[dr, "m_1"] & \\ L \arrow[ur, "l_1"] \arrow[dr, "l_2"'] & & P \\ & M_2 \arrow[ur, "m_2"'] & \end{tikzcd}$$

is a pullback diagram in the category S , then it is also a pushout diagram.

Lemma 5,1: Let F be a separating functor with $card\ F(1) = 1$, $\chi^1_F = 2$. Then the following properties of F are equivalent:

(i) F preserves subdirect products;

(ii) there exists an elementary epitransformation

$\nu : Q_{M_1} \vee Q_{M_2} \to F$;

(iii) F is a subfunctor of some Q_L .

Proof: (i) $\Longrightarrow$ (ii): I. There exists an epitransformation $\nu : Q_{M_1} \vee Q_{M_2} \to F$ satisfying a) b) from Definition 5,1. This follows easily from Lemma 4,5, Lemma 4,6 and Note 3,3. We prove c).

II. Let X be the set of all cardinal numbers m with $m \leqq card\,(M_1 \vee M_2)$. Denote by $\mathcal{Y}$ the set of all couples $\langle \gamma_1, \gamma_2 \rangle$ where $\gamma_1 : M_1 \to Z$, $\gamma_2 : M_2 \to Z$ are mappings onto some $Z \in X$ and $\nu_Z(\gamma_1) = \nu_Z(\gamma_2)$. Then $\mathcal{Y} \neq \emptyset$ because $card\ F(1) = 1$. Let $\overset{1}{\sim}$ or $\overset{2}{\sim}$ be equivalences defined on M_1 or on M_2, respectively as follows:

$a_1 \overset{1}{\sim} a_1' \Longleftrightarrow \gamma_1(a_1) = \gamma_1(a_1')$ for all $\langle \gamma_1, \gamma_2 \rangle \in \mathcal{J}$;

$a_2 \overset{2}{\sim} a_2' \Longleftrightarrow \gamma_2(a_2) = \gamma_2(a_2')$ for all $\langle \gamma_1, \gamma_2 \rangle \in \mathcal{J}$.

Put $S_1 = M_1/\overset{1}{\sim}$, $S_2 = M_2/\overset{2}{\sim}$, let $\sigma_1 : M_1 \to S_1$, $\sigma_2 : M_2 \to S_2$ be the projections. Let $i_1 : S_1 \to S_1 \vee S_2$, $i_2 : S_2 \to$ $\to S_1 \vee S_2$ be embeddings. We can suppose that i_1, i_2 are inclusions. Let $s_1 \in S_1$, $s_2 \in S_2$; we put $s_1 R s_2$ if and only if $s_1 = \sigma_1(a_1)$, $s_2 = \sigma_2(a_2)$ for some $a_1 \in M_1$, $a_2 \in M_2$ with $\gamma_1(a_1) = \gamma_2(a_2)$ for all $\langle \gamma_1, \gamma_2 \rangle \in \mathcal{J}$. Let R^* be the smallest equivalence on $S_1 \vee S_2$ containing R. Put $P = S_1 \vee S_2/R^*$, let $\pi : S_1 \vee S_2 \longrightarrow P$ be the projection. Put $m_1 = \pi \circ i_1 \circ \sigma_1$, $m_2 = \pi \circ i_2 \circ \sigma_2$. We prove that P, m_1, m_2 have the required properties.

III. Now we prove $\nu_P(m_1) = \nu_P(m_2)$. It is easy to see that for every $\iota = \langle \gamma_1, \gamma_2 \rangle \in \mathcal{J}$, $\gamma_1 : M_1 \to Z_\iota$, $\gamma_2 : M_2 \longrightarrow Z_\iota$ there exists a mapping $\varphi_\iota : P \to$ $\to Z_\iota$ with $\varphi_\iota \circ m_1 = \gamma_1$, $\varphi_\iota \circ m_2 = \gamma_2$ and $\langle P; \{\varphi_\iota ; \iota \in \mathcal{J}\} \rangle$ is a subdirect product. Then

$[F(\varphi_\iota)](\nu_P(m_1)) = \nu_{Z_\iota}(\varphi_\iota \circ m_1) = \nu_{Z_\iota}(\gamma_1) = \nu_{Z_\iota}(\gamma_2) =$ $= \nu_{Z_\iota}(\varphi_\iota \circ m_2) = [F(\varphi_\iota)](\nu_P(m_2))$. Consequently $\nu_P(m_1) = \nu_P(m_2)$ and $\nu_Y(p \circ m_1) = \nu_Y(p \circ m_2)$ for every $p : P \longrightarrow Y$.

IV. Evidently $m_1(M_1) \cup m_2(M_2) = P$. Since $m_1(M_1) = m_2(M_2)$ (use Lemma 3.5), m_1 and m_2 are surjections.

V. Let $\varphi_1 : M_1 \longrightarrow Y$, $\varphi_2 : M_2 \longrightarrow Y$ be mappings, $\nu_Y(\varphi_1) = \nu_Y(\varphi_2)$. Then $\varphi_1(M_1) = \varphi_2(M_2)$. Let $Z \in \mathcal{X}$, $t : Z \to Y$, $\varkappa : Y \to Z$ satisfy $t(Z) = \varphi_1(M_1)$, $\varkappa \circ t = e_Z$. Then $\iota_0 = \langle \varkappa \circ \varphi_1, \varkappa \circ \varphi_2 \rangle \in \mathcal{J}$, hence $\varphi_1 = t \circ \rho_{\iota_0} \circ m_1$, $\varphi_2 = t \circ \rho_{\iota_0} \circ m_2$.

(ii) $\Longrightarrow$ (iii). Put $H = Q_{M_1} \vee Q_{M_2}$. Let

$$\begin{tikzcd}
 & M_1 \arrow[dr, "m_1"] & \\
L \arrow[ur, "l_1"] \arrow[dr, "l_2"'] & & P \\
 & M_2 \arrow[ur, "m_2"'] &
\end{tikzcd}$$

be the pull-back-push-out-diagram. Let $\lambda : H \to Q_L$ be the natural transformation with $\lambda_{M_1}(e_{M_1}) = l_1$, $\lambda_{M_2}(e_{M_2}) = l_2$. Then for $\varphi_1 \in Q_{M_1}(Y)$, $\varphi_2 \in Q_{M_2}(Y)$ the equality $\lambda_Y(\varphi_1) = \lambda_Y(\varphi_2)$ holds if and only if $\nu_Y(\varphi_1) = \nu_Y(\varphi_2)$. Consequently there exists a monotransformation $\mu : F \longrightarrow Q_L$ with $\lambda = \mu \circ \nu$.

(iii) $\Longrightarrow$ (i) is evident.

<u>Theorem 5.1</u>: The following properties of a functor F are equivalent:

(i) F preserves subdirect products;

(ii) $F \simeq C_{P,\pi,M}$, where π is a surjection, or $F \simeq C_{0,m} \vee G$, where G is small, separating and if $G \neq C_0$, then for every $X \neq \emptyset$, $x, y \in G(X)$ the functor $G_{\langle \{x,y\}, X \rangle}$ is a subfunctor of some $Q_{M_1} \vee Q_{M_2}$;

(iii) $F \simeq C_{P,\pi,M}$, where π is a surjection, or

$F \simeq C_{0,m} \vee G$, where $G = \bigvee_{x \in G(1)} G^{x}$ and every G^{*} satisfies the following assertions:

1) $G^{x}(1) = \{x\}$;

2) there exists an epitransformation $\nu : \bigvee_{\iota \in \mathcal{J}} Q_{M_\iota} \rightarrow G^{x}$ with $card\, \mathcal{J} \geq 2$ and for every $\iota, \iota' \in \mathcal{J}$, $\iota \neq \iota'$ the domain restriction $\nu_{\iota,\iota'} : Q_{M_\iota} \vee Q_{M_{\iota'}} \rightarrow G^{x}$ of ν is elementary.

Note 5,3: The assertion about G from (iii) can be formulated as follows: G is a direct limit of a special sort of a diagram composed from functors Q. The description of the sort of the diagram is easy, use Note 5,1.

Proof: (i) $\Rightarrow$ (ii): Let $F = F_d \vee G$, where G is separating and F_d has no non-trivial separating subfunctor. F_d and G are small (Lemma 4,1), $F_d \simeq C_{P,p,M}$ (lemma 4,3) where either $P = \emptyset$ or p is a surjection. If $G \neq C_0$ then necessarily $P = \emptyset$. Let $X \neq \emptyset$, $j : X \rightarrow 1$, $x, y \in G(X)$, $x \neq y$. If $[G(j)](x) = [G(j)](y)$ then $G_{\langle\{x,y\},X\rangle}$ is a subfunctor of some Q_L (Lemma 5,1); in the other case $G_{\langle\{x,y\},X\rangle} \simeq Q_{M_1} \vee Q_{M_2}$.

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are easy to prove.

Note 5,4: The notion of the preservation of subdirect products is near to the one of the preservation of separating systems. A couple $\langle X; \{\varsigma_\alpha ; \alpha \in A\}\rangle$, where X is a set, $\varsigma_\alpha : X \rightarrow X_\alpha$ are mappings, is said to be a separating system if the assertion a) from

Definition 4,1 is satisfied. The definition of functors preserving separating systems is evident. Denote by S^* the category of all non-void sets and all their mappings. It is easy to see:

1) A functor $F : S^* \longrightarrow S^*$ preserves separating systems if and only if it preserves subdirect products.

2) If a functor $F : S^* \longrightarrow S^*$ preserves subdirect products, then the functor $G : S \longrightarrow S$ with $G^* = F$, $G(\emptyset) = \emptyset$, preserves subdirect products.

3) If a functor $F : S \longrightarrow S$ preserves separating systems then either $F = C_0$ or F^* is some of the functors G^*, where G satisfies the conditions from the Theorem 5,1.

Thus we receive the following characterization of functors preserving separating systems:

A functor F preserves separating systems if and only if $F \simeq C_{P,\pi,M} \vee G$, where π is an injection, $G(\emptyset) = \emptyset$ and G preserves subdirect products.

Note 5,5: In [3] functors filtrafing products are considered. These are precisely functors preserving separating systems but not preserving products. Thus, the characterization of functors filtrating products follows easily from Note 5,4 and Theorem 4,1.

Examples 5,1:

a) The functor F in the Example 2,1 preserves difference kernels, intersections of finite collections and products of finite collections. It does not preserve intersections and subdirect products.

b) The functor F in the Example 2,2 c) is separating, preserves intersections but it does not preserve difference kernels.

c) Now we describe a separating functor F which preserves difference kernels and intersections but which does not preserve subdirect products: Let N be the set of all natural numbers, Q_N^1 and Q_N^2 be two copies of Q_N. Points of $Q_N^1(X)$ or $Q_N^2(X)$ will be denoted by $\langle x_1, x_2, \dots \rangle^1$ or $\langle x_1, x_2, \dots \rangle^2$, respectively. We receive F from $Q_N^1 \vee Q_N^2$ by identification of every $\langle x_1, x_2, \dots \rangle^1$ with $\langle x_1, x_2, \dots \rangle^2$ whenever there exists $m \in N$ such that $x_n = x_m$ for all $n \geq m$.

6.

Every subfunctor of some Q_M preserves subdirect products. The converse is not true. The discussion of the problem is given in the present part.

Lemma 6.1: Let F be a separating functor which preserves subdirect products. Let $\varphi : 2 \to 2$ be the mapping with $\varphi(0) = 1$, $\varphi(1) = 0$. Then $F(\varphi)$ has no fix-point.

Proof: The set of all fix-points of $F(\varphi)$ is the difference kernel of $F(e_2)$ and $F(\varphi)$. The difference kernel of e_2 and φ is empty and F preserves difference kernels.

Lemma 6.2: Let F be a separating functor which preserves subdirect products, let ${}^2\chi_F = 2$,

$card\, F(1) = 1$. Then there exists an epitransformation $\nu : \bigvee_{\iota \in \mathcal{J}} Q_{2_\iota} \longrightarrow F$ such that

1) all domain restrictions $\nu_\iota : Q_{2_\iota} \longrightarrow F$ of ν are monotransformations;

2) if $\iota, \iota' \in \mathcal{J}$, $\iota \neq \iota'$, then $\nu_X(\varphi) = \nu_X(\varphi')$, with $\varphi \in Q_{2_\iota}(X)$, $\varphi' \in Q_{2_{\iota'}}(X)$, if and only if φ and φ' both are the constant mappings on a point $x \in X$.

<u>Proof</u>: Let $\langle A, 2 \rangle$ be a reaching couple of F. Let B be the set of all $a \in A$ with ${}^a 2 = 2$. Then $\langle B, X \rangle$ is also a reaching couple of F. Choose $\mathcal{J} \subset B$ such that if $\varphi : 2 \longrightarrow 2$ is the mapping with $\varphi(0) = 1, \varphi(1) = 0$, then

1) for every $b \in B$ either $b \in \mathcal{J}$ or $[F(\varphi)](b) \in \mathcal{J}$;

2) for no $\iota \in \mathcal{J}$ there is $[F(\varphi)](\iota) \in \mathcal{J}$.

Then the epitransformation $\nu : \bigvee_{\iota \in \mathcal{J}} Q_{2_\iota} \longrightarrow F$ with $\nu_2(e_{2_\iota}) = \iota$ has the required properties.

<u>Theorem 6,1</u>: Let the first character or the second one of a separating functor F be less than or equal to 2. Then F is a subfunctor of some Q_M if and only if it preserves subdirect products and $card\, F(0) = 1$.

<u>Proof</u>: The case ${}^1\chi_F \leq 2$ follows easily from Lemma 5,1.

Let ${}^2\chi_F \leq 2$, $card\, F(1) = 1$ and F preserve subdirect products. If ${}^2\chi_F = 1$, then evidently $F \simeq I$. If ${}^2\chi_F = 2$, use the epitransformation $\nu : \bigvee_{\iota \in \mathcal{J}} Q_{2_\iota} \longrightarrow F$ satisfying the assertions 1) 2) from Lemma 6,2. Then it is easy to see that F is a subfunctor of Q_M, where

$M = \lfloor Y, 2 \rfloor$.

Example 6.1:

Now we give an example of a separating functor F preserving subdirect products, $\operatorname{card} F(1) = 1$ and such that F is not a subfunctor of any Q_M. The character of F is $\langle 3, 3 \rangle$:

Denote by Q_3^1, Q_3^2, Q_3^3 three different copies of the functor Q_3. Points of $Q_3^i(X)$ will be denoted by $\langle x_1, x_2, x_3 \rangle^i$. In $Q_3^1 \vee Q_3^2 \vee Q_3^3$ we make the following identifications:

$$\langle x_1, x_1, x_2 \rangle^1 \sim \langle x_1, x_2, x_2 \rangle^2 ,$$

$$\langle y_1, y_1, y_2 \rangle^2 \sim \langle y_1, y_2, y_2 \rangle^3 ,$$

$$\langle z_1, z_1, z_2 \rangle^3 \sim \langle z_1, z_2, z_2 \rangle^1 .$$

Theorem 5,1 implies easily that the functor F, received by these identifications, preserves subdirect products. Now we prove that F is not a subfunctor of any Q_X. Suppose that it is and let $\mu : F \to Q_X$ be a monotransformation. Then necessarily $\operatorname{card} X \geq 3$. Choose three different points a, b, c of X, put $A = \{a, b, c\}$. The points $\langle a, b, c \rangle^i$, $i = 1, 2, 3$ are three different points of $F(X)$; put $\alpha_i = \mu(\langle a, b, c \rangle^i)$, i.e. $\alpha_i = X \to X$.

1) We prove that $\alpha_i(X) = A$: If $\sigma : X \to X$ is a mapping with $\sigma(x) \neq x$ exactly for $x = a$ or $x = b$ or $x = c$, respectively, then $[F(\sigma)](\langle a, b, c \rangle^i) \neq \langle a, b, c \rangle^i$, consequently $\alpha_i \neq [Q_X(\sigma)](\alpha_i) = \sigma \circ \alpha_i$.

Now if $\sigma: X \to X$ is a mapping with $\sigma(x) \neq x$ exactly for $x \notin A$, then $[F(\sigma)](\langle a, b, c\rangle^i) = \langle a, b, c\rangle^i$, consequently $\alpha_i = \sigma \circ \alpha_i$.

2) Now let $\varphi, \sigma, \tau : X \to X$ be mappings with $\varphi(a) =$

$= \sigma(a) = \varphi(b) = x_1, \varphi(c) = \sigma(b) = \sigma(c) = x_2, \tau(a) = x_2,$

$\tau(b) = \tau(c) = x_1, \; x_1 \neq x_2$. Then

$$[F(\varphi)](\langle a, b, c\rangle^1) = [F(\sigma)](\langle a, b, c\rangle^2),$$

$$[F(\varphi)](\langle a, b, c\rangle^2) = [F(\sigma)](\langle a, b, c\rangle^3),$$

$$[F(\varphi)](\langle a, b, c\rangle^3) = [F(\tau)](\langle a, b, c\rangle^1).$$

This implies

$(*)$ $\varphi \circ \alpha_1 = \sigma \circ \alpha_2, \; \varphi \circ \alpha_2 = \sigma \circ \alpha_3, \; \varphi \circ \alpha_3 = \tau \circ \alpha_1$.

Choose $x \in X$ with $\alpha_1(x) = a$. The assertion $(*)$ implies easily: $\sigma \circ \alpha_2(x) = x_1$, consequently $\alpha_2(x) = a$, hence $\sigma \circ \alpha_3(x) = x_1$; thus $\alpha_3(x) = a$. But $\varphi \circ \alpha_3(x) = x_1, \; \tau \circ \alpha_1(x) = x_2$, which is a contradiction.

Note 6.1: The following characterization of subfunctors of Q-functors can be proved easily:

A functor F is a subfunctor of some Q_M if and only if either $F = C_0$ or $F \simeq C_{0,1}$ or $F \simeq C_1$ or there exists an epitransformation $\nu : \bigvee_{\iota \in J} Q_{M_\iota} \to F$ such that

1) all M_ι are non-void sets; the domain restrictions $\nu_\iota : Q_{M_\iota} \to F$ of ν are monotransformations;

2) there exists a set M and surjections $m_\iota : M \to M_\iota$

such that the equality $\nu_Y(\varphi) = \nu_Y(\varphi')$ with $\varphi \in$ $\in Q_{M_\iota}(Y)$, $\varphi' \in Q_{M_{\iota'}}(Y)$ holds if and only if $\varphi \circ m_\iota = \varphi' \circ m_{\iota'}$.

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