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HIGHER ORDER DIFFERENTIABILITY OF NONLINRAR OPERATORS ON NORMED SPACES - Part II

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This paper is a continuation of [55]. The numbering system for equations, definitions, theorems and references is sequential to Part I. The purpose of the entire paper was stated in Section 1.

For motivational purposes, we have so far restricted our discussion to second order Fréchet differentials. Various notions of mth order Fréchet differentials may be considered similarly. One may also consider mixed higher order differentials. For example, we may introduce a third order differential obtained by first finding the Fréchet differential of F, then the pointwise Fréchet differential of $dF(x; h)$, and finally the partial Fréchet differential $d_x \partial^2 F(x; h_1, h_2, h_3)$. Properties of mixed differentials may be obtained from the properties of the various differentials.

3.3. Notions of mth Order Gâteaux Differentials

Let $F: X \to Y$, where X is an open subset of the product space Π. The Gâteaux partial differential at $u_1 \dots u_n$ of F with respect to x_i exists if and only if there exists a bounded linear operator $D_i F(u_1 \dots u_n; \cdot) : E_i \longrightarrow Y$ such that

$$F(u_1, \dots, u_{i-1}, u_i + h_i, u_{i+1}, \dots, u_n) - F u_1 \dots u_n =$$
$$= D_i F(u_1 \dots u_n; h_i) + R(u_1 \dots u_n; h_i)$$

where

$$(6) \qquad \lim_{t \to 0} \frac{R(u_1 \dots u_n; t h_i)}{t} = 0 .$$

$D_i F(u_1 \dots u_n; h_i)$ is called the Gâteaux partial differential. The operator F is said to be totally Gâteaux differentiable at x_0 if F, considered as a mapping on $X \subset \Pi$ into Y, is Gâteaux differentiable at x_0. This means that

$$F(u_1 + h_1, \dots, u_n + h_n) - F u_1 \dots u_n =$$
$$= L(u_1 \dots u_n; h_1 \dots h_n) + R(u_1 \dots u_n; h_1 \dots h_n)$$

where L is a continuous linear operator in $h = (h_1 \dots h_n)$, and

$$(7) \qquad \lim_{t \to 0} t^{-1} R(u_1 \dots u_n; t h_1, \dots, t h_n) = 0 .$$

Clearly F has a Fréchet partial (total) differential at x_0 if and only if F has a Gâteaux partial (total) differential at x_0 and (6), (respectively (7)), holds uniformly with respect to $h_i \in E_i$ on the set $\|h_i\| = 1$ ($h_1 \dots h_n$ on the set $\|h_1\| = \dots = \|h_n\| = 1$).

Suppose for $m \geq 2$ the Gâteaux differential of order $m-1$ has been defined as a continuous $(m-1)$-

-linear operator $D^{m-1}F(x;h_1\dots h_{m-1})$, $D^{m-1}F(x;\dots)\in$ $\in \mathcal{L}_{m-1}(E_1\times\dots\times E_{m-1};Y)$, for each $x\in X$.

Definition 4 G. If $D^{m-1}F(x;\cdot)$, considered as a mapping on X into $\mathcal{L}_{m-1}(E_1\times\dots\times E_{m-1};Y)$ has a Gâteaux derivative at x_o, we denote this derivative by $D^m F(x_o;\cdot)$ and call it the mth order Gâteaux derivative of F at x_o. Thus $D^m F(x_o;\cdot)\in\mathcal{L}_m(E_1\times\dots$ $\dots\times E_m;Y)$. The mth order Gâteaux differential is denoted by $D^m F(x_o;h_1\dots h_m)$, $h_i\in E_i$. We have by definition:

$$\lim_{t\to 0} t^{-1}\|D^{m-1}F(x_o+th_m;\cdot)-D^{m-1}F(x_o;\cdot)-D^m F(x_o;\dots,th_m)\|_{\mathcal{L}_{m-1}}=0.$$

Definition 5 G. F is said to have at x_o a pointwise Gâteaux differential of order m if F has a pointwise Gâteaux differential of order $m-1$ and there exists a continuous m-linear operator $L(x_o;\cdot)$ such that for each fixed $h_1\dots h_{m-1}$,

$$\lim_{t\to 0} t^{-1}\|D^{m-1}F(x_o+th_m;h_1\dots h_{m-1})-D^{m-1}F(x_o;h_1\dots$$
$$\dots h_{m-1})-L(x_o;h_1\dots h_m)\|=0.$$

$L(x_o;\cdot)$ is called the mth order pointwise Gâteaux derivative and is denoted by $\partial_G^m F(x_o;\cdot)$.

The mth order partial Gâteaux differential (Definition 6G) and the mth order total Gâteaux differential (Definition 7G) may be defined in an obvious way and are denoted by $D_x D^{m-1}F(x_o;h_1\dots h_m)$ and $DD^{m-1}F(x_o;$ $h_1\dots h_{m-1};\Delta x,\Delta h_1,\dots,\Delta h_{m-1})$ respectively. We remark that $D_x D^{m-1}F(x_o;h_1\dots h_m)$ is continuous and linear in h_m but is not required to be continuous or linear in $h_1, \ldots, h_{m-1}$. However, if $D^{m-1}F(x_o; h_1 \ldots h_{m-1})$ is assumed to be linear and continuous in $h_1, \ldots, h_{m-1}$, then $D^m F(x_o; h_1 \ldots h_m)$ is automatically linear (but not necessarily continuous) in $h_1, \ldots, h_{m-1}$.

<u>Definition 8G.</u> F is said to have at x_o a strong Gâteaux differential of mth order if there exists an operator $T(x_o) \in \mathcal{L}_m(E_1 \times \ldots \times E_m; Y)$ such that for each $\varepsilon > 0$, there exists $r > 0$, where

$$\| D^{m-1}F(x_o + tk; \cdot) - D^{m-1}F(x_o + sk; \cdot) -$$

$$- T(x_o)(\cdot; (t-s)k) \| \leq \varepsilon\, |t - s|$$

for each $k \in E_m$ and each pair of numbers s, t with $|s| \leq r$ and $|t| \leq r$. $T(x_o)$ is called the mth order strong Gâteaux derivative and is denoted by $D^m F^*(x_o; \cdot)$.

The continuity implications of these notions are stated in the following theorem. The proof is a trivial modification of the proof of Theorem 3 using appropriate results on continuous and directionally continuous multilinear operators discussed in Section 1.

<u>Theorem 5</u>. (a) If $D^m F(x_o; h_1 \ldots h_m)$ exists, then $D^{m-1}F(x; h_1 \ldots h_{m-1})$ is directionally continuous (Definition 3d) at x_o.
(b) If $D^m F(x; h_1 \ldots h_m)$ has a pointwise Gâteaux differential at x_o, then $D^m F(x; h_1 \ldots h_m)$ is directionally continuous at x_o; hence also point-

wise directionally continuous (Definition 1d).

(c) If $D^m F(x; h_1 \dots h_m)$ has a partial Gâteaux differential at x_0, then $D^m F(x; h_1 \dots h_m)$ is jointly directionally continuous (Definition 2d) at $(x_0; h_1 \dots h_m)$.

(d) If $D^m F(x; h_1 \dots h_m)$ has a total Gâteaux differential at x_0, then $D^m F(x; h_1 \dots h_m)$ is jointly directionally continuous at $(x_0; h_1 \dots h_m)$.

(e) If $D^m F(x, \cdot)$ has a strong Gâteaux differential at x_0, then for some $M \geq 0$,

$$\| D^m F(x_0 + tk; \cdot) - D^m F(x_0 + sk; \cdot) \|_{\mathcal{L}_m} \leq M |t - s|$$

for all t, s in some neighborhood of zero.

<u>Theorem 6</u>. (a) $D^m F(x; \cdot)$ has a Gâteaux differential at x_0 if and only if $D^m F(x; h_1 \dots h_m)$ has a pointwise Gâteaux differential at x_0 and

$$(8) \quad \lim_{t \to 0} t^{-1} \{ D^m F(x_0 + tk; h_1 \dots h_m) -$$

$$- D^m F(x_0; h_1 \dots h_m) \} = \partial_G^{m+1} F(x_0; h_1 \dots h_m k)$$

holds uniformly with respect to k on the set $\| k \| = 1$. Consequently, $D^m F(x; \cdot)$ has a Fréchet differential at x_0 if and only if (8) holds uniformly with respect to k and $h_1 \dots h_m$ on the set $\| k \| = \| h_1 \| = \dots = \| h_m \| = 1$.

(b) $D^m F(x; h_1 \dots h_m)$ has a pointwise Gâteaux differential at x_0 if and only if $D^m F(x; h_1 \dots h_m)$ has a Gâteaux partial differential at x_0 and the latter is jointly continuous in $h_1 \dots h_m$

(c) If $D^m F(x\,;\, h_1 \ldots h_m)$ has a pointwise Gâteaux differential at x_0, then $D^m F(x\,;\, h_1 \cdots h_m)$ is totally differentiable at x_0 and the total differential is given by

$$\partial_G D^m F(x_0\,;\, h_1 \cdots h_m\,;\, \Delta x) + D^m F(x_0\,;\, \Delta h_1, h_2, \ldots, h_m)$$

$$+ \ldots + D^m F(x_0\,;\, h_1, \cdots, \Delta h_m)\,.$$

(d) If $D^m F(x\,;\, \cdot\,)$ has a strong Gâteaux differential at x_0, then $D^{m+1} F(x\,;\, \cdot\,)$ exists.

The proof of Theorem 6 is a simple modification of the proof of Theorem 4.

3.4. Strong Differentials.

Definitions 8 and 8G dealt with higher order Fréchet and Gâteaux strong differentials. One may also define strong pointwise differentials, strong partial differentials, etc. The notion of the strong derivative of a function of a real variable was first introduced by Peano [35] who felt it "portrayed the concept of the derivative used in the physical sciences more closely than does the usual derivative". Let f be a real function whose domain is an open subset of the real line. Then f is strongly differentiable at x_0 if

$$\lim_{\substack{(x_1, x_2) \to (x_0, x_0) \\ x_1 \neq x_2}} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

exists. This notion has been recently reconsidered and generalized by Leach [36], Esser and Shisha [37] and Nashed [38]. Relationships among the Gâteaux and Fréchet strong differentials and side conditions under which a Gâteaux variation or differential is a strong differential are given in [38]. We remark that the distinction between Gâteaux strong differential and Fréchet strong differential is only pointwise, i.e., the existence of a strong Gâteaux differential on an open set implies the existence of a strong differential on that set. The notion of strong differential is also closely connected with uniform differentials as defined by Vainberg [4]. Among other results on differentiability of convex functionals it is shown in [38] that if the one-sided Gâteaux variation $V^{+} f(x_0 ; h)$ of a convex functional f defined on an open subset X is a Fréchet differential, then $V^{+} f(x_0 ; h)$ is automatically a strong differential.

Strong differentiability at a given point x_0 implies a smoothness condition in a neighborhood of x_0 which is a desirable property in many applications, for example inverse function theorems and Newton's method [36].

Let D denote the subset of X on which $d^m F(x; h_1 \dots h_m)$ has a Fréchet strong pointwise differential and let D' denote the subset of X on which $d^m F(x; h_1 \dots h_m)$ has a Fréchet pointwise differential. Clearly $D \subset D'$ and in general either or both of these sets may be empty.

<u>Theorem 7</u>. If $d^m F(x; h_1 \dots h_m)$ is strongly pointwise Fréchet differentiable at x_0, then for each $h_1 \dots h_{m+1}$,

$$\lim_{\substack{x \to x_0 \\ x \in D}} \partial^{m+1} F^*(x; h_1 \dots h_m h_{m+1}) =$$

$$= \lim_{\substack{x \to x_0 \\ x \in D'}} \partial^{m+1} F(x; h_1 \dots h_{m+1}) =$$

$$= \partial^{m+1} F(x_0; h_1 \dots h_{m+1}) = \partial^{m+1} F^*(x_0; h_1 \dots h_{m+1}),$$

whenever both limits are meaningful.

Proof. Let $\varepsilon > 0$ be given. Since $d^m F(x; \cdot)$ is strongly pointwise differentiable at x_0, for each fixed $h_1 \dots h_m$ there exists $\delta > 0$ such that $\| x_1 - x_0 \| < \delta$ and $\| x_2 - x_0 \| < \delta$ and $x_1 \neq x_2$ imply that $x_1, x_2 \in X$ and

(9) $$\| d^m F(x_2; h_1 \dots h_m) - d^m F(x_1; h_1 \dots h_m) - \partial^{m+1} F^*(x_0; h_1 \dots h_m; x_2 - x_1) \| \leq \varepsilon \| x_2 - x_1 \| .$$

Let $x \in D'$ where $\| x - x_0 \| \leq \frac{\delta}{2}$. Then x is a permissible value for x_1 in (9). Now let k be any fixed nonzero element of E_{m+1}. Then if $|t| < \frac{\delta}{2 \| k \|}$ it follows that $\| x + tk - x_0 \| < \delta$ and hence $x + tk$ is a permissible value for x_2 in (9) provided $|t| < \frac{\delta}{2 \| k \|}$. Then

$$\| d^m F(x + tk; h_1 \dots h_m) - d^m F(x; h_1 \dots h_m) - \partial^{m+1} F^*(x_0; h_1 \dots h_m; tk) \| \leq \varepsilon \| tk \| .$$

But since $\partial^{m+1} F^*(x_0; \cdot)$ is homogeneous, this is equivalent to

$$\| \frac{1}{t} \{ d^m F(x + tk; h_1 \dots h_m) - d^m F(x; h_1 \dots h_m)$$

$- \partial^{m+1} F^*(x_0; h_1 \dots h_m k) \} \| \leq \epsilon \| k \|$,

from which it follows that

$\| \partial_G d^m F(x; h_1 \dots h_m k) - \partial^{m+1} F^*(x_0; h_1 \dots h_m k) \| \leq \varepsilon \|k\|.$

But $x \in D'$ so that $\partial^{m+1} F(x; h_1 \dots h_m k)$ exists and is equal to $\partial_G d^m F(x; h_1 \dots h_m k)$. Thus $x \in D'$ and $\| x - x_0 \| < \frac{\delta}{2}$ imply that

$\| \partial^{m+1} F(x; h_1 \dots h_m k) - \partial^{m+1} F(x_0; h_1 \dots h_m k) \| \leq \varepsilon \|k\|.$

Thus for each k,

$$(10) \quad \lim_{\substack{x \to x_0 \\ x \in D'}} \partial^{m+1} F(x; h_1 \dots h_m k) = \partial^{m+1} F(x_0; h_1 \dots h_m k).$$

Now assuming that $\lim_{\substack{x \to x_0 \\ x \in D}} \partial^{m+1} F(x; h_1 \dots h_m k)$ exists,

$$\lim_{\substack{x \to x_0 \\ x \in D}} \partial^{m+1} F^*(x; h_1 \dots h_m k) = \lim_{\substack{x \to x_0 \\ x \in D}} \partial^{m+1} F(x; h_1 \dots h_m k)$$

since $\partial^{m+1} F^*$ and $\partial^{m+1} F$ are identical on D.

But then it follows that

$$(11) \quad \lim_{\substack{x \to x_0 \\ x \in D}} \partial^{m+1} F^*(x; h_1 \dots h_m k) = \lim_{\substack{x \to x_0 \\ x \in D'}} \partial^{m+1} F(x; h_1 \dots h_m k)$$

since $D \subset D'$. (10) and (11) give the desired result.

<u>Theorem 8</u>. (a) If $d^m F(x; h_1 \dots h_m)$ is pointwise Fréchet differentiable at x_0 and $\partial^{m+1} F(x; h_1 \dots h_m h_{m+1})$ is jointly continuous, then $d^m F(x; h_1 \dots h_m)$ has a strong pointwise Fréchet differential at x_0.

(b) Let S_r denote the ball $\| x \| < r$. Then $d^m F(x; h_1 \dots h_m)$ is strongly pointwise Fréchet differentiable on S_r if and only if $\partial^{m+1} F(x; h_1 \dots h_{m+1})$

is jointly continuous.

(c) $d^m F(x; h_1 \dots h_m)$ is strongly Fréchet differentiable on S_r if and only if $d^m F(x; \cdot)$ has in S_r a locally uniform Fréchet differential [4] and that $d^{m+1} F(x; \cdot)$ be locally bounded in S_r.

Proof. (a) follows easily from the inequality

$$\| d^m F(x_2; h_1 \dots h_m) - d^m F(x_1; h_1 \dots h_m) - \partial^{m+1} F(x_0; h_1 \dots h_m, x_2 - x_1) \| \leq \| d^m F(x_2; h_1 \dots h_m) - d^m F(x_1; h_1 \dots h_m) - \partial^{m+1} F(x_1; h_1 \dots h_m, x_2 - x_1) \| +$$

$$+ \| \partial^{m+1} F(x_1; h_1 \dots h_m, x_2 - x_1) - \partial^{m+1} F(x_0; h_1 \dots h_m, x_2 - x_1) \|.$$

(b) follows from part (a) and Theorem 7.

(c) follows from Theorem 4.1 in Vainberg [4], and an obvious modification of part (b).

3.5. Remarks on Higher Order Hadamard Variations and Differentials

The notion of Hadamard differential which was introduced in [39] takes the following form in the setting of normed spaces. A continuous operator $F: X \to Y$ is said to have an Hadamard differential at x_0 if there exists a linear mapping $u: E \to Y$ such that for any continuous mapping $g: [0, 1] \to X$ for which $g(0) = x_0$ and $g'(0^+)$ exists, the mapping $S(t) = F[g(t)]$ is differentiable at $t = 0^+$ and $S'(0^+) = u(g'(0^+))$. The mapping u is called the Hadamard derivative of F at x_0. Variants and generalizations of Hadamard differentials have been studied by many authors [2,pp.151-152;40,41,42,43,44,45]. Connections between Hadamard differentiability and other notions of differentiability are readily available. If F is Hadamard differentiable at x_0, then F is Gâteaux differentiable at x_0 and the two derivatives are equal, but the converse is not necessarily true. Let $f(x,y) = \frac{x^3 y}{x^4 + y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Then f is Gâteaux differentiable at $(0,0)$, $DF(0,0; h,k) = 0$. Define $g(t) = (t, t^2)$ for $t \in [0,1]$. Then $g(0) = (0,0)$, $g'(0^+) = (1,0)$ and $S(t) = f[g(t)] = t^2 + \frac{3}{2}$. Then $S'(0^+) = \frac{3}{2}$ while $u[g'(0^+)] = 0$. Thus f is not Hadamard differentiable at $(0,0)$. Fréchet differentiability implies Hadamard differentiability. The converse is also true if the space E has finite dimension.

A continuous map g on the subset $[0,1] \times \ldots \times [0,1]$ of R_n into X is called an n-trajectory if $g(0,\ldots,0) = x_0$ and the Gâteaux derivative Dg of g exists at $(0^+,\ldots,0^+)$. We say that F has nth order Hadamard variation at x_0 if there exists a mapping $u : E \times \ldots \times E \to Y$ which is homogeneous in each of the variables such that for any n-trajectory g,

$$\frac{\partial^n F[g(t_1,\ldots,t_n)]}{\partial t_1 \ldots \partial t_n} \text{ exists at } t_1 = \ldots = t_n = 0^+$$

and is equal to $u[Dg(0^+,\ldots,0^+)]$. It is easy to show that if F has nth order Hadamard variation at x_0, then F has nth order Gâteaux variation at x_0. Various notions of higher order Hadamard differentials may be introduced and related to the notions of higher order differentials studied in this paper. This will be undertaken in a subsequent note.

Several notions of variations may be obtained also by varying the "difference quotient" in the definition of the Gâteaux variation. We introduce, for instance, the variation analogs of the Sindalovskii [46] and Murav'ev [47] derivatives. Let σ be an operator defined in a neighborhood of zero such that $\sigma(th) \to \theta$ as $t \to 0$ for each fixed h. We define the Sindalovskii variation by $\lim_{t \to 0} t^{-1}\{F[x-\sigma(th)]-F[x-\sigma(th)-th]\}$. Let G be any continuous operator. We define the Murav'ev variation of F (with respect to G) by $\lim_{t\to 0} t^{-1}\{F[x+th\,G(x)]-F(x)\}$. The Gâteaux variation is obtained as a special case of the Sindalovskii and Murav'ev variations. Other properties of these variations follow readily.

For an excellent survey of various notions and properties of derivatives of a function of a real variable, see [48].

4. Peano and Taylor Variations and Differentials

The existence of an m-th order Gâteaux variation or of an m-th order Gâteaux or Fréchet differential of an operator F at a point x_0, provides a local representation of the operator in a neighborhood of x_0. Such a representation gives, in this neighborhood, an approximation of the operator by a sum of homogeneous operators or by a polynomial operator. The sense of the approximation depends on the notion of variation or differential that is used, and is an order condition on the "remainder".

The usual Taylor theorem in functional analysis (cf. [2,5]) gives such a representation for $F(x_0+h)$ when F has an m-th order differential along the segment $\{x_0+th: 0\leq t\leq 1\}$. What we need here, however, is a representation theorem which requires only the existence of an n-th order variation or differentials at a point. This is available in a variant of Taylor's theorem, which was given for real functions by W.H. Young [49]. The generalization to operators is stated in the following theorem which can be easily established by induction.

Theorem 9. Let $F: X\to Y$, when X is an open subset of E.

(a) If $\delta^n F(x_0; h)$ exists, then the following expansion holds.

$$F(x_0+h)=F(x_0)+\delta F(x_0;h)+\dots+ \tag{12}$$
$$+\frac{1}{n!}\delta^n F(x_0;h)+R_n(x_0;h)\ ,$$

where $\lim\limits_{t\to 0}\dfrac{R_n(x_0;th)}{t^n}=0$.

(b) If the n-th order Gâteaux differential $D^n F(x_0;h)$ exists, then the expansion (12) holds, where the variations $\delta^k F(x_0;h)$ are now replaced by $D^k(x_0;h)$ for $k=1,2,\dots,n$.

(c) If the n-th order Fréchet differential $d^n F(x_0;h)$

exists, then the following expansion holds

$$F(x_0+h) = F(x_0) + dF(x_0; h) + \dots +$$
$$+\frac{1}{n!} d^n F(x_0; h) + R_n(x_0; h) ,$$

where

$$\lim_{h \to \theta} \frac{R_n(x_0; h)}{\|h\|^n} = 0 .$$

W.H. Young [49] states that if a function has an expansion such as

$$f(a+h) = f(a) + c_1 h + \dots + c_n h^n + R_n(a; h)$$

where $\lim_{h \to 0} \frac{R_n(a, h)}{h^n} = 0$, then the n-th order derivative of f at a exists. Graves [9] was first to observe that this is not necessarily true by giving a simple counterexample: $f(x) = x^3 \sin \frac{1}{x}$, $x \neq 0$; $f(0) = 0$.

The failure of the converse to Theorem 9 to hold motivates the following definitions.

Definition 9. An operator $F: X \to Y$ is said to have an n-th order Peano variation at $x_0 \in X$ if there exists a homogeneous form $Q_n(x_0; h) = \sum_{i=1}^{n} \frac{1}{i!} H_i(x_0; h)$, where $H_i(x_0; \alpha h) = \alpha^i H_i(x_0; h)$ for each real number α such that for all h with $x_0 + h \in X$,

$$F(x_0+h) - F(x_0) = Q_n(x_0; h) + W_n(x_0; h) , \tag{13}$$

and

$$\lim_{t \to 0} \frac{\|W_n(x_0; th)\|}{t^n} = 0 . \tag{14}$$

Note that if a representation such as (13) - (14) exists, it is unique. $H_n(x_0; h)$ is called the Peano variation of order n of F at x_0.

Definition 10. An operator $F: X \to Y$ is said to have an n-th order Peano differential at x_0 if there exists a polynomial $P_n(x_0; h) = \sum_{i=1}^{n} \frac{1}{i!} L_i h^i$,

such that for all h with $x_0 + h \in X$,

(15) $F(x_0 + h) - F(x_0) = P_n(x_0; h) + W_n(x_0; h)$

where

$$\lim_{h \to 0} \frac{\| W_n(x_0; h) \|}{\| h \|^n} = 0. \tag{16}$$

$L_n h^n$, which is uniquely determined, is called the Peano differential of order n. The Peano directional differential is defined by replacing the condition (16) for the representation (15) by the weaker condition

$$\lim_{t \to 0} \frac{\| W_n(x_0; t h) \|}{t^n} = 0. \tag{17}$$

The existence of a first order Peano variation (Peano differential) is equivalent to the existence of the Gâteaux variation (Fréchet differential). However, this is not necessarily true for second or higher order Peano variation or differentials. For example, let $f(x) = 0$ if x is rational and $f(x) = x^{n+1}$ if x is irrational. Then f has n-th order Peano differential at 0: $P_n(0; h) = 0$ for all h, while $f''(0)$ does not exist.

In the definition of Peano variation of order n, we have not stipulated the existence of any Gâteaux variation of order greater than one. This motivates the following:

Definition 11. Let $F : X \to Y$ have at x_o a Gâteaux variation of order $n-1$. If

$$(18) \quad \lim_{t \to 0} \frac{n!}{t^n} [F(x_o + th) - F(x_o) - \sum_{k=1}^{n-1} \frac{1}{k!} \delta^k F(x_o ; th)]$$

exists, we call it the n-th order Taylor variation of F at x_o and denote it by $V_T^n F(x_o ; h)$. The n-th order Taylor and Taylor directional differentials are defined in an obvious way.

Several notions of Taylor differentials may be obtained by using the various notions of n-th order Fréchet differentials introduced in Section 3.

It is easy to show that the n-th order Peano variation is homogeneous in h of degree n. Implication - relations among the Peano variation, Peano differential and Peano directional differential follow readily.

Theorem 10. (a) If the n-th Taylor variation (differential) exists, then the n-th Peano variation (differential) exists and the two are equal.
(b) If the n-th Gâteaux variation (Gâteaux differential, Fréchet differential) exists, then the n-th Taylor variation (directional Taylor differential, Taylor differential) exists and the two are equal.
(c) If F has n-th Peano variation (differential) at x_o and if the $(n-1)$-st Gâteaux variation (Fréchet differential) of F at x_o exists, then the n-th Taylor variation (differential) at x_o exists.

<u>Proof</u>: (a) From Definition 11,

$$F(x+h) = F(x) + \delta F(x;h) + \dots + \frac{1}{(n-1)!}\,\delta^{n-1}F(x;h)$$
$$+\frac{1}{n!}\,V_T^n F(x;h) + W_n(x;h)\ ,$$

where

$$\lim_{t\to 0} \frac{\|W_n(x;th)\|}{t^n} = 0\ .$$

From the uniqueness of the representation (13) - (14), it follows that the n-th Peano variation exists and is equal to $V_T^n(x;h)$.

(b) follows from Theorem 9.

(c) follows from the uniqueness of the representation (13) - (14).

<u>Remark</u>. The Peano and Taylor variations and differentials provide a different approach to higher order differentiability that is most useful in approximation theory and the calculus of variations. They may be used to derive sufficiency conditions and higher order necessary conditions (that is, other than the Euler equations) in the calculus of variations. The second order directional Taylor differential suffices for the purpose of the theory of the second variation in the calculus of variations. Many of the results of [50], for instance, may be derived using this approach. Necessary and sufficient conditions for Peano and Taylor differentials to be Fréchet differentials corresponding to

the various notions studied in Section 3, as well as connections with difference-differential (including Riemann differentials, smooth operators and direct differentials) are given in [1].

Remark. Oliver [51] showed that a Peano derivative of a function of a real variable which exists at every point of an interval is of Baire class 1, possesses the Darbeaux property and is equal to the ordinary n-th derivative on a dense open set. If the n-th Peano derivative is bounded either above or below, then it coincides everywhere with the n-th ordinary derivative. In the case of a function of a real variable, the Peano derivatives were referred to by Denjoy [52] as differential coefficients. The notion also occurs in the work of de la Vallée Poussin [53]. Taylor derivatives of functions are also studied by Butzer [54].

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