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AN ELEMENTARY CHARACTERIZATION OF THE CATEGORY OF
RELATIONAL SYSTEMS

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Introduction

In [2] F. Lawvere characterized the category of sets by elementary axioms, using a language, with one sort of variable symbols (mappings) and two unary functions symbols (domain and codomain ¹) and one ternary relation, composition. $A \xrightarrow{f} B$ means f is a map with domain A and codomain B . D. Schlomiuk, [5], presented a method of characterizing the category of topological spaces by using the full subcategory of discrete spaces; the fact that the functor, inclusion, from sets to topological spaces has a left adjoint, together with additional axioms on a certain constant (the two-point space $(\{a, b\}, \{a\}, \{a, b\})$). Lawvere also characterized algebraic categories [1], using the special properties of a certain constant, the free object on one generator, and an adjointness condition. It is conjectured that what one needs is a previously characterized reflective subcategory C and finitely many

¹ An object is a map which is a domain and a codomain.

objects $a_1 \dots a_m$ determined up to isomorphism, such that the full subcategory on $Cu \cup a_i$, is in the transitive closure of "adequate". We shall adopt this technique in the present paper to the category of relational systems.

In [3],[4], Hedrlín, Pultr and Trnková defined morphisms to form the category of relational systems of type Δ , $R(\Delta)$, and showed the usefulness in representation of abstract concrete categories as full subcategories of a category of relational systems. The definition of the category of relational systems of Δ is the following:

Let $\Delta = \{n_i\}_{i \in I}$ where the n_i are cardinals, and I is a set. Then the objects are pairs $(X, \{R_i\}_{i \in I})$ where $R_i \subseteq X^{n_i}$.

$(X, \{R_i\}_{i \in I}) \xrightarrow{f} (Y, \{S_i\}_{i \in I})$ is a morphism iff f is a function from X to Y such that $n_i f(R_i) \subseteq S_i$ for all $i \in I$; where if $n_i \xrightarrow{g} X \in X^{n_i}$, $n_i f_i(g) = gf$, $n_i \xrightarrow{gf} Y$.

Although the present author was not able to find a complete characterization of $R(\Delta)$ such that every complete (categorically) model was naturally equivalent to the $R(\Delta)$ of Hedrlín and Pultr, he did find an axiom system denoted by R_I such that if C is a category satisfying the axioms of R_I , and C is complete then C is naturally equivalent to $R(\Delta)$ where

$\Delta = \{n_i\}_{i \in I}$ and one can determine the values of the n_i by a simple test in C , whenever I is a finite set. For I infinite one needs infinitely many a_i to form the category in the transitive closure of "adequate" for the generalization.

We shall start by stating the axioms for R_1 (where 1 is a one point set) and then develop from this the axioms for R_I .

Characterization of R_1 .

Axiom 1. R_1 has an initial (0), and terminal (1) object, equalizers, coequalizers of pairs of maps, products and sums of pairs of objects.

Axiom 2. There is an object $1'$ such that if G is any object, $G \neq 0 \Rightarrow \exists f, 1' \xrightarrow{f} G$.

Definition: An object G is discrete $\Leftrightarrow \exists v, G \xrightarrow{v} 1'$.

Axiom 3. $G \xrightarrow{v} 1'$ and $G \xrightarrow{u} 1' \Rightarrow u = v$.

Definition: $X \in A$ (or X is an element of A) $\Leftrightarrow 1' \xrightarrow{x} A$.

(Note the definition $1 \xrightarrow{x} A$ given by Lawvere, [1], and Schlomiuk [5] is not applicable here.)

Axiom 4. $1'$ is a projective generator.

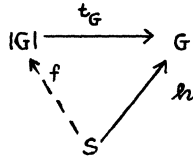
It is clear that if $A \xrightarrow{f} B$ then f is epi $\Leftrightarrow \forall x \in B, \exists y \in A \quad \forall y f = x$; f is mono \Leftrightarrow for every pair of elements $x, y \in A$, $x \neq y, xf \neq yf$.

Definition: f is a bijection $\iff f$ is *mono* and *epi*.

Axiom 5. Every non zero object has elements.

Axiom 6. Every element of a sum can be factored through exactly one of the injections.

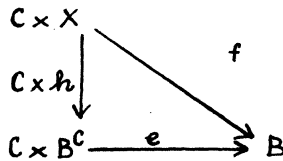
Axiom 7. For each object G there exists a discrete object $|G|$ and a bijection t_G such that for each discrete object S and each map $S \xrightarrow{h} G$ there exists a unique f such that



commutes.

It is clear that \parallel is a functor, and $|f| \text{ mono} \implies f \text{ mono}$, $|f| \text{ epi} \implies f \text{ epi}$.

Axiom 8. If C, A are discrete then there exists a discrete object B^C and a map $C \times B^C \xrightarrow{e} B$ such that for every discrete object X and mapping $C \times X \xrightarrow{f} B$ there exists a unique mapping $X \xrightarrow{h} B^C$ such that



commutes.

It is clear that the elements of B^C are in one-to-one correspondence with the maps from B to C in the following way:

Let $\eta \in B^C$, define (η) to be the unique map

for which

$$\begin{array}{ccc}
 C \times 1' & \xrightarrow{\pi_C} & C \\
 \downarrow C \times \eta_f & & \downarrow (\eta_f) \\
 C \times B^c & \xrightarrow{e} & B
 \end{array}$$

commutes; if

$C \xrightarrow{f} B$ let $[f]$ be the unique map such that

$$\begin{array}{ccc}
 C \times 1' & \xrightarrow{\pi_C} & C \\
 \downarrow C \times [f] & & \downarrow f \\
 C \times B^c & \xrightarrow{e} & B
 \end{array}$$

commutes.

It is clear B^C can be extended to a functor of two variable contravariant in the exponent.

Axiom 9. There exists a discrete object N and maps $1' \xrightarrow{o} N, N \xrightarrow{i} N$ such that for every discrete object $X, x_0 \in X$, and each map $X \xrightarrow{\mu} X$ there exists a unique map $N \xrightarrow{x} X$ such that

$$\begin{array}{ccc}
 & N & \xrightarrow{i} & N \\
 & \swarrow o & & \downarrow x \\
 1' & & & X \\
 & \searrow x_0 & & \downarrow \mu \\
 & X & \xrightarrow{\mu} & X
 \end{array}$$

commutes.

Axiom 10. If C, B are discrete and C has elements then for every map $C \xrightarrow{f} B$ there exists $B \xrightarrow{g} C$ such that $f \circ g = f$.

We now have the following theorem schema: If Φ is a theorem of the elementary theory of the category

of sets and Φ' is obtained from Φ by replacing "set" by discrete object then Φ' is a theorem of R_1 .

Axiom 11. There is an object A of R such that

- (1) $A \xrightarrow{f} A \Rightarrow f = A$,
- (2) $|A| \neq 0$,
- (3) $\exists f, A \xrightarrow{f} G \iff G$ is not discrete.

Definition: $|A| = n$.

We note that if $A \xrightarrow{f} G$ then $n \xrightarrow{|f|} |G|$ and thus there exists uniquely $1' \xrightarrow{[|f|]} |G|^n$.

Axiom 12. \forall objects G there exists a monomorphism $R \xrightarrow{m} |G|^n$ such that for all $f, A \xrightarrow{f} G \exists \mu$ such that

$$\begin{array}{ccc}
 & & |G|^n \\
 & \nearrow^{[|f|]} & \uparrow m \\
 1' & & \\
 & \searrow_{\mu} & R
 \end{array}$$

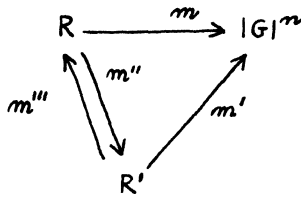
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and if m' is any other monomorphism with this property there exists a unique monomorphism m'' such that

$$\begin{array}{ccc}
 R & \xrightarrow{m} & |G|^n \\
 & \searrow_{m''} & \nearrow_{m'} \\
 & & R'
 \end{array}$$

commutes.

Thus if m, m' both satisfy the two above properties then we have



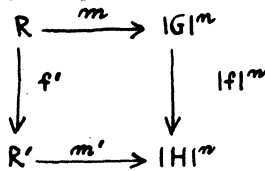
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thus $m'''m = m'$, $m''m' = m$ thus $m''m'''m''m' = m$, which by uniqueness gives $m''m'''m'' = m''$ and as m'' is mono, $m'''m'' = 1_{R'}$. Similarly $m''m''' = 1_R$.

Furthermore if h is an isomorphism $R \xrightarrow{h} R'$ and m satisfies the conditions of axiom 11 so does hm . Thus if we define $(|G|, m)$ represents G to mean that $(|G|, m)$ satisfies the properties of axiom 12 with respect to G , then if $(|G|, n)$ is another representation then $n = hm$ where h is an isomorphism.

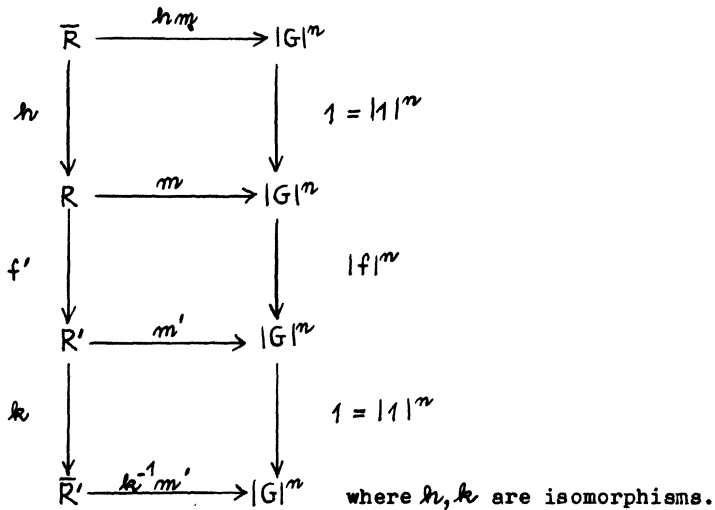
Axiom 13. In every discrete object H and every monomorphism $R \xrightarrow{m} H^n$ there exists an object G such that $|G| = H$ and (H, m) represents G , furthermore if G' is represented by (H, m) , G is isomorphic to G' .

Axiom 14. If $G \xrightarrow{f} H$ is any morphism and G is represented by $(|G|, m)$ and H by (H, m') then there exists a unique map f' such that

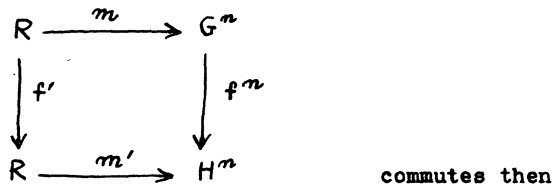


commutes.

It is clear that this is independent of the choice of representatives m, m' as can be seen from the commutative diagram



Axiom 15. If G, H are discrete and m, m' are monomorphisms $R \xrightarrow{m} G^n, R' \xrightarrow{m'} H^n$ and f is a map $G \xrightarrow{f} H$ such that there exists a unique map f' for which



if G^*, H^* are represented by $(G, m), (H, m')$ respectively, there exists uniquely a map $G^* \xrightarrow{f^*} H^*$ such that $|f^*| = f$.

Definition: The pair category P_1 is the category whose objects are pairs (G, m) where G is

discrete and $R \xrightarrow{m} G^n$ is a monomorphism; a morphism $(G, m) \xrightarrow{f} (H, m')$ is a map $G \xrightarrow{f} H$ for which there exists a unique map f' such that

$$\begin{array}{ccc}
 R & \xrightarrow{m} & G^n \\
 \downarrow f' & & \downarrow f^n \\
 R' & \xrightarrow{m'} & H^n
 \end{array}$$

commutes.

Meta-theorem I: Let \mathcal{A} be any model of R_1 and \mathcal{B} be the pair category constructed from \mathcal{A} . Then there exists a natural equivalence $\mathcal{A} \xrightarrow{T} \mathcal{B}$ with the property that H is discrete $\iff T(H) = (H, 0_H)$ where $0 \xrightarrow{0_H} H^n$; and $T(A)$ can be chosen to be $(n, [1_A])$.

Proof: Axioms 12 - 15 insure the existence of a natural equivalence. We need only show that the equivalence can be chosen with the two given properties.

Let G be discrete and (G, m) represent G . If $m \neq 0$ then $R \neq 0$ which implies R has elements; thus there exists x such that $1' \xrightarrow{x} R \xrightarrow{m} G^n$. Now as G is discrete there are no maps from $A \xrightarrow{f} H$ thus the following diagram commutes vacuously:

$$\begin{array}{ccc}
 & [1_A] & \rightarrow & |H|^n \\
 1' & \nearrow & & \uparrow 0 \\
 & \searrow u & & 0
 \end{array}$$

By axiom 12 there is a monomorphism m' from R to 0 ; thus we have $1' \xrightarrow{x} R \xrightarrow{m'} 0$, i.e. 0 has elements. Thus $R = 0$

and m must be the unique map $0 \xrightarrow{0_N} H^n$. The converse is similar.

Since A is determined by its definition up to isomorphism, and $|A| = n$ by definition, we need only that $(m, [1_A])$ satisfies the three defining properties of A .

(1) If $(m, [1_A]) \xrightarrow{f} (m, [1_A]) \Rightarrow f = 1_m$.

Let f be such a morphism, then we have the following commutative diagram:

$$\begin{array}{ccc}
 & [1_A] & \\
 1' & \xrightarrow{\quad} & m^n \\
 \downarrow 1' & & \downarrow f^n \\
 1' & \xrightarrow{\quad} & m^n
 \end{array}$$

As $1'$ is the only

map $1' \xrightarrow{1'} \cdot$. This implies that

$$\begin{array}{ccc}
 m & \xrightarrow{m} & m \\
 \downarrow m & & \downarrow f \\
 n & \xrightarrow{m} & n
 \end{array}$$

commutes i.e.

$$f = 1_m = [1_A]$$

(2) $| (m, [1_A]) | \neq 0$ is obvious as $m \neq 0$.

(3) $\exists f, | (m, [1_A]) | \xrightarrow{f} (G, m) \iff m \neq 0$.

Let (G, m) be an object in B . If $m = 0$, and $(m, [1_A]) \xrightarrow{f} (G, m)$, there exists f' such that

$$\begin{array}{ccc}
 1' & \xrightarrow{[1_A]} & n^n \\
 \downarrow f' & & \downarrow f \\
 0 & \longrightarrow & G^n
 \end{array}$$

commutes, but there

are no maps from $1'$ to 0 . Thus (G, m) discrete \Rightarrow there are no maps from $(n, [1_A])$ to (G, m) . If $m \neq 0$ and, $R \xrightarrow{m} G^n$, then R has elements. Let $1 \xrightarrow{x'} R \xrightarrow{m} G^n$. Then $n \xrightarrow{(x'm)} G$. It is claimed that $(x'm)$ is a morphism in \mathcal{B} from $(n, [1_A])$ to (G, m) . This will be true if the following diagram commutes:

$$\begin{array}{ccc}
 1' & \xrightarrow{[1_A]} & n^n \\
 \downarrow x' & & \downarrow (x'm)^n \\
 R & \xrightarrow{m} & G^n
 \end{array}$$

This is equivalent to the commutativity of

$$\begin{array}{ccc}
 & & n \\
 & \nearrow 1_m & \downarrow x'm \\
 n & \xrightarrow{x'm} & G
 \end{array}$$

which is trivial.

Meta-lemma I: In \mathcal{B} (as in meta-theorem I),
 $|(G, m)|_{\mathcal{B}} = (G, 0)$.

Proof: It has been shown that the discrete objects of B are those of the form (G, o) . It is clear that $(G, o) \xrightarrow{G} (G, m)$ is a bijection and the following diagram commutes

$$\begin{array}{ccc}
 (G, o) & \xrightarrow{G} & (G, m) \\
 \uparrow f' = f & \nearrow f & \\
 (H, o) & &
 \end{array}$$

; and f' is uniquely determined by f .

Meta-lemma 2: Let B, C be complete models of P_1 . Then there are functors $B^d \xrightarrow{Law} C^d$, and $C^d \xrightarrow{vere} B^d$ such that $Lawvere \cong 1$ and $vereLaw \cong 1$ where B^d and C^d are the full subcategories of discrete objects of B and C respectively.

Proof: This is a restatement of the principal result of [2].

Meta-theorem 2: Let B, C be complete models of R , let B', C' be pair categories constructed from B, C respectively. Let $|A|_B = n_1$ and $|A|_C = n_2$ then B' and C' are naturally equivalent iff $Law(n_1) \cong n_2$. If B' and C' are naturally equivalent the equivalence can be given by $Law'(G, m) = (Law(G), Law(m))$. Hence B and C are naturally equivalent $\Leftrightarrow n_1 \cong n_2$.

Proof: If $\Phi: B \rightarrow C$ is a natural equivalence

then $\Phi|_{B_d}$ is a natural equivalence from B^d to C^d

and thus there is a natural equivalence $\Phi' : B' \rightarrow C'$

and $\Phi'(H, 0) = |\Phi(H)|$, $\Phi'(|A|, 0) \cong \Phi'|A| \cong n_1$,

but $|A| = n_2$ thus $n_1 \cong n_2$.

Conversely, we have $Law : B^d \rightarrow C^d$ and define $Law'(G, m) = (Law(G), Law(m))$. The fact that Law' is an equivalence can be seen from the commutativity of the following diagram and its inverse.

($vere'$ is defined similarly to Law' .)

$$\begin{array}{ccccc}
 R \xrightarrow{m} G^n & \xrightarrow{\theta_G^n} & Lawvere(G)^n & \xrightarrow{\theta_n} & Lawvere(G)^{Lawvere(m)} \\
 \uparrow \theta_R^{-1} & \nearrow \theta_{G^n} & & & \uparrow \\
 R & & & & R \\
 \downarrow \theta_R & \searrow \theta_{G^n}^{-1} & & & \downarrow \\
 Lawvere(R) & \xrightarrow{Lawvere(m)} & Lawvere(G)^n & \xrightarrow{\begin{bmatrix} \theta_{G^n}^{-1} & \theta_{G^n} \end{bmatrix} \theta_n} & Lawvere(G)^{Lawvere(m)}
 \end{array}$$

where θ is the natural isomorphism given by the equivalence. If B' and C' are naturally equivalent then so are B and C .

Remark 1. If n is finite one can characterize $R(n)$ completely by changing the axiom 11 - (2) to $|A| = n \cdot ' = ' + ' + ' \dots ' (n \text{ times})$.

Remark 2. If $R(n)$ is the usual category of relational systems of type $\{m\}$, and B is any complete model of R in which $|A| = n$,

$Law'(G, m) = (\{x | ' \xrightarrow{x} |G|, m'\})$ where m' is defi-

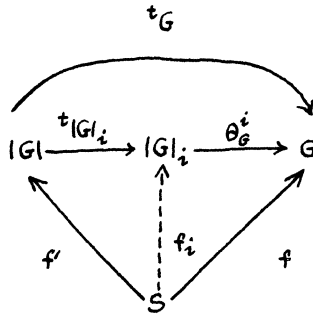
ned by $m'(x) = xm'$, $x \in G^n$.

The Characterization of R_I , I finite

Let $I = \{0, 1, \dots, n-1\}$. Then we have the following axiom system for R_I .

Axioms 1 - 10 are the same as for R .

Axiom 11. For each object G , there exists objects $|G|_0, |G|_1, \dots, |G|_n$, bijections $t_{|G|_i}, \theta_G^i$ such that for every discrete object S , and any map $S \xrightarrow{f} G$ there exists unique maps f_i such that



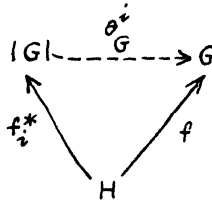
commutes where t, θ

are as in axiom 7.

Definition: G is an i -object $\iff \theta_i = 1$

Axiom 12. If $\theta_i = \theta_j$ $i \neq j$ then $\theta_i = \theta_j$ for all i, j and furthermore $t_G = 1_G$ i.e. G is discrete.

Axiom 14. For any i object H and any map $H \xrightarrow{f} G$ there exists uniquely a map f^* such that



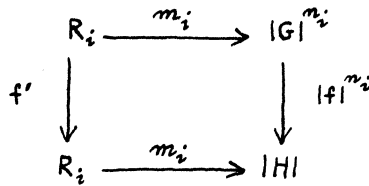
commutes; moreover

$$|f|' = |f_i^*|.$$

Axioms 15_i - 19_i. The i -objects satisfies axiom 11 - 14 of R .

Let G be any object, let $(|G|, m_i)$ represent $|G|_i$, $(||G|_i| = |G|)$ as in axiom 14_i. Thus we may say $(|G|, m_i)_{i < n}$ represents G . If $G \xrightarrow{f} H$ then we have $|G|_i \xrightarrow{|f|_i} |H|_i$, and $||f|_i| = |f|$,

thus we have the following commutative diagrams



where $m_i = |A_i|$ where A_i is the object A in the copy of R associated with i .

Axiom 20. $(|G|, m_i)_{i < n}$ represents G , and $(|G|, m'_i)_{i < n}$ represents G , then $m'_i = h_i m_i$ where the h_i are isomorphisms.

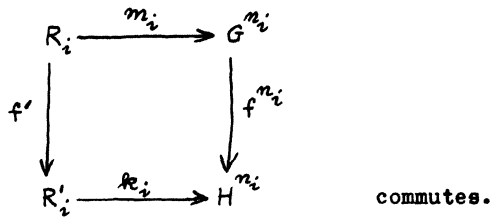
Axiom 21. If G is discrete and m_i are monomorphisms $R_i \xrightarrow{m_i} |G|^{n_i}$ then there exists uniquely up to isomorphism on object H , $|H| = G$ and $(G, m_i)_{i < n}$ represents H .

Axiom 22. If $(G, m_i)_{i < n}$, $(H, m'_i)_{i < n}$ represent G^* , H^* respectively, and furthermore if $G \xrightarrow{f} H$ such that $\exists ! f'_i$ for which

$$\begin{array}{ccc}
 R_i & \xrightarrow{m_i} & G^{n_i} \\
 \downarrow f'_i & & \downarrow f^{m_i} \\
 R'_i & \xrightarrow{m'_i} & H^{n_i}
 \end{array}$$

commutes then there exists f^* , $G^* \xrightarrow{f^*} H^*$, $|f^*| = f$; furthermore $(G, k_i^j)_{i < n}$, $(H, k_i^j)_{i < n}$ represent $|G^*|_j$ and $|H^*|_j$ respectively then f represents $|f^*|_j$.

Definition: The n -tuple category N_n is the category where objects are n -tuples $(G, m_i)_{i < n}$ where $R \xrightarrow{m_i} |G|^{n_i}$, $|A_i| = n_i$ and G is discrete and $(G, m_i)_{i < n} \xrightarrow{f} (H, k_i)_{i < n} \iff G \xrightarrow{f} H$ and for all i , $\exists f'$



Definition: $\sigma_i^j(G, m_i)_{i < n} = (G^{m'_i})_{i < n}$ where
 $m'_i = 0 \quad i \neq j \quad m'_j = m_j$.

Meta-theorem I': Let \mathcal{A} be any model of R_I and let \mathcal{B} be the n -tuple category constructed from \mathcal{A} . Then there exists a natural equivalence $T: \mathcal{A} \rightarrow \mathcal{B}$ such that $T(|G|_i) = \sigma_{h_i}^i(T(G))$,

$$T(A_i) = (n_i, m_i) \quad \text{where } m_i = 0 \quad i \neq j \\
 m_j = [|A_j|].$$

Meta-theorem 2': Let B, C be complete models of R_I . Let B', C' be the pair categories constructed from them. Let $|A_i|_B = n_i$ and $|A_i|_C = m_i$ then B' and C' are naturally equivalent iff there exists a function $f: I \rightarrow I$ one-one, onto such that $\text{Law}(n_i) \cong m_{f(i)}$. If B' and C' are naturally equivalent then the equivalence can be given by $\text{Law}'(G, n_i) \cong (\text{Law}(G), \text{Law } m_{f(i)})$.

Hence \mathcal{B} and \mathcal{C} are equivalent iff an f with the above property exists.

Remark. One can make the same remarks modulo the appropriate changes about \mathcal{R}_I that one can make about \mathcal{R} .

R e f e r e n c e s

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