Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 11 (1970), No. 2, 271--284

Persistent URL: http://dml.cz/dmlcz/105278

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11, 2 (1970)

FREDHOLM ALTERNATIVE FOR NONLINEAR OPERATORS IN BANACH SPACES AND ITS APPLICATIONS TO THE DIFFERENTIAL AND INTEGRAL EQUATIONS

(Preliminary Communication)

Svatopluk FUČÍK, Praha

1. Introduction. This communication deals with the solving of nonlinear operators' equations in Banach spaces and with the nonlinear generalization of the Fredholm alternative. There are obtained the theorems of the following type: If T is an operator (generally nonlinear) defined on a Banach space X with values in a Banach space Y, then $TX = Y$ provided that the equation $Tx = \theta_y$ has the solution $x = \theta_x$ only and X, Y, T satisfy some additional conditions.

Similar results were obtained by S.I. Pochožajev [15] for the real Banach spaces and for the homogeneous operators and, by J. Nečas [11], for the complex Banach spaces and for the operators being "near to homogeneous" ones. Both preceding papers discourse on the operators the domain of which is a Banach space and the range in its dual space X^* only. Hence, the integral operators defined on $L_p(\Omega)$ ($p \neq 2$) with values in $L_p(\Omega)$ are not concluded in the abstract theory established in [11] and [15]. Such a problem is solved in Section 7 on the base of Section 3.

Sections 4 and 5 deal with sufficient conditions under which an operator T and Banach spaces X, Y possess the assumptions of the main theorems in Section 3.

2. Definitions

Definition 1. Let $K > 0$ be a real number, X and Y be Banach spaces, $\{X_n\}$, $\{Y_n\}$ be two sequences of finite dimensional subspaces such that $X_n \subset X$, $Y_n \subset Y$. Let Q_n be a bounded linear mapping from Y into Y_n, $Q_n^2 = Q_n$ (i.e. a linear projection) for each positive integer n.

We shall say that the couple $\langle X, Y \rangle$ has an approximation scheme $[\{X_n\}, \{Y_n\}, \{Q_n\}]_K$ for the operators from X into Y (shortly speaking, $\langle X, Y \rangle$ has an approximation scheme $[\{X_n\}, \{Y_n\}, \{Q_n\}]_K$) if the following conditions are satisfied:

(1) $X_1 \subset X_2 \subset \dots \subset X_n \subset X_{n+1} \subset \dots$,

(2) $Y_1 \subset Y_2 \subset \dots \subset Y_n \subset Y_{n+1} \subset \dots$,

(3) $\overline{\bigcup_{n=1}^{\infty} X_n} = X$,

(4) $\dim X_n = \dim Y_n$ ($\dim$ = dimension)

(5) $\|Q_n\|_{(Y\to Y)} \leq K$, where $(Y \to Y)$ is the space of all bounded linear operators from Y into Y,

(6) $\lim_{n\to\infty} \|Q_n y - y\|_Y = 0$ for each $y \in Y$.

Definition 2. Let X and Y be two Banach spaces, $\langle X, Y \rangle$ has an approximation scheme $[\{X_n\}, \{Y_n\}, \{Q_n\}]_K$ and let $T: X \to Y$ be an operator, the domain of T is X and the range is in Y. Then T is said to be an A-operator with respect to a given approximation scheme $[\{X_n\}, \{Y_n\}, \{Q_n\}]_K$ (shortly speaking, T is an A-operator) if for any sequence $\{n_j\}$ of positive integers with $n_j \to \infty$ and a bounded sequence $\{x_{n_j}\}$ with each $x_{n_j} \in X_{n_j}$ such that $\lim_{n_j \to \infty} \| Q_{n_j} T x_{n_j} - y \|_Y = 0$ for some $y \in Y$, there exists an infinite subsequence $\{n_{j(k)}\}$ and $x \in X$ such that $Tx = y$ and $\lim_{n_{j(k)} \to \infty} \| x_{n_{j(k)}} - x \|_X = 0$.

The concept of an A-operator and approximation scheme is a slight variant of the conditions of S.I. Pochožajev [15], W,V. Petryshyn [12,13,14], F.E. Browder-W.V. Petryshyn [2] and D.G.de Figueiredo [5,6,7].

Definition 3. Let X and Y be two Banach spaces, $T: X \to Y$, $\beta(t)$ a real-valued strictly increasing and continuous function defined on $\langle 0, \infty)$ with $\beta(0) = 0$ and $\lim_{t \to \infty} \beta(t) = \infty$.

a) T is said to be $\beta(t)$-homogeneous if $T(tu) = \beta(t) Tu$ for each $t \geq 0$ and all $u \in X$.

b) T is said to be $\beta(t)$-quasihomogeneous with respect to T_0, if there exists an operator $T_0 : X \to Y$, T_0 is $\beta(t)$-homogeneous and if $t_n \searrow 0$ ($t_1 \geq t_2 \geq \ldots \geq t_n \geq t_{n+1} \geq \ldots > 0$ real numbers and $\lim_{n \to \infty} t_n = 0$), $u_n \rightharpoonup u_0$ ($\rightharpoonup$ denotes the weak convergence in X), $\beta(t_n)T(\frac{u_n}{t_n}) \to q \in Y$, then $T_0 u_0 = q$ ($\to$ denotes the strong convergence).

c) T is said to be $\beta(t)$-strongly quasihomogeneous with respect to T_0, if there exists an operator $T_0 : X \to Y$, T_0 is $\beta(t)$-homogeneous and $t_n \searrow 0$, $u_n \rightharpoonup u_0$ imply $\beta(t_n)T(\frac{u_n}{t_n}) \to T_0 u_0$.

If $S : X \to Y$ is $\beta(t)$-strongly quasihomogeneous with respect to S_0, then S_0 is strongly continuous (i.e. $x_n \rightharpoonup x_0$ implies $S_0 x_n \to S_0 x_0$). If $\beta(t) = t^{\alpha} (\alpha > 0)$ and $T : X \to Y$ is $\beta(t)$-homogeneous, then T is $\beta(t)$-quasihomogeneous with respect to T provided that T is strongly closed (i.e. $x_n \rightharpoonup x_0$, $Tx_n \to y$ imply $Tx_0 = y$). T is $\beta(t)$-strongly quasihomogeneous with respect to T provided that T is strongly continuous.

<u>Definition 4</u>. Let X and Y be two Banach spaces, $T_0 : X \to Y$, $S_0 : X \to Y$ $\beta(t)$-homogeneous operators and $\lambda \neq 0$ a real number.

λ is said to be an eigenvalue for the couple (T_0, S_0) if there exists $u_0 \in X$, $u_0 \neq \theta_X$ (θ_X is the zero element of X) such that $\lambda T_0 u_0 - S_0 u_0 = \theta_Y$.

3. <u>Main theorems</u>

<u>Theorem 1</u>. Let X and Y be two reflexive real Banach spaces, $\langle X, Y \rangle$ have an approximation scheme, $T : X \to Y$ be an odd ($T(-x) = -Tx$ for each $x \in X$) $\beta(t)$-quasihomogeneous with respect to T_0,

demicontinuous (i.e. $x_n \rightharpoonup x_o$ implies $Tx_n \rightharpoonup Tx_o$) A -operator. Let $S: X \to Y$ be an odd completely continuous (i.e. S is continuous and it transforms every bounded subset of X on a compact subset of Y) and $\beta(t)$ -strongly quasihomogeneous with respect to S_o . Suppose that there exists a constant $c >$ > 0 such that $\| Tu \|_Y \geq c \cdot [\beta^{-1}(\frac{1}{\|u\|_X})]^{-1}$ for each $u \in X$, $u \neq \theta_X$. Let $\lambda \neq 0$ be a real number so that λ is not an eigenvalue for the couple (T_o, S_o) .

Then the operator $\lambda T - S$ transforms X onto Y .

Theorem 2. Let X and Y be two real reflexive Banach spaces, $\langle X, Y \rangle$ have an approximation scheme. Let $T: X \to Y$ be an odd $\beta(t)$ -homogeneous and continuous A -operator. Let $S: X \to Y$ be a completely continuous odd and $\beta(t)$-homogeneous operator. Let $\lambda \neq 0$ be a real number such that λ is not an eigenvalue for the couple (T, S) .

Then the operator $\lambda T - S$ is from X onto Y.

4. Approximation scheme

Let X be a Banach space with a Schauder basis. Then the couple $\langle X, X \rangle$ has an approximation scheme. Moreover, if X is a reflexive and Y is a separable Banach space, then the couples $\langle X, X^* \rangle$ and $\langle Y, X \rangle$ have an approximation scheme.

If the couple $\langle X, X \rangle$ has an approximation scheme, then under some additional conditions the Banach space X has a Schauder basis (see [10]).

Hence a separable Hilbert space, $C[0,1]$, $L_p(\Omega)$, $C^k[0,1]$, $C^1([0,1]^N)$ (see [17]) have the approximation scheme (they have a Schauder basis).

Let I be an open interval on the real line, $W_p^{(k)}(I)$, $\overset{\circ}{W}{}_p^{(k)}(I)$ the Sobolev spaces. Then $W_p^{(k)}(I)$ and $\overset{\circ}{W}{}_p^{(1)}(I)$ have a Schauder basis for $k \geq 1$ and $p \geq 1$ (k is an integer, p is a real number). I did not succeed to prove the Sobolev space $\overset{\circ}{W}{}_p^{(k)}(\Omega)$ has a Schauder basis.

5. A-operators

Theorem 3. Let X be a reflexive Banach space, Y a Banach space, $T: X \to Y$, $S: X \to Y$, $f: X \to E_1$, $\Phi: X \to Y^*$. Let $\langle X, Y\rangle$ have an approximation scheme $[\{X_n\}, \{Y_n\}, \{Q_n\}]_K$. Let S be a completely continuous operator, f a weakly upper semicontinuous functional, $f(\theta_X) = 0$ and let Φ be a weakly continuous operator (i.e. $x_n \rightharpoonup x_0$ implies $\Phi x_n \rightharpoonup \Phi x_0$), $\Phi(\theta_X) = \theta_{Y^*}$. Suppose that φ, γ, μ are continuous real-valued functions on $\langle 0, \infty)$ such that $\gamma(t_0) = 0$ implies $t_0 = 0$, $\gamma(t) \geq 0$, $\mu(t) > 0$ for each $t > 0$, $\varphi(t)$ is a strictly increasing function. Let $Q_n^* \Phi(x) = \Phi(x)$ for each positive integer n and all $x \in X_n$.

Then T is an A-operator provided that one of the following conditions is satisfied:

(5.1) T is continuous and

$$(\Phi(x-y), Tx - Ty) + f(x-y) \geq \gamma(\|x-y\|_X)$$

for each $x, y \in X$, where $(\cdot\,,\cdot)$ is the pairing between Y and Y^*.

(5.2) T is continuous and

$$(\Phi(x-y), Tx-Ty) + (\Phi(x-y), Sx-Sy) + f(x-y) \geq \gamma(\|x-y\|_X)$$

for each $x, y \in X$.

(5.3) T is demicontinuous, Φ is $\mu(t)$-homogeneous, $\overline{\Phi(X)} = Y^*$ and

$(\Phi(x-y), Tx - Ty) \geq \gamma(\|x-y\|_X)$ for each $x, y \in X$.

(5.4) T is demicontinuous, Φ is $\mu(t)$-homogeneous, $\overline{\Phi(X)} = Y^*$ and

$$(\Phi(x-y), Tx-Ty) + (\Phi(x-y), Sx - Sy) \geq \gamma(\|x-y\|_X)$$

for each $x, y \in X$.

(5.5) X has Property (H) (i.e. (1) $x_n \rightharpoonup x_o$, $\|x_n\|_X \longrightarrow \|x_o\|_X$ imply $x_n \longrightarrow x_o$ and (2) X is strictly convex), T is demicontinuous, Φ is $\mu(t)$-homogeneous, $\overline{\Phi(X)} = Y^*$ and

$$(\Phi(x-y), Tx-Ty) \geq (\varphi(\|x\|_X) - \varphi(\|y\|_X))(\|x\|_X - \|y\|_X)$$

for each $x, y \in X$.

(5.6) X has Property (H), T is demicontinuous, Φ is $\mu(t)$-homogeneous, $\overline{\Phi(X)} = Y^*$ and

$$(\Phi(x-y), Tx-Ty) + (\Phi(x-y), Sx - Sy) \geq$$
$$\geq (\gamma(\|x\|_X) - \gamma(\|y\|_X))(\|x\|_X - \|y\|_X)$$

for each $x, y \in X$.

Remarks. $L_p(\Omega)$, $\ell_p (p > 1)$ have Property (H). A Hilbert space has Property (H), too.

Let the approximation scheme $[\{X_n\}, \{Y_n\}, \{Q_n\}]_K$ of the couple $\langle X, Y\rangle$ have the following property: $Q_n y = y$ for each positive integer n and all $y \in Y_n$. Set $Y = X^*$ and Φ = the identity operator on X. Then Φ satisfies the assumptions of Theorem 3.

Let X be a Banach space with a weakly continuous duality mapping (see [5,6,7]), for example $\ell_p (1 < p < \infty)$. Set $Y = X$ and Φ = the duality mapping. Then Φ satisfies the assumptions of Theorem 3.

Theorem 4. Let X be a Banach space, $[\{X_n\}, \{X_n\}, \{Q_n\}]_K$ an approximation scheme for $\langle X, X\rangle$, $T: X \to X$, $T = I - S$, where I is the identity operator and S is a contraction mapping with a constant $0 \le \alpha < 1$ (i.e. $\| Sx - Sy \|_X \le \alpha \| x - y \|_X$). Let $\alpha K < 1$.

Then T is an A-operator.

Theorem 5. Let X, S, K, α satisfy the assumptions of Theorem 4. Let $U: X \to X$ be a completely continuous operator and X a reflexive space. Set $T = I - S - U$.

Then T is an A-operator.

6. **The set of eigenvalues**

Lemma 1 ([16]). Let X be a separable and reflexive Banach space, $G \subset X$ an open subset,

$f : G \rightarrow E_1$ a functional of the class C^m (i.e. there exists the Fréchet derivative $D^j f(x)$ continuous at x up to the order m - see [18]). Let the following conditions be satisfied:

(6.1) $m \geq \max(\ell, 2)$,

(6.2) $\sup_{x \in G} \dim \operatorname{Ker} D^2 f(x) = \ell < \infty$, where $\operatorname{Ker} D^2 f(x) = \{h; h \in X, (D^2 f(x), h, w) = 0$ for each $w \in X\}$.

(6.3) $D^2 f(x)(X)$ is a closed subset of the space $(X \rightarrow X^*)$ for each $x \in X$.

Set $M = \{x; x \in G, Df(x, h) = 0$ for each $h \in X\}$.

Then $\operatorname{meas} f(M) = 0$ (meas = the Lebesgue measure).

Theorem 6. Let X be a reflexive Banach space such that $\langle X, X^* \rangle$ has an approximation scheme. Let $T: X \rightarrow X^*$ be an odd A-operator and $\beta(t)$-quasihomogeneous with respect to T_0. Let $S: X \rightarrow X^*$ be an odd completely continuous operator and $\beta(t)$-strongly quasihomogeneous with respect to S_0. Suppose that there exists $c > 0$ such that $\| Tu \|_{X^*} > c \cdot [\beta^{-1}(\frac{1}{\|u\|_X})]^{-1}$

and $(T_0 u, u) \geq c \cdot \| u \|_X \cdot [\beta^{-1}(\frac{1}{\| u \|_X})]^{-1}$ for each $u \in X, u \neq \theta_X$.

Let $T_0 = \operatorname{grad} f$, $S_0 = \operatorname{grad} g$ (for the definition see [18]). Set $\varphi(x) = \frac{g(x)}{f(x)}$ for $x \neq \theta_X$ and suppose that the functional φ satisfies the assumptions of Lemma 1 on some neighborhood of the unit sphere in X.

Then there exists a set $N \subset E$, $meas\, N = 0$ such that $(\lambda T - S)X = X^*$ for each $\lambda \in E_1 - N$.

Theorem 7. Let X and Y be two reflexive Banach spaces such that $\langle X, Y \rangle$ has an approximation scheme. Let $T: X \longrightarrow Y$ be a t-quasihomogeneous with respect to T_0, demicontinuous and odd A-operator. Let $S: X \rightarrow Y$ be a t-strongly quasihomogeneous with respect to S_0 completely continuous operator. Suppose that there exists a constant $c > 0$ such that $\| Tu \|_Y \geq c \| u \|_X$ and $\| T_0 u \|_Y \geq c \| u \|_X$ for each $u \in X$. Let T_0 and S_0 be linear operators, $T_0 X = Y$.

Then there exists a set $N \subset E_1$, N is at most denumerable and if N has a limit point λ, then $\lambda = 0$ and N is such that $(\lambda T - S)X = Y$ for each $\lambda \in E_1 - N$.

7. Applications

a) Let Ω be a bounded open subset of E_N. Denote Δ the Laplace operator. Find the weak solution u of the Dirichlet problem

$$\begin{cases} -\lambda \Delta u - u \cdot \dfrac{|u|^s}{1 + |u|^s} = f \quad (s \geq 0, \lambda \neq 0) \\ u = 0 \quad \text{on } \partial\Omega , \end{cases}$$

i.e. let $\overset{\circ}{W}{}_2^{(1)}(\Omega)$ be a Sobolev space ($\overset{\circ}{W}{}_2^{(1)}(\Omega)$

is a Hilbert space) and $f \in (\overset{\circ}{W}{}_2^{(1)}(\Omega))^*$.
We seek $u \in \overset{\circ}{W}{}_2^{(1)}(\Omega)$ such that

$$\lambda \int_\Omega \sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx - \int_\Omega \frac{|u|^s}{1+|u|^s} u \cdot v \cdot dx = \int_\Omega f \cdot v \cdot dx$$

for each $v \in \overset{\circ}{W}{}_2^{(1)}(\Omega)$.

This equation has a solution for each $f \in (\overset{\circ}{W}{}_2^{(1)}(\Omega))^*$ provided that the equation

$$\lambda \int_\Omega \sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx - \int_\Omega u \cdot v \cdot dx = 0$$

for each $v \in \overset{\circ}{W}{}_2^{(1)}(\Omega)$ has the trivial solution only (see Theorem 1), i.e. for $\lambda \neq \frac{1}{\lambda_K}$, where $\{\lambda_K\}$ is a spectrum for the Dirichlet problem and the equation $-\Delta u - \lambda u = 0$.

To apply the main theorems to the more general partial differential equations one would have to prove that the Sobolev spaces $\overset{\circ}{W}{}_p^{(k)}(\Omega)$ have a Schauder basis (or an approximation scheme) for $p \neq 2$ (see Section 4).

b) Let $A: L_p(\Omega) \longrightarrow L_p(\Omega)$ $(p > 1)$ be a linear bounded operator with the norm $\|A\|$. Let $f \in L_\infty(\Omega)$ with the norm $\|f\|_\infty$ in $L_\infty(\Omega)$ and let $\alpha = \|A\| \cdot \|f\|_\infty \cdot \frac{9}{8} < 1$. Suppose that $K(x,y)$, $L(x,y)$ are continuous functions on $\overline{\Omega} \times \overline{\Omega}$ and $s \geq 0$. Set

$$Su = \frac{|\int_\Omega L(x,y)u(y)dy|^s}{1+|\int_\Omega L(x,y)u(y)dy|^s} \cdot \int_\Omega K(x,y)u(y)dy .$$

By Theorem 1 the equation

$$(1)\quad \lambda(u - A(f\frac{u^3}{1+u^2})) - Su = F \qquad (\lambda \neq 0)$$

has a solution $u \in L_p(\Omega)$ for an arbitrary $F \in L_p(\Omega)$ provided that the equation

$$(2)\quad \lambda(u - A(fu)) - \int_\Omega K(x,y)u(y)\,dy = 0$$

has a trivial solution only.

By Theorem 7 there exists a set $N \subset E_1$, N is at most denumerable and if λ is a limit point of N, then $\lambda = 0$ and N is such that (1) has a solution $u \in L_p(\Omega)$ for each $\lambda \in E_1 - N$ and all $F \in L_p(\Omega)$.

The proofs will appear later elsewhere.

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(Oblatum 13.1.1970)