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FREDHOLM ALTERNATIVE FOR NONLINEAR OPERATORS

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Introduction. Let X be a reflexive Banach space (real or complex), let A, T, S be nonlinear mappings of X into its dual X^*. This paper deals with the solution of the equations

$$A(u) = h, \quad T(u) - \lambda S(u) = h ,$$

where λ is a number (real or complex), $h \in X^*$. There is given a generalization of the results of J.Nečas [1] and S.I.Pochožajev [2], first of all the proofs of Fredholm alternative for nonlinear positive $\varkappa - *$ quasihomogeneous and strongly positive $\varkappa - *$ quasihomogeneous mappings (Theorems 5.3, 6.4). All the main results are contained in Paragraphs V. and VI. In IV, there are defined mappings with Properties (B) and (B′) and the fundamental assertions about these mappings are proved. These mappings are examined in the papers of F.E.Browder and in [1],[3], too. The basis of Paragraph V is [2]. But S.I.Pochožajev works only with a separable reflexive Banach space with Schauder basis and he supposes that T, S are positive $\varkappa$ -homogeneous mappings satisfying other assumptions than those given in this paper. The foundation of Paragraph VI is [1], where analogical theorems as here are proved. The difference is in the assumption about the mappings T and S : in [1] it is assumed that S is strongly continuous and T has Property (B′); here we suppose that T has Property (B) and S is only completely continuous.

1. Terminology and notations

Let X be a Banach space (real or complex). Then X^* denotes its dual (in complex case its antidual - see [4]), Λ denotes the system of all finite-dimensional subspaces of the space X whose dimension is larger than 1 . We suppose that the space X is infinite dimensional, hence the system Λ is nonempty. The pairing between $f \in X^*$ and $u \in X$ is denoted by (f,u). Let $F \in \Lambda$. For $g \in F^*$ we denote by $\|g\|_F$ the norm of g in the space F^* . The pairing between $g \in F^*$ and $v \in F$ is denoted by $(g, v)_F$. If $f \in X^*$, then we define the functional $f_F \in F^*$ by the formula $(f_F, v)_F = (f,v)$ for all $v \in F$. If A is a mapping of X into X^*, then we define the mapping A_F of F into F^* : for each $u \in F$ let $A_F(u) \in F^*$, $(A_F(u),v)_F = (A(u),v)$ for all $v \in F$. For $M \subset X$ the symbol $A(M)$ denotes the image of M under the mapping A . Further, we use the following notations: $D_R = \{u \in X ; \|u\| < R\}$, $S_R = \{u \in X; \|u\| = R\}$ for $R > 0$; if $M \subset X$, then $\overline{M}$ (resp. $\overline{M}^w$) is the closure of M in the strong (resp. weak) topology. The symbols $\longrightarrow$, $\rightharpoonup$ denote the strong and weak convergences. Let E_N (resp. C_N) be the real (resp. complex) N -dimensional Euclidean space. For $x = (x_1, \dots, x_N)$, $y = (y_1, \dots, y_N) \in E_N$ (resp. $\in C_N$) let $(x, y)_N = \sum_{i=1}^{N} x_i \bar{y}_i$, $\|x\|_N = (x, x)^{1/2}$.

Definition 1.1. Let X be a topological space, $M \subset X$. Then M is said to be compact, if each open covering of M contains a finite covering. M is said to be sequentially compact, if each sequence $\{u_n\} \subset M$ contains a subsequence which is convergent in X.

Definition 1.2. Let X be a topological space, $\mathfrak{M}$ a system of subsets of X. Then $\mathfrak{M}$ is said to be a filter, if for each its finite subsystem $\mathfrak{M}_0$ there is $\bigcap_{F \in \mathfrak{M}_0} F \neq \emptyset$.

Definition 1.3. Let X, Y be two Banach spaces; let A be a mapping of X into Y. Then A is said to be

(1) continuous if $u_n \to u_0$ in X implies $A(u_n) \to A(u_0)$ in Y;

(2) demicontinuous if $u_n \to u_0$ in X implies $A(u_n) \rightharpoonup A(u_0)$ in Y;

(3) hemicontinuous if $u, v \in X$, $t_n > 0$, $t_n \to 0$ implies $A(u + t_n v) \rightharpoonup A(u)$ in Y.

(4) bounded if for each bounded subset M of X the set $A(M)$ is bounded

(5) completely continuous if for each bounded subset M of X the set $A(M)$ is sequentially compact and A is continuous;

(6) closed if $u_n \to u_0$ in X, $A(u_n) \to f$ in Y implies $f = A(u_0)$;

(7) strongly closed if $u_n \rightharpoonup u_0$ in X, $A(u_n) \to f$ in Y implies $f = A(u_0)$;

(8) odd if $A(-u) = -A(u)$ for all $u \in X$.

Definition 1.4. Let X, Y be two Banach spaces; let A be a mapping of X into Y, $\varkappa > 0$. Then A is said to be

(9) positive $\varkappa$-homogeneous if $A(tu) = t^{\varkappa} A(u)$ for all $t > 0$, $u \in X$;

(10) positive $\varkappa - *$-quasihomogeneous if there exists a positive $\varkappa$-homogeneous mapping A_0 of X into Y such that

$\kappa_n > 0, \kappa_n \to 0, u_n \rightharpoonup u_0$ in X, $\kappa_n^{\varkappa} A\left(\frac{u_n}{\kappa_n}\right) \to f$ in Y

implies $A_0(u_0) = f$, $u_n \to u_0$;

(11) strongly positive $\varkappa - *$-quasihomogeneous if there exists a mapping A_0 of X into Y such that

$\kappa_n > 0, \kappa_n \to 0, u_n \rightharpoonup u_0$ in X implies there exist subsequences $\{\kappa_n'\}$, $\{u_n'\}$ of the sequences $\{\kappa_n\}$, $\{u_n\}$ such that $\kappa_n' = \kappa_k$ if and only if $u_n' = u_k$ and $\kappa_n'^{\varkappa} A\left(\frac{u_n'}{\kappa_n'}\right) \to f$, where $f \in Y$; if, in addition, $u_n \to$ $\to u_0$, then $f = A_0(u_0)$.

Definition 1.5. Let X be a Banach space, let A be a mapping of X into X^*. Then A is said to be

(12) coercitive if $\lim\limits_{\|u\| \to +\infty} \dfrac{|(A(u), u)|}{\|u\|} = +\infty$;

(13) regular surjective if the following two conditions are fulfilled:

(i) $A(X) = X^*$;

(ii) for each $R > 0$ there exists $\kappa > 0$ such that $f \in X^*$, $\|f\| \leq R$, $A(u) = f$, $u \in X$ implies $\|u\| \leq \kappa$.

Definition 1.6. Let X be a real (resp. complex) Banach space, T and S positive $\varkappa$-homogeneous mappings of X into X^*, where $\varkappa > 0$. Let λ be a real (resp. complex) number. Then λ is said to be an eigenvalue of T, S if there exists $u \in X$ such that $u \neq 0$, $T(u) - \lambda S(u) = 0$.

2. Local degree

Let G be an open and bounded subset of E_N, ∂G its boundary. Suppose that f is a continuous mapping of $\overline{G}$ into E_N, $x_0 \in E_N$, $x_0 \notin f(\partial G)$. We shall denote by $deg(f, G, x_0)$ the local degree of the mapping f. The degree has (see [7]) these properties:

(14) if $deg(f, G, x_0) \neq 0$, then there exists $z_0 \in G$ such that $f(z_0) = x_0$;

(15) if $\mathcal{F}$ is a continuous mapping of $\overline{G} \times \langle 0, 1 \rangle$ into E_N, $\mathcal{F}(x,t) \neq x_0$ for each $x \in \partial G$, $t \in \langle 0, 1 \rangle$, then $deg(\mathcal{F}(x,0), G, x_0) = deg(\mathcal{F}(x,1), G, x_0)$.

Theorem 2.1 ([8]). Let $R > 0$, $N \geq 2$, $G = \{x \in E_N ; \|x\| < R\}$. Suppose f is a continuous mapping of $\overline{G}$ into E_N, $f(x) \neq 0$ and $\frac{f(x)}{\|f(x)\|_N} \neq \frac{f(-x)}{\|f(-x)\|_N}$ for all $x \in \partial G$. Then $deg(f, G, 0)$ is an odd number.

3. Approximation of positive $\varkappa$-homogeneous completely continuous mappings

Theorem 3.1. Let X, Y be two Banach spaces, $\varkappa > 0$. Suppose A is a completely continuous and positive $\varkappa$-homogeneous mapping of X into Y, let $\varepsilon > 0$.

Then there exists a completely continuous and positive $\varkappa$-homogeneous mapping B of X into a finite dimensional subspace of the space Y such that $\|A(u) - B(u)\| \leq \varepsilon \|u\|^{\varkappa}$ for all $u \in X$. If, in addition, A is odd, then we can take B odd, too.

Proof. Let the mapping A be odd (otherwise see [1]). The mapping A is completely continuous, therefore $A(S_1)$ is a sequentially compact set ($S_1 = \{u \in X; \|u\| = 1\}$). Hence there exists a finite ε-net of $A(S_1)$. Let $y_1, \dots, y_p$ be this ε-net. For $i = 1, \dots, p$ define

$$r_i(u) = \begin{cases} \varepsilon - \|A(u) - y_i\| & \text{if } u \in S_1,\ \|A(u) - y_i\| < \varepsilon \\ 0 & \text{for the other } u \in S_1, \end{cases}$$

(16)

$$s_i(u) = \begin{cases} \varepsilon - \|A(u) + y_i\| & \text{if } u \in S,\ \|A(u) + y_i\| < \varepsilon \\ 0 & \text{for the other } u \in S_1. \end{cases}$$

For each $u \in S_1$ we have $\sum_{i=1}^{p} r_i(u) > 0$, $\sum_{i=1}^{p} s_i(u) > 0$ and we can define (for $u \in S_1$)

$$P(u) = \frac{\sum_{i=1}^{p} r_i(u) y_i}{2 \cdot \sum_{i=1}^{p} r_i(u)} - \frac{\sum_{i=1}^{p} s_i(u) y_i}{2 \cdot \sum_{i=1}^{p} s_i(u)}$$

and

$$B(u) = \|u\|^{\varkappa} \cdot P\left(\frac{u}{\|u\|}\right) \quad \text{for all } u \in X.$$

We obtain

$$\|A(u) - B(u)\| \leq \left(\frac{\sum_{i=1}^{p} r_i\left(\frac{u}{\|u\|}\right) \cdot \left\|A\left(\frac{u}{\|u\|}\right) - y_i\right\|}{2 \cdot \sum_{i=1}^{p} r_i\left(\frac{u}{\|u\|}\right)} + \frac{\sum_{i=1}^{p} s_i\left(\frac{u}{\|u\|}\right) \cdot \left\|A\left(\frac{u}{\|u\|}\right) + y_i\right\|}{2 \cdot \sum_{i=1}^{p} s_i\left(\frac{u}{\|u\|}\right)} \right) \cdot \|u\|^{\varkappa} \leq \varepsilon \cdot \|u\|^{\varkappa}.$$

The mapping A is odd, therefore $s_i(-u) = s_i(u)$.
Hence B is odd, too.

4. Mappings with Properties (B) and (B').

Definition 4.1. Let X be a reflexive Banach space, let A be a mapping of X into X^* . Then A is said to have Property (B), if there exists a mapping $\bar{A}$ of $X \times X$ into X^* such that the following conditions are valid:

(a) the restriction of A on any finite dimensional subspace of X is a demicontinuous mapping;

(b) $\bar{A}$ is bounded, for each $u \in X$ the mapping $\bar{A}(\cdot, u)$ is hemicontinuous on X and $\bar{A}(u,u) = A(u)$;

(c) $\operatorname{Re}(\bar{A}(u,u) - \bar{A}(v,u), u - v) \geq 0$ for each $u, v \in X$;

(d) $u_n \rightharpoonup u$ in X , $(\bar{A}(u_n,u_n) - \bar{A}(u,u_n), u_n - u) \to 0$ implies $\bar{A}(v,u_n) \rightharpoonup \bar{A}(v,u)$ for each $v \in X$ and $u_n \to u$;

(e) $u_n \rightharpoonup u$ in X, $v \in X$, $w^* \in X^*$, $\bar{A}(v,u_n) \rightharpoonup w^*$ in X^* implies $(\bar{A}(v,u_n), u_n) \to (w^*, u)$.

Remark 4.1. Let $\kappa(t)$ be a real-valued non-negative continuous function defined in the interval $(0, +\infty)$ such that $t_n > 0$, $\kappa(t_n) \to 0$ implies $t_n \to 0$. Suppose that there exists a mapping $\bar{A}$ of $X \times X$ into X^* such that (a),(b),(e) of Definition 4.1 and the following two conditions are valid:

(c') $\operatorname{Re}(\bar{A}(u,u) - \bar{A}(v,u), u - v) \geq \kappa(\|u - v\|)$ for each $u, v \in X$;

(d') $u_n \rightharpoonup u$ in X, $(\bar{A}(u_n,u_n) - \bar{A}(u,u_n), u_n - u) \to 0$ implies $\bar{A}(v,u_n) \rightharpoonup \bar{A}(v,u)$ for each $v \in X$.

Then the mapping A has Property (B).

Definition 4.2([1]) Let X be a reflexive Banach space, let A be a mapping of X into X^*. Then A is said to have Property (B′) if there exists a mapping $\bar{A}$ of $X \times X$ into X^* such that Conditions (a),(b),(c),(e) of Definition 4.1 and Condition (d′) of Remark 4.1 are fulfilled.

Lemma 4.1. Let X be a Banach space, let A be a mapping of X into X^* satisfying Condition (a) of Definition 4.1. Then for each $F \in \Lambda$ the mapping A_F (see 1.) is continuous.

Lemma 4.2. Let X be a reflexive Banach space, let A be a mapping of X into X^* with Property (B′). Let $u_0 \in X$, $h \in X^*$. Assume that for each $F \in \Lambda$ there exists a sequence $\{u_n\} \subset X$ (dependent on F) and a number $t_F \in \langle 0, 1 \rangle$ such that

$u_n \rightarrow u_0$, $(A(u_n), u_n) \rightarrow t_F(h, u_0)$, $(A(u_n), v) \rightarrow t_F(h, v)$

for all $v \in F$.

Then there exists $t_0 \in \langle 0, 1 \rangle$ such that $A(u_0) = t_0 h$.

Proof. Let $F \in \Lambda$ be arbitrary (but fixed) such that $u_0 \in F$, let $\{u_n\}$, t_F be the sequence and the number of the assumptions. The mapping $\bar{A}$ is bounded, therefore by Eberlein-Smuljan's Theorem there exists a subsequence $\{u_n'\}$ such that $\bar{A}(u_0, u_n') \rightharpoonup u^*$, where $u^* \in X^*$. By (e) we have

$$(17) \qquad (\bar{A}(u_n', u_n') - \bar{A}(u_0, u_n'),\ u_n' - u_0) \rightarrow$$
$$\rightarrow t_F(h, u_0) - (u^*, u_0) - t_F(h, u_0) + (u^*, u_0) = 0,$$

(d′) implies $\bar{A}(v, u_n') \rightharpoonup \bar{A}(v, u_0)$ for each $v \in F$ and by using (e) we obtain

(18) $(\bar{A}(u_n', u_n') - \bar{A}(v, u_n'), u_n' - v) \to (t_F h - \bar{A}(v, u_0), u_0 - v)$
for each $v \in F$.

The real part of the left side in (18) is non-negative by (c), hence $\mathrm{Re}\,(t_F h - \bar{A}(v, u_0), u_0 - v) \geq 0$ for each $v \in F$. If we write $v = u_0 - \lambda w$, where $\lambda > 0$, $w \in F$, then

(19) $\mathrm{Re}\,(t_F h - \bar{A}(u_0 - \lambda w, u_0), w) \geq 0$.

Moreover, by (b) we obtain (19) for $\lambda = 0$, too. That means $\mathrm{Re}\,(t_F h - A(u_0), w) \geq 0$ for each $w \in F$. In this inequality, we can write $(-w)$ or (iw) (in the complex case) instead of w , hence

(20) $(t_F h - A(u_0), w) = 0$ for each $w \in F$.

We can suppose $A(u_0) \neq 0$, because for $A(u_0) = 0$ the assertion of Lemma 4.2 is clear. Let $(A(u_0), w_0) \neq 0$, $w_0 \in X$ and assume $w_0 \in F$. It follows from (20) that $(h, w_0) \neq 0$ and $t_F = t_0$, where $t_0 = \dfrac{(A(u_0), w_0)}{(h, w_0)}$,
hence t_0 is independent of F . All preceding considerations are valid for each $F \in \Lambda$ such that $u_0, w_0 \in F$. But $\bigcup\limits_{\substack{F \in \Lambda \\ u_0, w_0 \in F}} F = X$, therefore $(t_0 h - A(u_0), w) = 0$ for each $w \in X$, that means $A(u_0) = t_0 h$. This concludes the proof.

Lemma 4.3. Let the assumptions of Lemma 4.2 be fulfilled, let A have Property (B). Then for each $F \in \Lambda$ and for the sequence $\{u_n\}$ from Lemma 4.2 we have $u_n \to u_0$.

Proof. By (17) in the proof of Lemma 4.2 and (d´) we obtain $u'_n \to u_o$, where $\{u'_n\}$ is a subsequence of $\{u_n\}$, $\bar{A}(u_o, u'_n) \rightharpoonup u^*$. Suppose, on the contrary, that there exist a subsequence $\{u''_n\}$ and a number $\varepsilon > 0$ such that $\|u''_n - u_o\| > \varepsilon$. By Eberlein-Smuljan´s Theorem we can suppose $\bar{A}(u_o, u''_n) \rightharpoonup w^*$, $w^* \in X^*$. Analogously as in (17) we obtain $(\bar{A}(u''_n, u''_n) - \bar{A}(u_o, u''_n), u''_n - u_o) \to 0$ and by (d) $u''_n \to u_o$. This is a contradiction, hence $u_n \to u_o$.

<u>Remark 4.2.</u> If $t_F = 1$ for each $F \in \Lambda$ in the assumption of Lemma 4.2 or 4.3, then $t_o = 1$. It follows from the proof of Lemma 4.2.

<u>Lemma 4.4.</u> Let X be a reflexive Banach space, let A be a mapping of X into X^* with Property (B´). Then A is a strongly closed mapping.

<u>Proof</u>. Let $v_n \rightharpoonup u_o$ in X, $A(v_n) \to h$ in X^*. Define for each $F \in \Lambda$ a sequence $\{u_n\}$ and a number t_F so: $u_n = v_n$, $t_F = 1$. Then the assumptions of Lemma 4.2 are fulfilled and, by Remark 4.2, we obtain $A(u_o) = h$.

<u>Lemma 4.5</u>. Let X be a reflexive Banach space, let A be a mapping of X into X^* with Property (B). Suppose $v_n \rightharpoonup u_o$ in X, $A(v_n) \to h$. Then $A(u_o) = h$, $v_n \to u_o$.

<u>Remark 4.3</u>. Let X be a reflexive Banach space, $\varkappa > 0$, let A be a positive $\varkappa$-homogeneous mapping of X into X^*. If A has Property (B), then A is positive $\varkappa - *$-quasihomogeneous. If A is completely continuous, then A is strongly positive $\varkappa - *$-quasihomogeneous.

Lemma 4.6. Let X be a real (resp. complex) reflexive Banach space, let T, S be mappings of X into X^*. Suppose T has Property (B) and S is completely continuous. Then for each real (resp. complex) number λ the mapping $A_\lambda = T - \lambda S$ has Property (B), where $\bar{A}_\lambda(u, v) = \bar{T}(u, v) - \lambda S(v)$.

5. Fredholm alternative for odd mappings

Lemma 5.1. Let X be a reflexive Banach space, let A be a mapping of X into X^* with Property (B). Let $R > 0$, $h \in X^*$. Suppose $A(u) \neq t h$ for each $t \in \langle 0, 1 \rangle$, $u \in S_R$. Then there exists $F_0 \in \Lambda$ such that $F \in \Lambda$, $F \supset F_0$ implies $A_F(u) \neq t h_F$ for each $t \in \langle 0, 1 \rangle$, $u \in S_R \cap F$.

Proof. Assume that our assertion is not true. Then for each $F \in \Lambda$ the set

$N_F = \{u \in S_R \cap F';\ A_{F'}(u) = t h_{F'},\ F' \in \Lambda,\ F' \supset F,\ t \in \langle 0, 1 \rangle\}$

is non-empty. Let us prove that the system $\{\bar{N}_F^w\}_{F \in \Lambda}$ is a filter: let Λ_0 be an arbitrary finite subsystem of the system Λ, let F_1 be the linear hull of the set $\bigcup_{F \in \Lambda_0} F$; then $F_1 \in \Lambda$, $N_{F_1} \neq \emptyset$, $N_{F_1} \subset N_F$ for each $F \in \Lambda_0$, hence $\{\bar{N}_F^w\}_{F \in \Lambda}$ is a filter. The sets $\bar{N}_F^w$ are weakly closed; $\bar{N}_F^w \subset \bar{D}_R$, where $\bar{D}_R$ is weakly compact (see [4],p.200). Therefore there exists $u_0 \in \bigcap_{F \in \Lambda} \bar{N}_F^w$. By Eberlein-Smuljan's Theorem the sets $\bar{N}_F^w$ are weakly sequentially compact; hence for each $F \in \Lambda$ there exists a sequence $\{u_n\} \subset N_F$ (dependent of F) such that $u_n \rightarrow u_0$

(see [6],p.52). By definition N_F there exist sequences $\{F_n\} \subset \Lambda$, $\{t_n\} \subset \langle 0, 1\rangle$ (dependent of F) such that $u_n \in F_n$, $F_n \supset F$, $A_{F_n}(u_n) = t_n h_{F_n}$. The set $\langle 0, 1\rangle$ is compact, therefore we ean assume $t_n \to t_F$, $t_F \in \langle 0, 1\rangle$. We obtain

$$(A(u_n), u_n) = (A_{F_n}(u_n), u_n)_{F_n} = \\ = t_n(h_{F_n}, u_n)_{F_n} = t_n(h, u_n) \to t_F(h, u_0),$$

$$(A(u_n), v) = (A_{F_n}(u_n), v)_{F_n} = \\ = t_n(h_{F_n}, v)_{F_n} = t_n(h, v) \to t_F(h, v)$$

for each $v \in F$, hence the assumptions of Lemmas 4.2 and 4.3 are fulfilled. Thus, there exists $t_0 \in \langle 0, 1\rangle$ such that $A(u_0) = t_0 h$, $u_n \longrightarrow u_0$. Hence $u_0 \in S_R$ and we have obtained a contradiction. This concludes the proof.

Lemma 5.2. Let X be a real reflexive Banach space, let A be an odd mapping of X into X^* with Property (B). Let $h \in X^*$, $R > 0$. Assume $\|A(u)\| > \|h\|$ for each $u \in S_R$. Then there exists $F_0 \in \Lambda$ such that:

for each $F \in \Lambda$, $F \supset F_0$ there exists $u_F \in D_R \cap F$ satisfying the equation $A_F(u_F) = h_F$.

Proof. For $F \in \Lambda$ let E_F denote a homeomorphism and isomorphism between F^* and F. The mapping $E_F A_F$ of F into F is continuous by Lemma 4.1, and odd. Lemma 5.1 implies that there exists $F_0 \in \Lambda$ such that $A_F(u) \neq t h_F$ for each $F \in \Lambda$, $F \supset F_0$, $u \in S_R \cap F$, $t \in \langle 0, 1\rangle$. That means $E_F A_F(u) - t E_F h_F \neq 0$ for each $u \in S_R \cap$ $\cap F$, $t \in \langle 0, 1\rangle$. Theorem 2.1 and Property (15) of the local degree imply

$deg(E_F A_F - E_F h_F, D_R, 0) = deg(E_F A_F, D_R, 0) \neq 0.$
By (14) there exists $u_F \in D_R \cap F$ such that
$E_F A_F(u_F) - E_F h_F = 0$, i.e. $A_F(u_F) = h_F$.

Theorem 5.1. Let X be a real reflexive Banach space, let A be an odd mapping of X into X^* with Property (B). Let $h \in X^*$, $R > 0$. Suppose $\|A(u)\| > \|h\|$ for each $u \in S_R$. Then there exists $u \in D_R$ such that $A(u) = h$.

Proof. By Lemma 5.2 the set $M_F = \{u \in D_R \cap F';\ A_{F'}(u) = h_{F'},\ F' \in \Lambda, F' \supset F\}$ is non-empty for each $F \in \Lambda$. Analogously, as in the proof of Lemma 5.1 we obtain: $\{\overline{M}_F^w\}_{F \in \Lambda}$ is a filter; the sets $\overline{M}_F^w$ are weakly closed; $\overline{M}_F^w \subset \overline{D}_R$, $\overline{D}_R$ is weakly compact, therefore there exists $u_0 \in \bigcap_{F \in \Lambda} \overline{M}_F^w$; the sets $\overline{M}_F^w$ are weakly compact, therefore for each $F \in \Lambda$ there exists a sequence $\{u_n\}$ (dependent of F) such that $u_n \rightharpoonup u_0$. By definition M_F there exists a sequence $\{F_n\} \subset \Lambda$ such that $u_n \in F_n$, $F_n \supset F$, $A_{F_n}(u_n) = h_{F_n}$. Lemmas 4.2, 4.3 and Remark 4.2 imply $A(u_0) = h$, $u_0 \in \overline{D}_R$. We have $\|A(u)\| > \|h\|$ for $u \in S_R$, hence $u_0 \in D_R$. This completes the proof.

Theorem 5.2. Let X be a real reflexive Banach space, let A be an odd mapping of X into X^* with Property (B). Suppose $\lim\limits_{\|u\| \to +\infty} \|A(u)\| = +\infty$. Then the mapping A is regular surjective.

Proof. Theorem 5.1 implies $A(X) = X^*$. Assume that (ii) of Definition 1.5 is not valid. Then there exist $R > 0$ and sequences $\{u_n\} \subset X$, $\{f_n\} \subset X^*$ satisfying the conditions $A(u_n) = f_n$, $\|f_n\| \leq R$, $\|u_n\| \to +\infty$. Simultaneously, by the assumption $\|A(u_n)\| \to +\infty$. This contradiction concludes the proof.

Theorem 5.3. Let X be a real reflexive Banach space, let T, S be two odd mappings of X into X^*, $\varkappa > 0$. Suppose T is a positive $\varkappa - *$ -quasihomogeneous mapping with Property (B), S is strongly positive $\varkappa - *$ -quasihomogeneous and completely continuous. Suppose that λ is not an eigenvalue of T_0, S_0, where T_0 and S_0 are the mappings of Definition 1.4, (10),(11). Then the mapping $A_\lambda = T - \lambda S$ is regular surjective.

Proof. By Lemma 4.6 the mapping A_λ has Property (B). The mapping A_λ is odd, therefore it is sufficient to prove $\lim\limits_{\|u\| \to +\infty} \|A_\lambda(u)\| = +\infty$ and to use Theorem 5.2. Assume that the condition $\lim\limits_{\|u\| \to +\infty} \|A_\lambda(u)\| = +\infty$ is not fulfilled. Then there exist a sequence $\{u_n\} \subset X$ and a number $K > 0$ such that $\|u_n\| \to +\infty$, $\|A_\lambda(u_n)\| \leq K$. If we write $v_n = \frac{u_n}{\|u_n\|}$, then we can assume $v_n \rightharpoonup v$ in X. The mapping S is strongly positive $\varkappa - *$ -quasihomogeneous, hence we may assume without loss of generality that

$$(21) \qquad \left(\frac{1}{\|u_n\|}\right)^{\varkappa} S \left(\frac{v_n}{\|u_n\|^{-1}}\right) \to g, \text{ where } g \in X^*.$$

We know $\|A_\lambda(u_n)\| \cdot \left(\frac{1}{\|u_n\|}\right)^{\varkappa} \to 0$, therefore

$$(22) \quad \left(\frac{1}{\|u_n\|}\right)^{\varkappa} T \left(\frac{v_n}{\|u_n\|^{-1}}\right) \to f, \text{ where } f - \lambda g = 0,\ f \in X^*.$$

The mapping T is positive $\varkappa - \varkappa$ -quasihomogeneous, hence $f = T_0(v)$, $v_n \to v$. From here $g = S_0(v)$ and we obtain

$$(23)\quad \left(\frac{1}{\|u_n\|}\right)^{\varkappa} A_\lambda\left(\frac{v_n}{\|u_n\|^{-1}}\right) \to T_0(v) - \lambda S_0(v) = 0, \ \|v\| = 1.$$

This is a contradiction with the assumption that λ is not an eigenvalue of T_0, S_0. Hence, $\lim_{\|u\| \to +\infty} \|A_\lambda(u)\| = +\infty$ and Theorem 5.2 has proved Theorem 5.3.

<u>Theorem 5.4</u>. Let X be a real reflexive Banach space. Let T, S be two odd mappings of X into X^*. Suppose T, S are positive $\varkappa$-homogeneous, $\varkappa > 0$, T has Property (B) and S is completely continuous. Then for each real number λ one and only one of the following two conditions is fulfilled:

(α) λ is an eigenvalue of T, S;

(β) the mapping $A_\lambda = T - \lambda S$ is regular surjective.

<u>Proof</u>. Let Condition (α) be fulfilled, $u \in X$, $u \neq 0$, $A_\lambda(u) = 0$. Then $A_\lambda(tu) = 0$ for each $t > 0$, hence Condition (ii) of Definition 1.5 is not fulfilled, i.e. (β) is not valid. Now suppose λ is not an eigenvalue of T, S. By Remark 4.3, the assumptions of Theorem 5.3 are fulfilled, hence (β) is valid.

<u>Theorem 5.5.</u> Let X be a real reflexive Banach space. Let T, S be odd mappings of X into X^*, $\varkappa > 0$. Suppose T, S are positive $\varkappa$-homogeneous, T has Property (B), $T(u) \neq 0$ for all $u \in S_1$ and S is completely continuous. Let λ be an arbitrary real number. Then there exists an odd, positive a-homogeneous and completely continuous mapping B of X into a finite

dimensional subspace of the space X^* such that $T-\lambda S = T_0 - \lambda B$, where the mapping $T_0 = T-\lambda(S-B)$ is regular surjective.

Proof. The condition $T(u) \neq 0$ for all $u \in S_1$ and Lemma 4.5 imply $d = \inf_{u \in S_1} \| T(u) \| > 0$. From here we obtain $\lim_{\|u\| \to +\infty} \| T(u) \| = +\infty$, because T is positive $\varkappa$-homogeneous. If $\lambda = 0$, then define $B(u) = 0$ for each $u \in X$.

By Theorem 4.2 the mapping $T = T_0$ is regular surjective. Now assume $\lambda \neq 0$. Let $0 < \varepsilon < \frac{d}{|\lambda|}$. By Theorem 3.1 there exists an odd, positive $\varkappa$-homogeneous and continuous mapping B of X into a finite dimensional subspace of the space X^* such that $\|S(u)-B(u)\| \leq \varepsilon \|u\|^{\varkappa}$. By Lemma 4.6 the mapping $T_0 = T - \lambda(S-B)$ has Property (B). For all $u \in S_1$ we have

$$\| T_0(u) \| = \| T(u) - \lambda(S(u) - B(u)) \| \geq \| T(u) \| - |\lambda| \cdot \| S(u) - B(u) \| \geq d - \varepsilon |\lambda| > 0 .$$

From here we obtain $\lim_{\|u\| \to +\infty} \| T_0(u) \| = +\infty$, because the mapping T_0 is positive $\varkappa$-homogeneous. Theorem 5.2 implies that T_0 is regular surjective. This completes the proof.

Theorem 5.6. Assume the assumptions of Theorem 5.5 are fulfilled, let $A_\lambda = T - \lambda S$. Then there exists a finite dimensional subspace F of the space X^* with the following property:

for each $f \in X^*$ there exist $f_1 \in A_\lambda(X)$, $f_2 \in F$ such that $f = f_1 + f_2$

Proof. Let B be the mapping of Theorem 5.5, $B(X) \subset F$, where F is the finite dimensional subspace of the space X^*. For each $f \in X^*$ there exists $u \in X$ such that $T_0(u) = T(u) - \lambda(S(u) - B(u)) = f$ (by Theorem 5.5). It is sufficient to write $f_1 = A_\lambda(u)$, $f_2 = \lambda B(u)$.

Lemma 5.3. Let X be a real reflexive Banach space, let A be a mapping of X into X^* with Property (B), $R > 0$. Suppose

(24) $\|A(u)\| \neq 0$ and $\dfrac{A(u)}{\|A(u)\|} \neq \dfrac{A(-u)}{\|A(-u)\|}$ for all $u \in S_R$.

Then there exists $F_0 \in \Lambda$ such that $F \in \Lambda, F \supset F_0$ implies $\|A_F(u)\|_F \neq 0$, $\dfrac{A_F(u)}{\|A_F(u)\|_F} \neq \dfrac{A_F(-u)}{\|A_F(-u)\|_F}$ for all $u \in S_R \cap F$.

Proof. By Lemma 5.1 there exists $H_0 \in \Lambda$ such that $F \in \Lambda$, $F \supset H_0$ implies $A_F(u) \neq 0$ for all $u \in S_R \cap F$. Suppose that the assertion of Lemma 5.3 is not valid. Then for each $F \in \Lambda$ the set

$$M_F = \{u \in S_R \cap F';\ \|A_{F'}(u)\|_{F'} \neq 0 \neq \|A_{F'}(-u)\|_{F'}\ ,\ \frac{A_{F'}(u)}{\|A_{F'}(u)\|_{F'}} \neq \frac{A_{F'}(-u)}{\|A_{F'}(-u)\|_{F'}}\ ,\ F' \in \Lambda\ ,\ F' \supset F\}$$

is non-empty. Analogously as in the proof of Lemma 5.1 for $\{\overline{N}_F^w\}_{F \in \Lambda}$ we obtain: $\{\overline{M}_F^w\}_{F \in \Lambda}$ is a filter, the sets $\overline{M}_F^w$ are weakly closed, $\overline{M}_F^w \subset \overline{D}_R$, $\overline{D}_R$ is weakly compact; therefore there exists $u_0 \in \bigcap_{F \in \Lambda} \overline{M}_F^w$; the sets $\overline{M}_F^w$ are weakly compact, therefore for each $F \in \Lambda$ there exists a sequence $\{u_m\} \subset M_F$ (dependent of F) such that $u_m \rightharpoonup u_0$. That means by definition M_F that there exists a sequence $\{F_n\} \subset \Lambda$ (dependent of F) such that $u_n \in F_n$, $F_n \supset F$ and $\|A_{F_n}(u_n)\|_{F_n} \neq 0 \neq \|A_{F_n}(-u_n)\|_{F_n}$, $\dfrac{A_{F_n}(u)}{\|A_{F_n}(u_n)\|_{F_n}} = \dfrac{A_{F_n}(-u_n)}{\|A_{F_n}(-u_n)\|_{F_n}}$. The sequence $\{u_n\}$ is bounded, hence by (b) there exists $K_F > 0$ such that $\|A(u_n)\| \leq K_F$, $\|A(-u_n)\| \leq K_F$, therefore $\|A_{F_n}(u_n)\|_{F_n} \leq K_F$, $\|A_{F_n}(-u_n)\|_{F_n} \leq K_F$. Let us write $b_n(F) = \|A_{F_n}(u_n)\|_{F_n}$, $c_n(F) = \|A_{F_n}(-u_n)\|_{F_n}$. We can suppose $b_n(F) \to b(F)$, $c_n(F) \longrightarrow c(F)$, where $b(F), c(F) \in \langle 0, K_F \rangle$, because the interval $\langle 0, K_F \rangle$ is compact. Let us prove this assertion:

(25) there exists $H_1 \in \Lambda$ such that $u_0 \in H_1$, $H_0 \subset H_1$ and $F \in \Lambda$, $F \supset H_1$ implies $b(F) \neq 0$.

Let (25) be not valid, let $F \in \Lambda$. Then there exist $F' \in \Lambda$, $F' \supset F$ and sequences $\{u_n\} \subset S_R$, $\{F_n\} \subset \Lambda$ such that $u_n \rightharpoonup u_0$, $u_n \in F_n$, $F_n \supset F'$ and $b_n(F') = \|A_{F_n}(u_n)\|_{F_n} \to 0$. From here we obtain $(A(u_n), u_n) = (A_{F_n}(u_n), u_n)_{F_n} \to 0$ and $(A(u_n), v) = (A_{F_n}(u_n), v)_{F_n} \to 0$ for each $v \in F$. The assumptions of Lemmas 4.2, 4.3 are satisfying, where $h = 0, t_F = 1$; therefore $u_0 \in S_R$, $A(u_0) = 0$. This is a contradiction with (24), hence (25) is proved.

Analogously, we can prove that there exists $H_2 \in \Lambda$ such that $H_0 \subset H_2$ and $c(F) \neq 0$ for each $F \in \Lambda, F \supset H_2$. From here it is clear that

(26) there exists $F_0 \in \Lambda$ such that $u_0 \in F_0$ and $F \in \Lambda$, $F \supset F_0$ implies $b(F) \neq 0 \neq c(F)$, $\|A_F(u)\|_F \neq 0$ for all $u \in S_R \cap F$.

Let $F \in \Lambda$ be arbitrary such that $F_o \subset F$ (but fixed), let $\{u_n\}$, $\{F_n\}$ be the sequences of the preceding part of this proof. We shall make similar considerations as in the proof of Lemma 4.2:

We know $c_n(F)A_{F_n}(u_n)-\ell_n(F)A_{F_n}(-u_n) = 0$, that means $(c_n(F)A(u_n) - \ell_n(F)A_{F_n}(-u_n), v) = 0$ for all

$v \in F_n$. By (b) we can suppose $\bar{A}(u_o, u_n) \rightharpoonup u_1^*$, $\bar{A}(-u_o, -u_n) \rightharpoonup u_2^*$, (e) implies $(\bar{A}(u_o, u_n), u_n) \to (u_1^*, u_o)$, $(\bar{A}(-u_o, -u_n), u_n) \to (u_2^*, u_o)$.
From here we obtain

$$(27) \left\{ \begin{aligned} &(c_n(F)\bar{A}(u_n, u_n)-\ell_n(F)\bar{A}(-u_n, -u_n)-c_n(F)\bar{A}(u_o, \\ &u_n)+\ell_n(F)\bar{A}(-u_o, -u_n), u_n-u_o) \to -c(F)(u_1^*, u_o)+ \\ &+\ell(F)(u_2^*, u_o)+c(F)(u_1^*, u_o)-\ell(F)(u_2^*, u_o) = 0 . \end{aligned} \right.$$

By (c) we have $c_n(F)(\bar{A}(u_n, u_n)-\bar{A}(u_o, u_n), u_n-u_o) \geqq 0$, $\ell_n(F)(\bar{A}(-u_n, -u_n)-\bar{A}(-u_o, -u_n), u_o-u_n) \geqq 0$,
therefore, by using (27), $c_n(F)(\bar{A}(u_n, u_n)-\bar{A}(u_o, u_n), u_n-u_o) \to 0$, $\ell_n(F)(\bar{A}(-u_n, -u_n)-\bar{A}(-u_o, -u_n), u_o-u_n) \to 0$.
We know that $c_n(F) \to c(F)$, $\ell_n(F) \to \ell(F)$, $\ell(F) \neq 0 \neq c(F)$ (see (26)), hence

$$(\bar{A}(u_n, u_n)-\bar{A}(u_o, u_n), u_n-u_o) \to 0 ,$$
$$(\bar{A}(-u_n, -u_n)-\bar{A}(-u_o, -u_n), u_o-u_n) \to 0 .$$

From here by (d) $u_o \in S_R$ and $\bar{A}(v, u_n) \rightharpoonup \bar{A}(v, u_o)$, $\bar{A}(-v, -u_n) \rightharpoonup \bar{A}(-v, -u_o)$ for all $v \in X$. By using (e) we obtain for each $v \in F$

$(c_n(F)\bar{A}(u_n, u_n)-\ell_n(F)\bar{A}(-u_n, -u_n)-c_n(F)\bar{A}(v, u_n)+$

$+ b_n(F)\bar{A}(-v, -u_n), u_n - v) \to (-c(F)\bar{A}(v, u_0) +$
$+ b(F)\bar{A}(-v, -u_0), u_0 - v)$.

The condition (c) implies the last expression is nonnegative. If we write $v = u_0 - \lambda w$, $\lambda > 0$, $w \in F$, then we obtain analogously as (19),(20) in the proof of Lemma 4.2, that

(28) $(b(F)A(-u_0) - c(F)A(u_0), w) = 0$ for each $w \in F$.

Assumption (24) implies $A(u_0) \neq 0$, because $u_0 \in S_R$. Let $w_0 \in X$, $(A(u_0), w_0) \neq 0$. We can suppose $w_0 \in F_0$. Then we have $w_0 \in F$, therefore (28) implies $(A(-u_0), w_0) \neq 0$. The number $a = \dfrac{(A(u_0), w_0)}{(A(-u_0), w_0)}$ is independent of F and by (28) we obtain $a = \dfrac{b(F)}{c(F)}$ and

(29) $(a A(-u_0) - A(u_0), w) = 0$ for each $w \in F$.

All preceding considerations are valid for each $F \supset F_0$, $F \in \Lambda$. We have $\bigcup_{\substack{F \supset F_0 \\ F \in \Lambda}} F = X$, hence (29) is valid for all $w \in X$, i.e. $a \cdot A(-u_0) - A(u_0) = 0$. From here $\dfrac{A(u_0)}{\|A(u_0)\|} \quad \dfrac{A(-u_0)}{\|A(-u_0)\|}$, $u_0 \in S_R$. We have obtained a contradiction with (24). This completes the proof.

Lemma 5.4. Let X be a real reflexive Banach space, let A be a mapping of X into X^* with Property (B). Let $h \in X^*$, $R > 0$. Suppose that (24) of Lemma 5.3 is valid and $\|A(u)\| > \|h\|$ for all $u \in S_R$. Then there exists $F_0 \in \Lambda$ such that for each $F \in \Lambda, F \supset F_0$ there exists $u \in F \cap D_R$ satisfying the equation $A_F(u) = h_F$.

Proof. As Lemma 5.2 but by using Lemma 5.1 and Lemma 5.3.

Theorem 5.7. Let the assumptions of Lemma 5.4 be fulfilled. Then there exists $u_0 \in X$ such that $A(u_0) = h$.

Proof. As Theorem 5.1 but by using Lemma 5.4.

Theorem 5.8. Let X be a real reflexive Banach space, let A be a mapping of X into X^* with Property (B) satisfying the conditions $\lim\limits_{\|u\| \to +\infty} \|A(u)\| = +\infty$ and $\frac{A(u)}{\|A(u)\|} \neq \frac{A(-u)}{\|A(-u)\|}$ for all $u \in X$, $\|u\| \geq R$, where $R > 0$. Then the mapping A is regular surjective.

Proof. As Theorem 5.2 but by using Theorem 5.7.

6. Fredholm alternative for coercitive mapping

Theorem 6.1([1],[3]). Let X be a reflexive Banach space (real or complex), let A be a coercitive mapping of X into X^* with Property (B'). Then A is a regular surjective mapping.

Theorem 6.2. Let X be a complex reflexive Banach space, let T, S be two mappings of X into X^*, let λ_1, λ_2 be real numbers, $\lambda_2 \neq 0$, $\lambda = \lambda_1 + i\lambda_2$. Suppose the mapping T is coercitive with Property (B), the mapping S is completely continuous. Suppose the following condition is fulfilled:

(f) $(T(u), u)$, $(S(u), u)$ are real numbers for all $u \in X$.

Then the mapping $A_\lambda = T - \lambda S$ is regular surjective.

Proof. By Lemma 4.6 A_λ has Property (B). We have for each $\varepsilon > 0$, $a, b \in E_1$

$$2|\lambda_1| a b \geq -\varepsilon |\lambda_1| a^2 - \frac{|\lambda_1|}{\varepsilon} b^2 ,$$

$$-2|\lambda_1| a b \geq -\varepsilon |\lambda_1| a^2 - \frac{|\lambda_1|}{\varepsilon} b^2 .$$

From here by (f) we obtain for each $u \in X$, $\varepsilon > 0$

$$|(A_\lambda(u), u)|^2 = (T(u), u)^2 - 2\lambda_1 (S(u), u)(T(u), u) + \\ + \lambda_1^2 (S(u), u)^2 + \lambda_2^2 (S(u), u)^2 \geq (T(u), u)^2 \cdot \\ \cdot (1 - \varepsilon |\lambda_1|) + (S(u), u)^2 \cdot (\lambda_1^2 + \lambda_2^2 - \frac{|\lambda_1|}{\varepsilon}) .$$

If $\lambda_1 \neq 0$ then there exists $\varepsilon_0 > 0$ such that

$\frac{|\lambda_1|}{\lambda_1^2 + \lambda_2^2} < \varepsilon_0 < \frac{1}{|\lambda_1|}$, hence for $u \in X$, $\|u\| \neq 0$,

$$(30) \quad \frac{|(A_\lambda(u), u)|}{\|u\|} \geq \sqrt{1 - \varepsilon_0 |\lambda_1|} \cdot \frac{|(T(u), u)|}{\|u\|} ,$$ where

$1 - \varepsilon_0 |\lambda_1| > 0$.

Moreover, (30) is valid for $\lambda_1 = 0$,too. Therefore A_λ is coercitive, because T is coercitive and it is sufficient to use Theorem 6.1.

Theorem 6.3. Let X be a complex reflexive Banach space, T and S positive $\varkappa$-homogeneous mappings of X into X^*, where $\varkappa > 0$. Suppose the mappings T, S satisfy the condition (f) of Theorem 6.2, T is coercitive. Then all eigenvalues of T, S are real and different of zero.

Theorem 6.4. Let X be a complex reflexive Banach space, let T, S be mappings of X into X^* satisfying Condition (f), $\varkappa > 0$. Suppose T is a coercitive and positive $\varkappa - \varkappa$ -quasihomogeneous mapping with Property (B), $\mathcal{S}$ is completely continuous and strongly positive $\varkappa - \varkappa$ -quasihomogeneous. Assume that a complex number λ is not an eigenvalue of T_0, $\mathcal{S}_0$, where T_0, $\mathcal{S}_0$ are the mappings of Definition 1,4,(10), (11). Then the mapping $A_\lambda = T - \lambda \mathcal{S}$ is regular surjective.

<u>Proof</u>. It is sufficient to prove this Theorem for λ real (see Theorem 6.2). Let us prove this assertion:

(31) for each $R > 0$ there exists $\kappa > 0$ such that $f \in X^*, \|f\| \leq R, 0 \leq \lambda_2 < 1, u \in X, A_{\lambda + i\lambda_2}(u) = T(u) - (\lambda + i\lambda_2)\mathcal{S}(u) = f$ implies $\|u\| \leq \kappa$.

Assume that (31) is not valid. Then there exists a number $R > 0$ and sequences $\{f_n\} \subset X^*$, $\{u_n\} \subset X$, $\{\lambda_n\} \subset \langle 0, 1)$ such that $\|u_n\| \to +\infty, A_{\lambda + i\lambda_n}(u_n) = f_n, \|f_n\| \leq R$.

Let us write $v_n = \dfrac{u_n}{\|u_n\|}$. We may suppose $v_n \rightharpoonup v$ in X, $\lambda_n \to \lambda_0$, $\lambda_0 \in \langle 0, 1 \rangle$. Suppose $\lambda_0 > 0$. Then we may assume $\lambda_n \geq \delta$, $\delta > 0$. There exists ε_0 such that $\dfrac{|\lambda|}{\lambda^2 + \delta^2} < \varepsilon_0 < \dfrac{1}{|\lambda|}$, hence $\dfrac{|\lambda|}{\lambda^2 + \lambda_n^2} < \varepsilon_0 < \dfrac{1}{|\lambda|}$.

Analogously as (30) we obtain

$$\frac{|(A_{\lambda + i\lambda_n}(u_n), u_n)|}{\|u_n\|} \geq \sqrt{1 - \varepsilon_0 |\lambda|} \cdot \frac{|(T(u_n), u_n)|}{\|u_n\|}, \quad \sqrt{1 - \varepsilon_0 |\lambda|} > 0.$$

Simultaneously,

$$\frac{|(A_{\lambda + i\lambda_n}(u_n), u_n)|}{\|u_n\|} \leq \|f_n\| \leq R,$$

but this gives a contradiction, because T is coercitive.

Hence $\lambda_0 > 0$ is impossible, i.e. $\lambda_0 = 0$. We have

$$\left(\frac{1}{\|u_n\|}\right)^{\varkappa} A_{\lambda+i\lambda_n}\left(\frac{v_n}{\|u_n\|^{-1}}\right) = \frac{f_n}{\|u_n\|^{\varkappa}} \longrightarrow 0 .$$

From here we obtain (analogously as (21),(22),(23) in the proof of Theorem 5.3) $\left(\frac{1}{\|u_n\|}\right)^{\varkappa} S\left(\frac{v_n}{\|u_n\|^{-1}}\right) \to S_0(v)$, $\left(\frac{1}{\|u_n\|}\right)^{\varkappa} T\left(\frac{v_n}{\|u_n\|^{-1}}\right) \to T_0(v)$, $\left(\frac{1}{\|u_n\|}\right)^{\varkappa} A_{\lambda+i\lambda_n}\left(\frac{v_n}{\|u_n\|}\right) \to$

$\to T_0(v) - \lambda S_0(v) = 0$, $\|v\| = 1$, because T is positive $\varkappa$-$\varkappa$-quasihomogeneous, S is strongly positive $\varkappa$-$\varkappa$-quasihomogeneous. We have obtained a contradiction, because λ is not an eigenvalue of T_0, S_0. This contradiction proves (31).

Now, suppose $f \in X^*$ is arbitrary, $0 < \lambda_n < 1$, $\lambda_n \to 0$. By Theorem 6.2 there exists $\{u_n\} \subset X$ such that $A_{\lambda+i\lambda_n}(u_n) = f$, (34) implies $\|u_n\| \leq r$, $r > 0$. Hence we may assume $u_n \to u$ in X and $S(u_n) \to u^*$ in X^*, because S is completely continuous. From here $T(u_n) \to f + \lambda u^*$. Lemma 4.5 implies $T(u) = f + \lambda u^*$, $u_n \to u$, hence $S(u) = u^*$. That means $A_{\lambda+i\lambda_n}(u_n) \to A_\lambda(u) = f$. Now we know $A_\lambda(X) = X^*$ and that means together with (31) that A_λ is regular surjective.

Theorem 6.5. Let X be a complex reflexive Banach space, let T, S be positive $\varkappa$-homogeneous mappings of X into X^* satisfying Condition (f). Suppose T is a coercitive mapping with Property (B), S is completely continuous. Then for each complex number λ one and only one of the following two conditions is fulfilled:

(α) λ is an eigenvalue of T, S ;

(β) the mapping $A_\lambda = T - \lambda S$ is regular surjective.

Theorem 6.6. Let X be a complex reflexive Banach space, let T, S be two positive $\varkappa$-homogeneous mappings of X into X^* satisfying Condition (f). Let T have Property (B), let S be completely continuous, λ a complex number. Suppose $(T(u), u) \geq c_1 \|u\|^{1+\varkappa} - c_2$ for all $u \in X$, $c_1 > 0$, $c_2 \geq 0$. Then there exists a completely continuous and positive $\varkappa$-homogeneous mapping B of X into a finite dimensional subspace of the space X^* such that $T - \lambda S = T_0 - \lambda B$, where $T_0 = T - \lambda(S - B)$ is a regular surjective mapping.

Proof. If $\lambda = 0$, then define $B(u) = 0$ for all $u \in X$. The mapping $T_0 = T$ is regular surjective by Theorem 6.1. Assume $\lambda \neq 0$, $0 < \varepsilon < \frac{c_1}{|\lambda|}$. Let B be the mapping of Theorem 3.1, $\|S(u) - B(u)\| \leq \varepsilon \|u\|^{\varkappa}$. By Lemma 4.6 the mapping $T_0 = T - \lambda(S - B)$ has Property (B). For each $u \in X$ we have

$$|(T(u) - \lambda(S(u) - B(u)), u)| \geq c_1 \|u\|^{\varkappa+1} - c_2 - \varepsilon \cdot |\lambda| \cdot \|u\|^{\varkappa+1}.$$

From here we see that T_0 is coercitive and Theorem 6.1 proves our assertion.

Theorem 6.7. Let the assumptions of Theorem 6.6 be fulfilled, $A_\lambda = T - \lambda S$. Then there exists a finite dimensional subspace F of the space X^* with the following property:

for each $f \in X^*$ there exist $f_1 \in A_\lambda(X)$, $f_2 \in F$

such that $f = f_1 + f_2$.

Proof. As Theorem 5.6 but by using Theorem 6.6.

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