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REMARKS ON MONOTONE OPERATORS

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1. Introduction. This note deals with the different properties of monotone operators in the sense of F.E. Browder and G.J.Minty (see for instance [1],[2],[3],[6]).

The purpose of Section 3 is to give some conditions for the relations among demicontinuity, continuity, strong continuity, boundedness and surjectivity.

Some examples of monotone operators with pathological behaviour are given in Section 4.

Analogous theorem as the Fredholm alternative for the problem $I - T$ (T is a linear completely continuous operator) is proved in Section 5 for the linear monotone operators (see Theorem 2).

A well-known problem in the functional analysis is to find all isometries in a metric space. A similar problem for monotone operators is solved in Section 6. Solving this problem, we obtained a nonlinear characterization of Hilbert spaces (see Theorem 3).

2. Terminology, notations and definitions

Let X be a real Banach space with the norm $\| \cdot \|_X$, θ_X its zero element; X^* denotes the adjoint (dual) space of all bounded linear functionals on X. The pairing between $x^* \in X^*$ and $x \in X$ is denoted by (x^*, x). The Euclidean N-space is denoted E_N. The pairing in Hilbert space is the inner product. We shall use the symbols " $\longrightarrow$ ", " $\rightharpoonup$ " to denote the strong convergence in X (or in X^*) and weak convergence in X (or in X^*), respectively.

Let H be a Hilbert space and A a bounded linear operator defined on H with values in H. Then A^* denotes an adjoint operator.

Let $M \subset X$. Then $\bar{M}$ denotes a closure in the norm topology. Let $R > 0$ be a real number. Then K_R denotes an open ball with the center in origin and the radius R. $M \subset X$ is said to be compact (resp. weakly compact) if for each sequence $\{x_n\}$, $x_n \in M$ there exist $x_0 \in X$ and a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \longrightarrow x_0$ (resp. $x_{n_k} \rightharpoonup x_0$).

The following theorem is well-known: X is a reflexive Banach space if and only if each bounded subset of X is weakly compact.

Let F be a mapping with the domain $D \subset X$ and values in X^*. Then

(1) F is said to be hemicontinuous on D if for each $x_0 \in D$ and $w \in X$ we have $F(x_0 + tw) \rightharpoonup F(x_0)$ (as $t \longrightarrow 0+$).

(2) F is said to be demicontinuous on D if $x_n \longrightarrow x_0$ implies $Fx_n \rightharpoonup Fx_0$.

(3) F is said to be continuous on D if $x_n \longrightarrow x_0$ implies $Fx_n \longrightarrow Fx_0$.

(4) F is said to be strongly continuous on D if $x_n \rightharpoonup x_0$ implies $Fx_n \longrightarrow Fx_0$.

(5) F is said to be compact on D if for each bounded subset $M \subset D$, $F(M)$ is compact.

(6) F is said to be completely continuous on D if F is compact and continuous.

(7) F is said to be bounded on D if for each bounded subset $M \subset D$, $F(M)$ is bounded.

(8) F is said to be surjective if for every $x^* \in X^*$ there exists $x_0 \in D$ such that $Fx_0 = x^*$ (i.e. $F(D) = X^*$).

(9) F is said to be D -maximal monotone if for $u_0 \in D$, $w_0 \in X^*$ the inequality $(w_0 - Fu, u_0 - u) \geq 0$ for all $u \in D$ implies that $w_0 = Fu_0$.

(10) $F \in M_1(D)$ if $(Fx - Fy, x - y) \geq 0$ for each $x, y \in D$.

(11) $F \in M_2(D)$ if $(Fx - Fy, x - y) > 0$ fμr each $x, y \in D$, $x \neq y$.

(12) $F \in M_3(D)$ if there exists $c > 0$ þuch that

$$(Fx - Fy, x - y) \geq c \| x - y \|^2$$

for each $x, y \in D$.

3. Continuity, boundedness, surjectivity and monotonicity

Proposition 1 ([4]). Let D be an open subset of a reflexive Banach space X and $f \in M_1(D)$ be a hemicontinuous mapping. Then f is demicontinuous.

Proposition 2. Let X, Y be Banach spaces and $A: X \to Y$ a linear demicontinuous mapping. Then A is continuous.

Proof. We suppose the contrary. Then there exist a sequence $\{x_n\}$, $x_n \in X$, $x_n \neq \theta_X$, $x_n \to \theta_X$ and $\varepsilon > 0$ such that $\|Ax_n\|_Y \geq \varepsilon$ for each positive integer n. Set $t_n = \|x_n\|^{-\frac{1}{2}}$ and $y_n = t_n x_n$. Then $y_n \to \theta_X$ and $\|Ay_n\|_Y = t_n \|Ax_n\|_Y \geq \varepsilon t_n \to \infty$ contradicts $Ax_n \rightharpoonup \theta_Y$.

Proposition 3. Let X be a reflexive Banach space, $A \in M_1(X)$ a linear mapping. Then A is a continuous mapping.

Proof. Any linear mapping is hemicontinuous on X. Proposition 1 implies demicontinuity of A and Proposition 2 implies continuity of A on X.

Proposition 4. Let X be a reflexive Banach space and $T \in M_1(X)$. Then

(1) T is X-maximal monotone provided T is demicontinuous on X.

(2) T is demicontinuous on X provided T is X-maximal monotone and bounded on X.

Proof. For part (1) see [1],[2]. If $x_n \to x_0$, then there exists a subsequence $\{x_{n_k}\}$ and $w \in X^*$ such that $Tx_{n_k} \rightharpoonup w$. Letting k tend to infinity in

$(Tx_{n_k} - Ty, x_{n_k} - y) \geq 0$, we obtain

$(w - Ty, x_0 - y) \geq 0$ for each $y \in X$

and using the X-maximal monotonicity, one obtains $w = Tx_0$. By contradiction we conclude $Tx_n \to Tx_0$.

Proposition 5. Let X be a reflexive Banach space, $T \in M_1(X)$ be a demicontinuous and compact mapping on X. Then T is strongly continuous on X.

Proof. By Proposition 4 T is X-maximal monotone provided $T \in M_1(X)$ and T is demicontinuous on X. Using [5, Theorem 1] T is strongly continuous on X.

Proposition 6. Let X be a reflexive Banach space, $T \in M_3(X)$ be a bounded operator with the range $R(T)$ weakly closed. Then T is demicontinuous on X.

Proof. If $x_n \to x_0$, then there exists a subsequence $\{x_{n_k}\}$ and $w \in X$ such that $Tx_{n_k} \rightharpoonup Tw$. Letting k tend to infinity in

$$(Tx_{n_k} - Ty, x_{n_k} - y) \geq c \| x_{n_k} - y \|^2$$

one obtains

$$(Tw - Ty, x_0 - y) \geq c \| x_0 - y \|^2$$

for each $y \in X$. Set $y = w$. Then $x_0 = w$ and by contradiction we conclude $Tx_n \rightharpoonup Tx_0$.

Proposition 7. Let X be a reflexive Banach space and $T \in M_3(X)$. Then T is demicontinuous if and only if T is surjective.

Proof. See [1],[2],[3],[7].

Proposition 8. Let $K_{R+\varepsilon} \subset E_N$ and $T \in M_1(K_{R+\varepsilon})$. Then $T(\overline{K}_R)$ is a bounded set in E_N.

Proof. Suppose that there exists a sequence $\{x_n\}$, $x_n \in \overline{K}_R$ such that $x_n \to x_0$ and $\|Tx_n\|_{E_N} \to \infty$. Then there exists a subsequence $\{x_{n_k}\}$ and $w \in E_N$ such that

$$\frac{Tx_{n_k}}{\|Tx_{n_k}\|_{E_N}} \longrightarrow w .$$ We have

$$\left(\frac{Tx_{n_k}}{\|Tx_{n_k}\|_{E_N}} - \frac{Ty}{\|Tx_{n_k}\|_{E_N}} , x_{n_k} - y\right) \geq 0 ,$$

and letting k tend to infinity one obtains $(w, x_0) \geq$ $\geq (w, y)$ for each $y \in K_{R+\varepsilon}$. Set $y = x_0 + \frac{\varepsilon}{2} w$. The last inequality gives $w = 0$, a contradiction with $\|w\|_{E_N} = 1$.

Proposition 9. Let $T \in M_1(E_N)$ be a surjective mapping. Then

$$\lim_{\|x\|_{E_N} \to \infty} \|Tx\|_{E_N} = \infty .$$

Proof. Suppose that there exist $u, w \in E_N$ and a sequence $\{x_n\}$, $x_n \in E_N$ such that

$\|x_n\|_{E_N} \to \infty$, $\dfrac{x_n}{\|x_n\|_{E_N}} \to u$ and $Tx_n \to w$. Letting

n tend to infinity in

$$\left(Tx_n - Tv , \frac{x_n}{\|x_n\|_{E_N}} - \frac{v}{\|x_n\|_{E_N}}\right) \geq 0$$

one obtains that $(w, u) \geq (Tv, u)$ for each $v \in E_N$. For each positive integer n there exists $v_n \in E_N$ such that $Tv_n = nu$. We have $(w, u) \geq n(u, u)$. The last inequality implies $u = \Theta_{E_N}$ which is a contradiction with $\|u\|_{E_N} = 1$.

<u>Proposition 10.</u> Let X be a reflexive Banach space, $T \in M_1(X)$ and $x_0 \in X$.
Suppose that there exists a linear differential Gâteaux $DT(x_0, h)$ (resp. a linear differential Fréchet $dT(x_0, h)$). Then there exists a Gâteaux-derivative (resp. a Fréchet-derivative) at the point $x_0 \in X$
(For definitions see [9].)

<u>Proof.</u> For each $h \in X$ and $t \in E_1$, $t \neq 0$ it is

$$t^2\left(\frac{T(x_0+th) - T(x_0)}{t}, h\right) \geq 0, \text{ i.e.}$$

$$\left(\frac{T(x_0+th) - T(x_0)}{t}, h\right) \geq 0 .$$

Letting t tend to zero one obtains $(DT(x_0, h), h) \geq$ ≥ 0 for each $h \in X$ and using Proposition 3, we conclude this proof.

4. <u>Examples</u>

<u>Example 1.</u> Let H be a separable Hilbert space, $\{y_n, n = 0, \pm 1, \pm 2, \dots\}$ be an orthonormal basis for H and define the operator

$$Bx = \sum_{n=-\infty}^{+\infty} a_n y_{n+1} \quad \text{for } x = \sum_{n=-\infty}^{\infty} a_n y_n .$$

Let I be the identity operator in H and set $Tx =$ $= 3Ix - \|x\|_H y_0 + Bx$. Then $T \in M_3(H)$, T is continuous and T is not weakly continuous.

<u>Example 2.</u> There exists a t-positive homogene-

ous operator T (i.e. $T(tu) = tTu$ for each $u \in X$ and $t > 0$) which is continuous, $T \in M_3(H)$, $TH = H$ and T is not a linear operator.

<u>Proof</u>. See the operator T in Example 1. ($TH = H$ follows from Proposition 7.)

<u>Example 3.</u> There exists an operator $T \in M_2(E_2)$, T is surjective and T is not continuous (see Proposition 6 and 7).

<u>Proof</u>. Define

$$f(x) = \begin{cases} x & \text{for } x < 0 , \\ x+1 & \text{for } x \geq 0 \end{cases}$$

and for each $[x, y] \in E_2$ set

$$T[x, y] = [y + f(x), - x] .$$

<u>Example 4</u>. There exists $T \in M_3(\overline{K}_1)$, $K_1 \subset E_N$ $(N \geq 2)$ such that $T(\overline{K}_1)$ is not bounded (see Proposition 8).

<u>Proof</u>. Let $\{x_n\}$, $x_n \in E_N$ be a sequence such that $\|x_n\|_{E_N} = 1$, $x_n \neq x_m$ for $n \neq m$ and $x_n \longrightarrow x_0$. Set

$$Tx = \begin{cases} x & \text{for } x \in \overline{K}_1 ,\ x \neq x_m ,\ m = 1, 2, \ldots , \\ x_n + n x_n & \text{for } x = x_n ,\ n = 1, 2, \ldots . \end{cases}$$

Then $T \in M_3(\overline{K}_1)$ and $T(\overline{K}_1)$ is not bounded.

<u>Example 5</u>. There exists $T \in M_3(X)$, T is continuous and $T(\overline{K}_1)$ is not bounded (see Proposition 8).

<u>Proof</u>. Set $X = \ell_2$ and define

$$f_n(t) = \begin{cases} 0 & \text{for } t \leq \frac{1}{2} , \\ nt - \frac{n}{2} , & \text{for } t > \frac{1}{2} , \end{cases}$$

for each positive integer n .

For $x = \{a_1, a_2, \dots\} \in \ell_2$ set

$$Tx = \{f_1(a_1), f_2(a_2), \dots\} + \{a_1, a_2, \dots\} .$$

Then $T \in M_3(\ell_2)$ and $T(\overline{K}_1)$ is not bounded since for $\{0, \dots, 0, \frac{1}{n}, 0, \dots\} \in \overline{K}_1$ we have $\|Tx_n\|_{\ell_2} = \frac{n}{2} + 1$.

Moreover, T is a continuous mapping.

Example 6. One can easily prove that if $f \in M_1(\langle -1, 1\rangle), f(-1) \in \langle -1, 1\rangle, f(1) \in \langle -1, 1\rangle$ (f is not generally continuous), then f has a fixed point at the interval $\langle -1, 1\rangle$ (i.e. there exists $x_0 \in \langle -1, 1\rangle$ such that $f(x_0) = x_0$) .

This assertion is not valid for greater dimensions.

Proof. Let $N \geq 2$ be an integer and

$$T[x_1, \dots, x_N] = \begin{cases} [\frac{x_1}{\|x\|}, \frac{x_2}{\|x\|}, 0, \dots, 0], & x \neq \theta_{E_N} , \\ [1, 0, \dots, 0] , & x = \theta_{E_N} , \end{cases}$$

$$A[x_1, \dots, x_N] = [-x_2, x_1, 0, \dots, 0] ,$$

$$Ux = \frac{1}{3}(x + Tx + Ax) .$$

Then $U \in M_3(\overline{K}_1)$, $U(\overline{K}_1) \subset \overline{K}_1$ and for each $x \in \overline{K}_1$ it is $Ux \neq x$.

5. Linear monotone operators

Theorem 1. Let X be a Banach space and $A \in M_1(X)$ a linear operator such that $\overline{R(A)} = X^*$ ($R(A)$ is the range of A). Then A is one-to-one.

Proof. Suppose that there exists $y \in X$, $y \neq \theta_X$ such that $Ay = \theta_{X^*}$. Then there exists $x_0 \in X$ such that

$(Ax_0, y) \geq \frac{1}{2} \|y\|_X$. We have

$$\frac{\lambda}{2}\|y\|_X \leq (A(x_0 - \lambda y), x_0 - \lambda y) + \lambda (Ax_0, y) =$$
$$= (Ax_0, x_0) \leq \|Ax_0\|_{X^*} \cdot \|x_0\|_X$$

for each $\lambda > 0$. Letting λ tend to infinity, one obtains a contradiction.

Theorem 2. Let H be a real Hilbert space, $A \in M_1(H)$ be a linear mapping. Then A is one-to-one if and only if $\overline{R(A)} = H$.

Proof. One part of this theorem is included in Theorem 1. Suppose that A is one-to-one and denote by $N(A)$ (resp. $N(A^*)$) the null-space of the operator A (resp. A^*). Then

$H = \overline{R(A)} \oplus N(A^*) = \overline{R(A^*)} \oplus N(A) = \overline{R(A^*)}$

(see [8]). But $\overline{R(A^*)} = H$ and applying Theorem 1 on the operator A^*, we obtain $N(A^*) = \{\theta_H\}$ and hence $H = \overline{R(A)}$. (The operators A and A^* are continuous - see Proposition 3.)

Example 7. There exists $A \in M_1(H)$ such that A is one-to-one and $R(A) \neq H$.

Proof. Set $H = L_2[0,1]$ and $(Ax)(s) = \int_0^s x(t)dt$ for each $x \in L_2[0,1]$. A is continuous since

$$\|Ax\|_{L_2}^2 = \int_0^1 \left|\int_0^s x(t)dt\right|^2 ds \leq \int_0^1 \left(\int_0^1 |x(t)|dt\right)^2 ds \leq$$
$$\leq \|x\|_{L_2}^2$$

For $x \in C[0,1]$ we have

$$(Ax, x) = \int_0^1 (Ax)'(s) \cdot x(s)\, ds = \int_0^1 [(Ax)(s)] \cdot$$
$$\cdot [(Ax)(s)]\, ds = \frac{(Ax)^2(1)}{2} \geq 0 \quad ,$$

the density of $C[0,1]$ in $L_2[0,1]$ and the continuity of A imply $(Ax, x) \geq 0$ for each $x \in L_2[0,1]$. A is one-to-one and $A(L_2) \subset \{$ the set of all absolutely continuous functions$\} \neq L_2[0,1]$.

6. Nonlinear characterizations of Hilbert spaces.

Proposition 11. Let H be a real Hilbert space and $T: H \to H$ an operator. Suppose that there exists $c \in E_1$ such that

$$(Tx - Ty, x-y) = c\|x-y\|_H^2 \text{ for each } x, y \in H \ .$$

Then there exists a bounded linear operator S such that

$$Tx = Sx + T(\theta_H) \text{ for each } x \in H \ .$$

Proof. Set $Sx = Tx - T(\theta_H)$. Then

$(Sx - Sy, x-y) = c\|x-y\|_H^2$ and $(Sx, x) = c\|x\|_H^2$

for each $x, y \in H$. We have

$$\frac{(Sx, y) + (Sy, x)}{2} = c\,(x, y) \quad ,$$

$$(S(x+z), y) = (Sx, y) + (Sz, y) \ ,$$
$$(S(tx), y) = t\,(Sx, y)$$

for each $t \in E_1$ and all $x, y, z \in H$. From these equations we obtain a linearity of S.

Proposition 12. Let X be a real Banach space and $T: X \to X^*$ such that $Sx = Tx - T(\theta_X)$ is an odd operator (i.e. $S(-u) = -Su$ for each $u \in X$). Suppose that there exists $c \in E_1$, $c \neq 0$ such that

$$(Tx - Ty, x-y) = c\,\|x-y\|_X^2 \text{ for each } x, y \in X.$$

Then

a) X is a Hilbert space,

b) S is a bounded linear operator.

Proof. S satisfies the following identity

$$(Sx - Sy, x-y) = c\,\|x-y\|_X^2 ,$$

$$(Sx - S(-y), x+y) = c\,\|x+y\|_X^2$$

for each $x, y \in X$.

By an easy calculus we obtain

$$(Sx, y) + (Sy, x) = c(\|x\|_X^2 + \|y\|_X^2 - \|x-y\|_X^2) =$$

$$= c(\|x+y\|_X^2 - \|x\|_X^2 - \|y\|_X^2)$$

and

$$\frac{\|x+y\|_X^2 + \|x-y\|_X^2}{2} = \|x\|_X^2 + \|y\|_X^2 \quad \text{for each } x, y \in X.$$

The last identity implies that X is a Hilbert space.

Assertion b) follows from Proposition 11.

Theorem 3. Let X be a real Banach space. Then X is a Hilbert space if and only if there exists a mapping $T: X \to X^*$ and a real number $c \neq 0$ such that

$$(Tx - Ty, x - y) = c\|x - y\|_X^2$$

for each $x, y \in X$.

Proof. If X is a Hilbert space, set $T = I$. If there exists $c \neq 0$ and $T: X \to X^*$ such that

$$(Tx - Ty, x - y) = c\|x - y\|_X^2$$

for each $x, y \in X$.

Set

$$S u = \frac{T(u) - T(-u)}{2}$$

for each $u \in X$. S satisfies the hypotheses of Proposition 12 and Theorem 3 is proved.

R e f e r e n c e s

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