Commentationes Mathematicae Universitatis Carolinae

Aleš Pultr

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Commentationes Mathematicae Universitatis Carolinae, Vol. 11 (1970), No. 4, 619--639

Persistent URL: `http://dml.cz/dmlcz/105304`

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11,4 (1970)

TENSOR PRODUCTS IN THE CATEGORY OF GRAPHS

A. PULTR x) , Praha

Introduction. Given graphs (sets with binary relations) X, Y, there are different ways of providing the set of the compatible mappings from X into Y by a structure of graph . The present paper was, actually, stimulated by the fact that, working on a more general problem, Hedrlín and Sichler suggested a very reasonable way of doing this, in a moment when the author investigated another one. The Hom-functors and the corresponding tensor products satisfied in both cases certain conditions (T1 - T4 listed below). This gave rise to a question how many ways there really are. They are 201 - and the aim of the present paper is to prove it.

The category of sets with binary relations and the relation preserving mappings will be denoted by $\mathcal{R}$, the natural forgetful functor $\mathcal{R} \to$ Set by $\mathcal{U}$ (we shall, however, often write simply φ instead of $\mathcal{U}\varphi$ for the underlying mapping of a morphism φ). The objects of $\mathcal{R}$ will be denoted by capitals, the relation of X (subset of $\mathcal{U}X \times \mathcal{U}X$) by $\varkappa(X)$ (or, simply by $\varkappa$, if there will be no danger of confusion). The set of morphisms from X into Y is

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x) Support from the Canadian National Research Council and McMaster University is gratefully acknowledged.

denoted by $\langle X,Y\rangle$.

A covariant functor

$$\otimes : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$$

will be called a tensor product, if the following conditions are satisfied:

T1: There exist a functor $[\ ,\ ] : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$ such that $\mathcal{U}[X,Y] \approx \langle X,Y\rangle$ naturally in X,Y and $\langle X \otimes Y, Z\rangle \approx \langle X, [Y,Z]\rangle$ naturally in X,Y,Z.

T2: There exist an $X_0 \in \mathcal{R}$ such that $X \otimes X_0 \approx X$ naturally in X.

T3: $X \otimes Y \approx Y \otimes X$ naturally in X,Y.

T4: $(X \otimes Y) \otimes Z \approx X \otimes (Y \otimes Z)$ naturally in X,Y,Z.

A covariant functor $\otimes : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$ satisfying T1, T2 and T3 will be called a WKT-product (weak commutative tensor product).

$\otimes$ is said to be regular, if $\mathcal{U}(X \otimes Y) \approx \mathcal{U}X \times \mathcal{U}Y$, otherwise, $\otimes$ is said to be singular.

The results given in this paper may be summarized as follows:

Up to a natural equivalence,

there are exactly 256 regular and 149 singular WKT-products in $\mathcal{R}$,

there are exactly 52 regular and 149 singular tensor products in $\mathcal{R}$.

(Thus, every singular WKT-product is a tensor product.) All the products are described constructively.

Some of the regular tensor products from the 52 ones appear among operations on graphs frequently used in the literature. Thus, e.g. (in the notation of 3.3) we may find the product Ia (coinciding for symmetric graphs with Id) in Chapter 3 of [1] as the sum of graphs, the product II (coinciding for symmetric graphs with IVb) in Chapter 4 of [1]. These two operations (termed cartesian and strong product) are studied e.g. in [5] and [3]. It is worth noting that the operation designated by $\otimes$ in [3] is close to be a tensor product satisfying (as it is easy to prove) the properties T1 and T4 . On the other hand, some authors use the term tensor product for the categorial product, which does not appear in our list (it does not satisfy in $\mathcal{R}$ already the condition T1: Let us take a discrete X and a Y which is not discrete. If $[\ ,\ ]$ existed, we would have had $\langle X \times Y,\ X \times Y \rangle \approx \langle X, [Y, X \times Y] \rangle$, while the left-hand side contains the identity and the right-hand side is void).

Throughout the paper, the following notation of particular graphs and mappings is used:

P : $\quad UP = 1 (= \{0\}),\ \ \kappa(P) = \emptyset$,

A : $\quad UA = 2 (= \{0,1\}),\ \kappa(A) = \{(0,1)\}$,

$\alpha_i : \ P \longrightarrow A \ \ (i = 0,1) : \ \alpha_i(0) = i$.

§ 1. <u>How the WKT-products of graphs have to look like</u>

1.1 Denote by K the full subcategory of $\mathcal{R}$ generated by the objects P and A, by $\mathcal{L}$ the full subcategory of $\mathcal{R}^K$ generated by the functors f such that

$$f(\alpha_0)(f(P)) \cup f(\alpha_1)(f(P)) = f(A) .$$

Let us have $\otimes$ and $[\ ,\ ]$ given. By [4] (3.1) and by T1 there is a functor

$$H : \mathcal{R} \longrightarrow \mathcal{L}$$

such that

(1) $\otimes \approx \mathbb{L}(-, H-), [-,-] \approx \mathbb{R}(H-,-)$.

Both $\mathbb{L}$ and $\mathbb{R}$ are described in [4]. Let me here recollect the definition of $\mathbb{R}$, which will be used explicitly. We have

$$U\mathbb{R}(f, X) = \langle f(P), X \rangle ,$$

$(\mu_0, \mu_1) \in \kappa(\mathbb{R}(f, X))$ iff there is a $\mu : f(A) \to X$ with $\mu_i = \mu \bullet f(\alpha_i)$, for $\tau : g \to f$ and $\varphi : X \to Y$

$$\mathbb{R}(\tau, \varphi)(\mu) = \varphi \bullet \mu \circ \tau^P .$$

Now, given the H , define functors $F_P, F_A : \mathcal{R} \longrightarrow \mathcal{R}$ and transformations

(2) $\tau_0, \tau_1 : F_P \longrightarrow F_A$

by $F_N(X) = H(X)(N), F_N(\varphi) = H(\varphi)^N (N = P, A), \tau_i^X = H(X)(\alpha_i)$.

We have always:

(3) $\tau_0^X(F_P(X)) \cup \tau_1^X(F_P(X)) = F_A(X)$.

We see easily that thus a one-to-one correspondence between the functors $\mathcal{R} \to \mathcal{L}$ and couples of transformations (2) satisfying (3) is established. Moreover, H and H' are naturally equivalent iff the corresponding couples are. In the following, we shall replace functors and couples (3) of transformations by equivalent ones without further mentioning.

Using T1 and the definition of $\mathbb{R}$ we obtain

$$\langle F_P(X), Y \rangle \approx \langle X, Y \rangle$$

so that $F_P \approx 1_{\mathcal{R}}$.

Thus, the H from (1) corresponds to some couple of transformations

$$\tau_0, \tau_1 : 1_{\mathcal{R}} \to F$$

with

(4) $\tau_0^X(X) \cup \tau_1^X(X) = F(X)$ for all X .

1.2. Considering the definition of $\mathbb{R}$ and the consequent form of $[X,Y]$ we see that we may define

$$\theta^{XY} : \langle A, [X,Y] \rangle \to \langle F(X), Y \rangle$$

by

$$\theta^{XY}(\mu) \circ \tau_i^X = \mu(i) \ .$$

It is easy to prove the following

Proposition: θ is a natural equivalence.

1.3. Proposition: F is a left adjoint. In fact, $F \approx - \otimes A$.

Proof: We have, by 1.2, T1 and T3

$$\langle F(X), Y \rangle \approx \langle A, [X,Y] \rangle \approx \langle A \otimes X, Y \rangle \approx \langle X \otimes A, Y \rangle \approx \langle X, [A,Y] \rangle$$

naturally in X, Y. Thus, F and $- \otimes A$ have the same righ- adjoint.

Remarks: 1) Thus, by [4], $F \approx \mathbb{L}(-, f)$ for some $f \in \mathcal{L}$. Hence, F is determined by its values on K (where it coincides with f). Taking in account the condition (4) applied for $X = P, A$, we see that $f(P)$ has at most 2 points and $f(A)$ at most 4 points. Hence using only T1 and T3, we have already proved that there are only finitely many products.

2) We can put directly $F = - \otimes A$ (see 1.1) and we shall do it from now on.

1.4 Proposition: If $- \otimes X_0 \approx 1_{\mathcal{R}}$, then $[X_0, -] \approx 1_{\mathcal{R}}$ and we have $X_0 = P$.

Proof: We have $\langle X,Y\rangle \approx \langle X\otimes X_0, Y\rangle \approx \langle X, [X_0, Y]\rangle$. Considering $UY = \langle X_0, Y\rangle$ we obtain $X_0 = P$.

Convention: To simplify the notation, we shall replace now the $\otimes$ by a naturally equivalent one such that actually

$$X \otimes P = P \otimes X = X \quad \text{for every } X .$$

1.5 We have in particular (see 1.3)

$$F(P) = P \otimes A = A \otimes P = A .$$

Proposition: $\tau_i^P = \alpha_i$.

Proof: By (4), we have either $\tau_i^P = \alpha_i$ or $\tau_i^P = \alpha_{1-i}$. In the second case, however, we would not have generally $Y = [P, Y]$.

1.6 Proposition: $\tau_i^A = 1_A \otimes \alpha_i$.

Proof: Consider the commutative diagram

$$\begin{array}{ccc}
P & \xrightarrow{\tau_i^P = \alpha_i} & A = P \otimes A = F(P) \\
\Big\downarrow \alpha_j & & \Big\downarrow \alpha_j \otimes 1_A = F(\alpha_j) \\
A = A \otimes P & \xrightarrow{\tau_i^A} & A \otimes A = F(A)
\end{array}$$

The commutativity obviously determines τ_i^A. On the other hand, we have $(\alpha_j \otimes 1_A) \circ \alpha_i = \alpha_j \otimes \alpha_i = (1_A \otimes \alpha_i) \circ \alpha_j$.

1.7 Define $A \square A$ by $U(A \square A) = 2 \times 2$, $\kappa(A \square A) =$
$= \{((0,0),(0,1)), ((0,0)(1,0)), ((0,1),(1,1)), ((1,0),(1,1))\}$,

$\alpha_i \square 1_A : A \to A \square A$ by $(\alpha_i \square 1)(j) = (i,j)$, $1_A \square \alpha_i : A \to A \square A$ by $(1 \square \alpha_i)(j) = (j,i)$.

(The sign $\square$ will be given a broader sense in the next paragraph.)

Define a mapping

$$\eta : A \,\square\, A \longrightarrow A \otimes A$$

by $\eta(i,j) = (\alpha_i \otimes \alpha_j)(0)$.

Proposition: η is a morphism in $\mathcal{R}$ and we have

$$\eta \circ (\alpha_i \,\square\, 1) = \alpha_i \otimes 1, \quad \eta \circ (1 \,\square\, \alpha_i) = 1 \otimes \alpha_i .$$

Proof: $\eta \circ (\alpha_i \,\square\, 1)(j) = \eta(i,j) = (\alpha_i \otimes \alpha_j)(0) =$

$= (\alpha_i \otimes 1) \cdot \alpha_j(0) = (\alpha_i \otimes 1)(j)$ and similarly the second equation. The equations yield immediately the first statement.

1.8 Proposition: $\eta(i,j) = \eta(k,l)$ iff $\eta(j,i) = \eta(l,k)$,

$(\eta(i,j), \eta(k,l)) \in \varkappa(A \otimes A)$ iff $(\eta(j,i), \eta(l,k)) \in \varkappa(A \otimes A)$.

Proof: Let $\varkappa$ be the equivalence from T3. The statements follow by the commutativity of the diagram

$$\begin{tikzcd}
P \arrow[r, equal, "\varkappa^{PP}"] \arrow[d, "\alpha_i \otimes \alpha_j"] & P \arrow[d, "\alpha_j \otimes \alpha_i"] \\
A \otimes A \arrow[r, "\varkappa^{AA}"] & A \otimes A
\end{tikzcd}$$

1.9 Summary: By 1.3, 1.7 and 1.8, the number of WKT-products on $\mathcal{R}$ (up to a natural equivalence) does not exceed the number of nonequivalent epimorphisms $\eta : A \,\square\, A \longrightarrow X$ satisfying the statements in 1.8. Since for a regular product $|A \otimes A| = 4$, we obtain:

There are at most 405 WKT-products, there are at most 256 regular WKT-products.

In the next paragraph, we shall show that the words "at most" may be omitted.

§ 2. How the WKT-products look like

From the lack of a better notation, we shall use throughout this paragraph the signs $\otimes$, $[\,,\,]$ for functors described below in 2.3, thus, in a sense different to that in § 1. We shall see soon, however, that if is not quite inconsistent.

2.1 Definition: Define functors

$$\square : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R} \quad ,$$
$$\{\,,\,\} : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$$

by

$$\mathcal{U}(X \square Y) = \mathcal{U}X \times \mathcal{U}Y \; ,$$

$((x_0, y_0), (x_1, y_1)) \in \kappa\,(X \square Y)$ iff either $(x_0, x_1) \in \kappa\,(X)$ and $y_0 = y_1$, or $x_0 = x_1$ and $(y_0, y_1) \in \kappa\,(Y)$,

$\mathcal{U}(\varphi \square \psi) = \mathcal{U}\varphi \times \mathcal{U}\psi$,

$\mathcal{U}\{X, Y\} = \langle X, Y \rangle$,

$(\mu_0, \mu_1) \in \kappa(\{X, Y\})$ iff $(\mu_0(x), \mu_1(x)) \in \kappa(Y)$ for every $x \in X$, $\{\varphi, \psi\}(\mu) = \psi \circ \mu \circ \varphi$.

Remark: Evidently, $(\mu_0, \mu_1) \in \kappa\,(\{X, Y\})$ iff there exists a $\mu : A \square X \longrightarrow Y$ with $\mu \circ (\alpha_i \square 1_X) = \mu_i$

2.2 Proposition: $\Theta : \langle - \square -, - \rangle \longrightarrow \langle -, \{-, -\} \rangle$ defined by $(\Theta^{XYZ}(\varphi))(x)(y) = \varphi(x, y)$ is a natural equivalence.

Proof: First, every $\Theta(\varphi)(x)$ is really in $\{Y, Z\}$, since if $(y_0, y_1) \in \kappa(Y)$, we have $((x, y_0), (x, y_1) \in$

$\in \varkappa(X \square Y)$. Now, if $(x_0, x_1) \in \varkappa(X)$, we have, for every y, $((x_0, y), (x_1, y)) \in \varkappa(X \square Y)$ and hence $(\varphi(x_0, y), \varphi(x_1, y)) \in \varkappa(Z)$, so that $(\theta(\varphi)(x_0), \theta(\varphi)(x_1)) \in$ $\in \varkappa(\{Y, Z\})$. On the other hand, define $\bar{\theta}: \langle -, \{-,-\}\rangle \rightarrow$ $\rightarrow \langle - \square -, - \rangle$ by $\bar{\theta}(\varphi)(x,y) = \varphi(x)(y)$. If $((x_0, y_0), (x_1, y_1)) \in \varkappa(X \square Y)$, we have either $x_0 = x_1$ and $(y_0, y_1) \in \varkappa(Y)$ or $(x_0, x_1) \in \varkappa(X)$ and $y_0 = y_1$. In both cases obviously $(\varphi(x_0)(y_0), \varphi(x_1)(y_1)) \in \varkappa(Z)$. Evidently θ is natural in all variables and $\theta\bar{\theta} = 1$, $\bar{\theta}\theta = 1$.

2.3 Definition: Let $\eta : A \square A \longrightarrow B$ be an epimorphism. An equivalence $e(\eta, X, Y)$ on $UX \times UY$ is generated by the relation e' defined by: $(x_i, y_i)\, e'\, (x_k, y_l)$ iff there exist morphisms $\varphi : A \longrightarrow X$,

$\psi : A \longrightarrow Y$ with $\varphi(m) = x_m$, $\psi(n) = y_n$ and $\eta(i,j) = \eta(k,l)$.

For morphisms $\varphi : A \rightarrow X$, $\psi : A \longrightarrow Y$ define a mapping

$$\varphi * \psi : UB \longrightarrow UX \times UY / e(\eta, X, Y)$$

by $(\varphi * \psi)(\eta(i,j)) = \overline{(\varphi(i), \psi(j))}$

(the bar denotes the class of equivalence and will be used further on often in this sense).

Further, define $\varepsilon^{XY}: UX \times UY \longrightarrow UX \times UY / e(\eta, X, Y)$ by

$$\varepsilon^{XY}(x,y) = \overline{(x,y)}.$$

Finally, define $X \otimes_\eta Y$ by

$$U(X \otimes_\eta Y) = UX \times UY / e(\eta, X, Y),$$

$$\varkappa(X \otimes_\eta Y) = (\varepsilon \times \varepsilon)(\varkappa(X \square Y)) \cup$$
$$\cup \bigcup \{((\varphi * \psi) \times (\varphi * \psi))(\varkappa(\mathrm{B})) \mid \varphi : A \to X, \psi : A \to Y\},$$

and $\varphi \otimes_\eta \psi : X \otimes_\eta Y \longrightarrow X' \otimes_\eta Y'$ for $\varphi : X \longrightarrow X'$, $\psi : Y \longrightarrow Y'$ by

$$\varphi \otimes_\eta \psi\,\overline{(x, y)} = \overline{(\varphi(x), \psi(y))}$$

(it is easy to see that this is correct). If there is no danger of confusion, the index η will be omitted. We check easily the following

Proposition: 1) $\otimes$ is a functor from $\mathcal{R} \times \mathcal{R}$ into $\mathcal{R}$.

2) The ε^{XY} defined above are morphisms in $\mathcal{R}$ and determine a transformation

$$\varepsilon : \square \longrightarrow \otimes \quad .$$

3) $\varphi * \psi$ defined above is a morphism $\mathrm{B} \to X \otimes Y$, $1_A * 1_A$ is an isomorphism and we have $\varphi * \psi = (\varphi \otimes \psi) \circ (1 * 1)$, $\varepsilon^{A,A} = (1 * 1) \circ \eta$.

Remark: By Proposition: 3), we may replace B by $A \otimes_\eta A$ and η by $(1 * 1) \circ \eta$ without changing the functor $\otimes$.

2.4 Definition: Define $[X, Y]_\eta$ by

$\mathcal{U}[X, Y]_\eta = \langle X, Y \rangle$,

$(\mu_0, \mu_1) \in \varkappa([X, Y]_\eta)$ iff there is a $\mu : A \otimes X \to Y$ with $\mu_i = \mu \circ (\alpha_i \otimes 1)$.

For $\varphi : X' \to X$, $\psi : Y \longrightarrow Y'$ define $[\varphi, \psi]_\eta : [X, Y] \to [X', Y']_\eta$ by $[\varphi, \psi]_\eta(\mu) = \psi \circ \mu \circ \varphi$.

We see easily that thus a functor $[\;,\;]_\eta : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$

is defined. The index η will be, in the following, often omitted.

Obviously, the formula

$$\nu^{XY}(\mu) = \mu$$

defines a transformation $\nu : [X,Y] \longrightarrow \{X,Y\}$.

2.5 <u>Proposition</u>: $\langle - \otimes -, - \rangle \approx \langle -, [-,-] \rangle$.

<u>Proof</u>: Let $\gamma : X \otimes Y \longrightarrow Z$ be a morphism. Consider

$$\theta(\gamma \cdot \varepsilon^{XY}) : X \longrightarrow \{X, Z\}$$

(θ from 2.2). Let $(x_0, x_1) \in \kappa(X)$. Thus, we have a $\mu : A \square Y \longrightarrow Z$ with

$$\theta(\gamma \cdot \varepsilon^{XY})(x_i) = \mu \cdot (\alpha_i \square 1_Y) .$$

Applying this equation for an $y \in Y$ and defining $\varphi : A \to \longrightarrow X$ by $\varphi(i) = x_i$ we obtain

$$\gamma \cdot \varepsilon^{XY} \cdot (\varphi \square 1_Y)(i, y) = \mu(i, y) .$$

Thus, $\mu = \gamma \cdot \varepsilon^{XY} \cdot (\varphi \square 1_Y) = \gamma \cdot (\varphi \otimes 1_Y) \cdot \varepsilon^{AY}$ and hence $\theta(\gamma \cdot \varepsilon^{XY})(x_i) = \mu \cdot (\alpha_i \square 1) = \gamma \cdot (\varphi \otimes 1) \cdot \varepsilon^{AY} \cdot (\alpha_i \square 1) = = \gamma \cdot (\varphi \otimes 1) \cdot (\alpha_i \otimes 1)$.

Since $(x_0, x_1) \in \kappa(X)$ was arbitrary, we see that there is (exactly one) $\tau(\gamma) : X \longrightarrow [Y, Z]$ with

$$\nu^{YZ} \cdot \tau(\gamma) = \theta(\gamma \cdot \varepsilon^{XY}) .$$

Now, let $\gamma : X \longrightarrow [Y, Z]$ be a morphism. Consider

$$\bar{\theta}(\nu^{YZ} \cdot \gamma) : X \square Y \to Z .$$

If $(x_i, y_j) \, \varepsilon' \, (x_k, y_\ell)$, we have $\varphi : A \longrightarrow X$, $\psi : A \longrightarrow Y$ with $\varphi(n) = x_n$, $\psi(n) = y_n$ and $\eta(i,j) = \eta(k,\ell)$. Since $(\varphi(0), \varphi(1)) \in \kappa(X)$, we have a $\mu : A \otimes Y \longrightarrow Z$ with

$$\gamma(\varphi(n)) = \mu \circ (\alpha_n \otimes 1) .$$

Hence, $\bar{\theta}(\nu^{YZ} \circ \gamma)(\varphi(i), \psi(j)) = (\gamma(\varphi(i)))(\psi(j)) =$

$= \mu \circ (\alpha_i \otimes 1_Y) \circ (1_P \square \psi)(j) = \mu \circ \varepsilon^{AY} \circ (\alpha_i \square 1_Y) \circ (1_P \square \psi)(j) =$

$= \mu \circ \varepsilon^{AY} \circ (1 \square \psi)(i,j) = \mu \circ (1_A \otimes \psi) \circ \eta(i,k)$

(see Remark in 2.3).

Thus,

$$\bar{\theta}(\nu^{YZ} \circ \gamma)(x_i, y_j) = \bar{\theta}(\nu^{YZ} \circ \gamma)(x_k, y_k) ,$$

so that $\bar{\theta}(\nu^{YZ} \circ \gamma)$ induces a mapping $\bar{\gamma} : U(X \otimes Y) \longrightarrow$ $\longrightarrow UZ$. Since we have (see above) $\bar{\gamma} \circ (\varphi \otimes \psi) = \mu \circ (1_A \otimes \psi) \circ \eta$, we see easily that $\bar{\gamma}$ preserves the relations. Thus, we have (exactly one) $\bar{\tau}(\gamma)$ such that

$$\bar{\tau}(\gamma) \circ \varepsilon^{XY} = \bar{\theta}(\nu^{YZ} \circ \gamma) .$$

Now, it is easy to see that $\tau\bar{\tau} = 1$, $\bar{\tau}\tau = 1$ and that τ is natural in all the three variables.

2.6 Proposition: Let η have the properties from 1.8. Then

$$X \otimes_\eta Y \approx Y \otimes_\eta X$$

naturally in X, Y.

Proof: It suffices to define $\kappa^{XY} : X \otimes Y \longrightarrow Y \otimes X$ by $\kappa^{XY}(\overline{x, y}) = (\overline{y, x})$ and to check the correctness of this definition by 2.3 (and the properties from 1.8).

2.7 Theorem: For every $\eta : A \square A \longrightarrow B$ satisfying the properties 1.8, $\otimes_\eta$ is a WKT-product. If $\otimes_\eta \approx \otimes_{\eta'}$ for some $\eta' : A \square A \longrightarrow B'$, then there is an isomorphism $\beta : B \longrightarrow B'$ with $\eta' = \beta \circ \gamma$.

Proof: T1 follows by 2.5 and definition 2.4, T2 is obvious (of course, $X_o = P$), T3 is proved in 2.6. Now, let $\varkappa : \otimes_\eta \longrightarrow \otimes_{\eta}$, be a natural equivalence. Put

$$\beta = (1_A *_{\eta'} 1_A)^{-1} \circ \varkappa^{AA} \circ (1_A *_\eta 1_A) .$$

Obviously, β is an isomorphosm and we have (see 2.3, Proposition 3))

$$\beta \circ \eta(i,j) = (1 *_{\eta'} 1)^{-1} \circ \varkappa \circ \varepsilon^{AA}(i,j) = (1 *_{\eta'} 1)^{-1} \circ \varkappa \circ (\alpha_i \otimes_\eta \alpha_j)(0) =$$

$$= (1 *_{\eta'} 1)^{-1}(\alpha_i \otimes_{\eta'} \alpha_j)(0) = (1 *_{\eta'} 1)^{-1} \circ \varepsilon'^{AA}(i,j) = \eta'(i,j) .$$

2.8 Comparing 2.7 with 1.9 we obtain

<u>Corollary</u>: Up to a natural equivalence, there exist exactly 256 regular and 149 singular WKT-products.

§ 3. <u>Tensor products</u>

The bars in the following statements have the same sense as in § 2.

3.1 <u>Lemma</u>: If $\overline{(y,z)} = \overline{(y', z')}$, then $\overline{(\overline{(x,y)},z)} = \overline{(\overline{(x,y')},z')}$ for any x . If $\overline{(x,y)} = \overline{(x',y')}$, then $\overline{(x,\overline{(y,z)})} = \overline{(x,\overline{(y',z)})}$.

<u>Proof</u>: We shall prove the first statement, the proof of the other one is analogous. We may suppose $(y,z)\ e'(y',z')$. Then there are $\varphi : A \longrightarrow Y$ and $\psi : A \longrightarrow Z$ with $\varphi(i) = x$, $\varphi(k) = x'$, $\psi(j) = y$, $\psi(l) = y'$ and $\eta(i,j) = \eta(k,l)$. It suffices to consider $\bar{\varphi} = const_x \otimes \varphi : A \longrightarrow X \otimes Y$ and ψ .

3.2 <u>Proposition</u>: Let there be an isomorphism

$$\varkappa : A \otimes (A \otimes A) \longrightarrow (A \otimes A) \otimes A$$

such that $\varkappa\overline{(i,\overline{(j,k)})} = \overline{(\overline{(i,j)},k)}$. Then, for general $X \otimes (Y \otimes Z)$ and $(X \otimes Y) \otimes Z$

$\overline{(x,\overline{(y,z)})} = \overline{(x',\overline{(y',z')})}$ iff $\overline{(\overline{(x,y)},z)} = \overline{(\overline{(x',y')},z')}$.

<u>Proof</u>: We shall prove only one of the implications, the other one is analogous.

Let $(x,\overline{(y,z)})\ e'(x',\overline{(y',z')})$. Thus, we have $\varphi : A \longrightarrow X$, $\mu : A \longrightarrow Y \otimes Z$ with $\varphi(i) = x$, $\varphi(k) = x'$, $\mu(j) = \overline{(y,z)}$, $\mu(\ell) = \overline{(y',z')}$ and $\eta(i,j) = \eta(k,\ell)$. Put $\mu(n) = \overline{(y_n, z_n)}$.

Since $(\mu(0), \mu(1)) \in \varkappa$, we have some of the following three possibilities:

I) $\overline{(y_n, z_n)} = \overline{(y^o_n, z^o_n)}$ with $y^o_0 = y^o_1$ and $\overline{(z^o_0, z^o_1)} \in \varkappa(Z)$. Thus, we have a $\chi : A \longrightarrow Z$ with $\chi(n) = z^o_n$. We obtain $\overline{(\overline{(x,y)},z)} = \overline{(\overline{(x',y')},z')}$ by lemma 3.1 considering $\varphi \otimes const_{y^o_0}$ and χ .

II) $\overline{(y_n, z_n)} = \overline{(y^o_n, z^o_n)}$ with $(y^o_0, y^o_1) \in \varkappa(Y)$ and $z^o_0 = z^o_1$. Thus, there is a $\psi : A \longrightarrow Y$ with $\psi(m) = y^o_m$. Using φ and ψ we obtain $\overline{(x, y^o_j)} = \overline{(x', y^o_\ell)}$ and $\overline{(\overline{(x,y)},z)} = \overline{(\overline{(x',y')},z')}$ follows again by 3.1.

III) $\mu(n) = (\psi \otimes \chi)\overline{(p_n, q_n)}$ with $\overline{(\overline{(p_0, q_0)}}$, $\overline{\overline{(p_1, q_1)})} \in \varkappa$. Thus, there is a $\pi : A \longrightarrow A \otimes A$ $(\pi(n) = \overline{(p_n, q_n)}$ satisfying

$$\mu = (\varphi \otimes \chi) \cdot \pi$$

considering 1_A and π (and $\eta(i,j) = \eta(k,\ell)$) we obtain

$$\overline{(i,\overline{(p_j,q_j)})}=\overline{(k,\overline{(p_\ell,q_\ell)})}$$

so that $\overline{(\overline{(i,p_j)},q_j)}=\overline{(\overline{(k,p_\ell)},q_\ell)}$. Hence (using 3.1 again)

$\overline{(\overline{(x,y)},z)}=((\varphi\otimes\psi)\otimes\chi)\overline{(\overline{(i,p_j)},q_j)}=$

$=((\varphi\otimes\psi)\otimes\chi\overline{(\overline{(k,p_\ell)},q_\ell)}=\overline{(\overline{(x',y')},z')}$.

3.3 Proposition: Let the assumption of 3.2 hold. Define mappings

$$X\otimes(Y\otimes Z\ \underset{\lambda^{XYZ}}{\overset{\varkappa^{XYZ}}{\rightleftarrows}}\ (X\otimes Y)\otimes Z$$

by $\varkappa\overline{(x,\overline{(y,z)})}=\overline{(\overline{(x,y)},z)}$, $\lambda\overline{(\overline{(x,y)},z)}=\overline{(x,\overline{(y,z)})}$.

Then $\varkappa^{XYZ}$ and λ^{XYZ} are mutually inverse isomorphisms and form a natural equivalence.

Proof: We shall show that $\varkappa$ preserves the relation, the remaining statements will be then evident. In view of 3.1 and 3.2 we may freely choose representatives of equivalence classes. This will be done without explicit mentioning.

Let $((x,\overline{(y,z)}),(x',\overline{(y',z')}))\in\varkappa$. We have the following possibilities:

I. $x=x'$ and $(\overline{(y,z)},\overline{(y',z')})\in\varkappa$ so that either 1) $y=y'$, $(z,z')\in\varkappa$ and hence $(\overline{(x,y)},z)$, $(\overline{(x,y)},z'))\in\varkappa$ or

2) $(y,y')\in\varkappa$, $z=z'$, so that $(\overline{(x,y)},\overline{(x,y')})\in\varkappa$ and hence $(\overline{(x,y)},z)$, $(\overline{(x,y')},z'))\in\varkappa$ or

3) there are $\varphi: A\longrightarrow Y$, $\psi: A\longrightarrow Z$ with $\varphi(i)=$ $=y$, $\varphi(k)=y'$, $\psi(j)=z$, $\psi(\ell)=z'$ and

$((\overline{i,j}),\ (\overline{k,\ell}))\in \varkappa$. It suffices to consider $const_x \otimes \varphi$ and ψ .

II. $(x,x')\in\varkappa,\ \overline{(y,z)} = \overline{(y',z')}$, which yields $(\overline{((\overline{x,y}),z)},\ \overline{((\overline{x',y'}),z')})\in\varkappa$ almost immediately.

III. There are $\varphi : A\longrightarrow X$ and $\mu : A\longrightarrow Y\otimes Z$ such that $\varphi(i)=x,\ \varphi(k)=x',\ \mu(j)=\overline{(y,z)},\ \mu(\ell)=\overline{(y',z')}$ and $((\overline{i,j}),(\overline{k,\ell}))\in\varkappa$. Put $\mu(n)=\overline{(y_n,z_n)}$ and define $\nu : A\longrightarrow A\otimes A$ by $\nu(0)=\overline{(i,j)},\ \nu(1)=\overline{(k,\ell)}$. We have either 1) $y_0 = y_1$ and $(z_0,z_1)\in\varkappa$. Define $\chi : A\longrightarrow Z$ by $\chi(n) = z_n$ and use $\varphi\otimes const_{y_0}$ and χ , or

2) $(y_0,y_1)\in\varkappa$ and $z_0 = z_1$. Define $\psi : A\longrightarrow Y$ by $\psi(n) = y_n$. Using φ and ψ we obtain $((\overline{x,y})$, $(\overline{x',y'}))\in\varkappa$, which yields (since $(z=z')$ $(\overline{(\overline{x,y}),z)}$, $\overline{((\overline{x',y'}),z')})\in\varkappa$, or

3) $\mu = (\psi\otimes\chi)\circ\pi$ for some $\pi : A\longrightarrow A\otimes A$. Putting $\alpha = (\varphi\otimes(\psi\otimes\chi))\circ(1\otimes\pi)\circ\nu$ we see that $\overline{(x,\overline{(y,z)})}=\alpha(0),\ \overline{(x',\overline{(y',z')})}=\alpha(1)$ and $\varkappa^{XYZ}(\alpha(i)) =$
$= ((\varphi\otimes\psi)\otimes\chi)\circ\varkappa\circ(1\otimes\pi)\circ\nu(i)$.

3.4 <u>Theorem</u>: $\otimes_\eta$ satisfies T4 iff there exists an isomorphism $\varkappa : A\otimes_\gamma (A\otimes_\gamma A)\longrightarrow (A\otimes_\gamma A)\otimes_\gamma A$ such that $\varkappa\,\overline{(i,\overline{(j,k)})} = \overline{(\overline{(i,j)},k)}$.

<u>Proof</u>: If such a $\varkappa$ exists, T4 holds by 3,3 (and 3.2). If there is an equivalence $\theta^{XYZ} : X\otimes(Y\otimes Z)\longrightarrow(X\otimes Y)\otimes Z$, put $\varkappa = \theta^{AAA}$ and consider the commutative diagram

$$\begin{tikzcd}
P \arrow[r, equal] \arrow[d, "\alpha_i \otimes (\alpha_j \otimes \alpha_k)"'] & P \arrow[d, "(\alpha_i \otimes \alpha_j) \otimes \alpha_k"] \\
A \otimes (A \otimes A) \arrow[r, "\theta^{AAA}"'] & (A \otimes A) \otimes A
\end{tikzcd}$$

3.5 Theorem: Every singular WKT-product is a tensor product.

Proof: If, for an $\eta : A \,\square\, A \longrightarrow B$, $\otimes_\eta$ is a singular WKT-product, we have some of the following cases

1) η is a constant,

2) $\eta(0,0) = \eta(0,1) = \eta(1,0) \neq \eta(1,1)$,

3) $\eta(0,0) \neq \eta(0,1) = \eta(1,0) = \eta(1,1)$,

4) $\eta(0,0) = \eta(1,1) \neq \eta(0,1) = \eta(1,0)$,

5) $\eta(0,0) = \eta(1,1) \neq \eta(0,1) \neq (1,0) \neq \eta(0,0)$,

6) $\eta(0,0) \neq \eta(0,1) = \eta(1,0) \neq (1,1) \neq \eta(0,0)$.

In the case 1) we have $A \otimes (A \otimes A) \approx (A \otimes A) \otimes A \approx \overline{P}$ where $U\overline{P} = 1$, $\varkappa(\overline{P}) = \{(0,0)\}$, so that the assumption of 3.2 and 3.3 is satisfied trivially. In the cases 2) - 5) either $(\eta \times \eta)(\varkappa(A \,\square\, A)) \subsetneqq \varkappa(B)$ and we see again that $A \otimes (A \otimes A) \approx \overline{P}$, or $(\eta \times \eta)(\varkappa(A \,\square\, A)) = \varkappa(B)$ and we easily check that the assumption of 3,2 and 3.3 is satisfied. The only less trivial case is the case 6). We may put $UB = 3$, $\eta(i,j) = i + j$. Necessarily, $(0,1), (1,2) \in \varkappa(B)$. We have

(1) $i + j + k = i' + j' + k' \Longrightarrow \overline{(i, \overline{(j,k)})} = \overline{(i', \overline{(j',k')})}$ and
$$\overline{(\overline{(i,j)}, k)} = \overline{(\overline{(i',j')}, k')} .$$

Really, let $i + j + k = i' + j' + k'$. Let us prove, e.g. the first equation on the right hand side. It is obvious if $i = i'$. If $i \neq i'$, we may assume $i = 0$ and $i' = 1$. Then, $j + k = j' + k' + 1$, so that $(j' + k', j + k) \in \kappa(B)$ and (see 2.3, Proposition 3)) hence $(\overline{(j', k')}, \overline{(j, k)}) \in \kappa(A \otimes A)$. Define $\mu: A \longrightarrow A \otimes A$ by $\mu(0) = \overline{(j', k')}$, $\mu(1) = \overline{(j, k)}$. Thus (see Definition 2.3), $\overline{(i, \overline{(j, k)})} = \overline{(i', \overline{(j', k')})}$ because of 1_A, μ and $\eta(0, 1) = \eta(1, 0)$.

By T3 (see also 2.6) we have

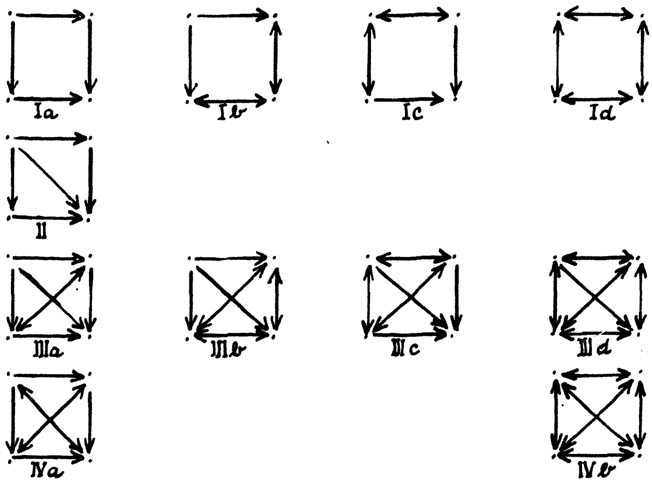
(2) $\overline{(\overline{(j, k)}, i)} = \overline{(\overline{(j', k')}, i')}$ iff $\overline{(i, \overline{(j, k)})} = \overline{(i', \overline{(j', k')})}$,

(3) $(\overline{(\overline{(j, k)}, i)}, \overline{(\overline{(j', k')}, i')}) \in \kappa$ iff $(\overline{(i, \overline{(j, k)})}, \overline{(i', \overline{(j', k')})}) \in \kappa$.

Combining (1) and (2), we see that we may define a mapping $\varkappa$ by $\varkappa\overline{(i, \overline{(j, k)})} = \overline{(\overline{(i, j)}, k)}$, by (1) and (3) we see that $\varkappa$ is an isomorphism.

3.6 <u>Theorem</u>: There are exactly 52 regular tensor products on $\mathcal{R}$.

<u>Description</u>: By 2.7 and 2.3, $\otimes$ is determined by $\varepsilon^{AA}: A \,\square\, A \longrightarrow A \otimes A$. In the following figures, the left upper points indicate the image of $(0, 0)$ the left lower points the image of $(0, 1)$ the right upper ones the image of $(1, 0)$ and the right lower one the image $(1, 1)$ under ε^{AA}. The arrows indicate the couples of distinct points in the relations (the couples of equal ones are discussed below).



Put $\Delta = \{(\xi,\xi) \mid \xi \in A \otimes A\}$, $L = \varkappa(A \otimes A) \cap \Delta$. By 2.6 we have

(1) $((0,1),(0,1)) \in L$ iff $((1,0),(1,0)) \in L$.

In the cases Ia and II any L satisfying (1) appears. Thus, we obtain in both cases eight tensor products. In the cases IIIa and IVa, necessarily $L = \emptyset$, hence we have only one tensor product both times. In the cases Ib and IIIb the L satisfying (1) and $((0,0),(0,0)) \notin L$ appear, in the cases Ic and IIIc the ones for which $((1,1),(1,1)) \notin L$ appear. Thus, every one of these four cases yields four tensor products. In each one of the remaining cases we have 6 tensor products, only $L = \{((0,0),(0,0))\}$ and $L = \{((1,1),(1,1))\}$ being prohibited.

<u>Report on the proof</u>: It would be tedious to give here the proof in full. In fact, it is not much more elegant then checking all the cases. On the other hand, the absence of

simple common features of the resulting tensor products seems to indicate that it hardly can be done in some tricky way.

Considering Definition 2.3 we see easily that

$((i,j,k),(i',j',k')) \in \varkappa(A \otimes (A \otimes A))$ iff $T_{\otimes}(i,j,k,i',j',k')$

$((i,j,k),(i',j',k')) \in \varkappa((A \otimes A) \otimes A)$ iff $T_{\otimes}(k,j,i,k',j',k')$,

where

$T_{\otimes}(i,j,k,i',j',k')$ is an abbreviation of the formula (we write $\binom{ij}{kl}$ for $((i,j),(k,l)) \in \varkappa(A \otimes A))$:

$$(\binom{io}{io} \& ((j,k)=(j',k')) \& (\exists(x,y),\binom{jk}{xy}))) \vee (\binom{i1}{i1} \& ((j,k) = $$
$$=(j',k')) \& (\exists(x,y),\binom{xy}{jk}))) \vee (\binom{io}{i1} \& \binom{jk}{j'k'}) \vee (\binom{i1}{i'o} \& \binom{j'k'}{jk})) .$$

Put, for $s=(i_0,i_1,\dots,i_5) \in 2^6$, $s'=(i_2,i_1,i_0,i_5,i_4,i_3)$.

Thus, by 3.4, a WKT-product $\otimes$ is a tensor product iff

$$\mathop{\&}_{s \in 2^6} (T_{\otimes}(s) \Longleftrightarrow T_{\otimes}(s')) .$$

The formula $\bigvee_{s \in 2^6} (T_{\otimes}(s) \not\Longleftrightarrow T_{\otimes}(s'))$ may be, after some computing, brought to the form (where $a=\binom{01}{00}$, $b=\binom{11}{01}$, $c=\binom{01}{10}$, $d=\binom{00}{11}$, $e=\binom{11}{00}$, $0=\binom{00}{00}$, $1=\binom{01}{01}$, $2=\binom{11}{11}$ and $\bar{x} = non\ x$) :

$$(c \& \bar{d}) \vee (\bar{c} \& e) \vee (a \& \bar{b} \& e) \vee (\bar{a} \& b \& e) \vee (a \& \bar{c} \& d) \vee (b \& \bar{c} \& d) \vee$$
$$\vee (a \& \bar{c} \& d \& \bar{e}) \vee (b \& \bar{c} \& d \& \bar{e}) \vee (0 \& \bar{a} \& b) \vee (0 \& \bar{a} \& c) \vee (2 \& a \& \bar{b}) \vee$$
$$\vee (2 \& \bar{b} \& c) \vee (2 \& \bar{b} \& e) \vee (\bar{0} \& \bar{1} \& 2 \& a) \vee (\bar{0} \& \bar{1} \& 2 \& e) \vee (0 \& \bar{1} \& \bar{2} \& b) \vee$$
$$\vee (0 \& \bar{1} \& \bar{2} \& e) \vee (1 \& \bar{a} \& \bar{b} \& c) \vee (1 \& \bar{a} \& \bar{b} \& e) .$$

Now, it suffices to exclude the WKT-products satisfiyng this formula.

5.1 Theorem: Let $\otimes$ be any of the 201 tensor products on $\mathcal{R}$. Define $e^X : P \otimes X \longrightarrow X$ by $e^X((\overline{x, 0})) = x$, $c^{XY} : X \otimes Y \longrightarrow Y \otimes X$ by $c^{XY}((\overline{x, y})) = (\overline{y, x})$, $a^{XYZ} : X \otimes (Y \otimes Z) \longrightarrow (X \otimes Y) \otimes Z$ by $a^{XYZ}((\overline{x, (\overline{y, z})})) = (\overline{(\overline{x, y}), z})$. Then e, c, a are natural equivalences (see 2.6 and 3.3) and $(\otimes, P, e, c, a)$ is a coherent multiplication in the sense of MacLane (see [2]).

Proof: follows immediately from the formulas for e, c, a and Theorem 5.1 in [2].

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Matematicko-fyzikální fakulta
Karlova universita
Sokolovská 83, Praha
Československo

(Oblatum 15.6.1970)