Stanislav Tomášek

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PROJECTIVELY GENERATED TOPOLOGIES ON TENSOR PRODUCTS

Stanislav TOMÁŠEK, Liberec

The purpose of this article is to introduce the inductive tensor topology in the class of all topological vector spaces (abbreviated TVS's) and to investigate its properties, especially, in connection with the projective tensor topology (cf.[11]). As to the methodology of this research, the present paper is a continuation of [10] and [11].

Let us note in advance and quite shortly, some general remarks concerning the subject of the following treatise. The taking of the tensor product under the inductive topology represents a bifunctor in the category $\mathcal{C}$ of all TVS's (or in the category of all topological abelian groups). Further, if E and F are, e.g., in $\mathcal{C}$, then $E \otimes F$ topologized in such a manner is an object in $\mathcal{C}$ - which, together with the bimorphism $(x, y) \longrightarrow x \otimes y$, realizes the tensor product defined axiomatically in [6],p.134,in the category $\mathcal{C}$.

The theory of the usual projective tensor topology possesses, in the class of all locally convex spaces, two different aspects - namely the metric and the projective ones. In what follows, we shall extend, as it was partially done also in [11], to the class of all TVS's, a part of the theory of topological tensor products which is based on the projective properties of the topology in question.

The metric properties of topological tensor products will be explored later in a separate paper.

1. Tensor products of abelian groups

In this section, we shall topologize the tensor product $E \otimes F$ of two abelian groups in a prescribed way as to make continuous certain families of Z-linear mappings (with respect to the module Z of all integers) on $E \otimes F$.

Suppose that E and F are two fixed topological abelian groups. Denote by $\mathcal{G}$ the category of all topological abelian groups and by $\mathcal{Y}(G)$, $G \in \mathcal{G}$ the collection of all Z-bilinear separately continuous mappings from $E \times F$ into G. Let $\mathcal{Y}$ be the union of all systems $\mathcal{Y}(G)$, $G \in \mathcal{G}$. If u is a Z-bilinear mapping of $E \times F$, then we often write $\tilde{u}$ for its associated Z-linear extension to $E \otimes F$.

Any such transformation $\tilde{u}$, $u \in \mathcal{Y}(G)$ determines a topology on $E \otimes F$, the so-called inverse image with respect to $\tilde{u}$ of the topology on G, compatible with its group structure. The least upper bound $T(\mathcal{Y})$ of all group topologies determined by the family $(\tilde{u}\,;\, u \in \mathcal{Y})$ is the projectively generated topology by the family $\mathcal{Y}$ and it will be termed the <u>inductive tensor group topology</u> on $E \otimes F$.

<u>Theorem 1</u>. If E and F are two abelian topological groups, then $T(\mathcal{Y})$ is the unique group topology on $E \otimes F$ with the following properties:

1^{o} The mapping $\varphi : (x, y) \longrightarrow x \otimes y$ from $E \times F$ into $(E \otimes F,\ T(\mathcal{Y}))$ is separately continuous.

2^{o} For any $G \in \mathcal{G}$ and for each $u \in \mathcal{Y}(G)$ the Z-linear extension $\tilde{u}$ i. ontinuous on $E \otimes F$ under the topology $T(\mathcal{Y})$.

Proof. First, we prove the separate continuity of φ . Let W be an arbitrary neighborhood of the zero element in $E \otimes F$ under the topology $T(\mathcal{Y})$. There are some $G_{\alpha_i} \in \mathcal{G}$, $u_{\alpha_i} \in \mathcal{Y}(G_{\alpha_i})$, $1 \leq i \leq m$, such that for suitable neighborhoods W_{α_i} in G_{α_i} $(1 \leq i \leq m)$ it holds

$$\bigcap_{i=1}^{m} \tilde{u}_{\alpha_i}^{-1}(W_{\alpha_i}) \subseteq W .$$

For a fixed y in F we take the neighborhoods V_{α_i} in E so that $u_{\alpha_i}(V_{\alpha_i}, y) \subseteq W_{\alpha_i}$ $(1 \leq i \leq m)$. Hence, for any x in $V = \bigcap_{i=1}^{m} V_{\alpha_i}$ we obtain $x \otimes y \in W$. This proves the continuity of $x \longrightarrow x \otimes y$. Similarly, one proves the continuity of $y \longrightarrow x \otimes y$. The property 2° follows immediately from the definition of $T(\mathcal{Y})$. The uniqueness of such a topology is clear.

Remark 1. (a) Let E_i , F_i $(i = 1, 2)$ be topological groups, u_i $(i = 1, 2)$ $\mathbb{Z}$ -linear continuous mappings from E_i into F_i $(i = 1, 2)$. Then the tensor mapping $u_1 \otimes u_2$ from $E_1 \otimes F_1$ into $E_2 \otimes F_2$ is continuous under the inductive tensor topologies.

(b) If E_i $(1 \leq i \leq m)$ and F are topological groups, then the spaces $(\prod_{i=1}^{m} E_i) \otimes F$ and $\prod_{i=1}^{m} (E_i \otimes F)$ are topologically isomorphic under the inductive tensor topologies,

The next task is to describe the structure of neighborhoods of the inductive tensor topology. First, let us consider a somewhat modificated situation of Theorem 1.

Let $\nu = (x_n ; n \in \mathbb{N})$ and $\mu = (y_n ; n \in \mathbb{N})$ be two, otherwise arbitrary, sequences of elements of E and

F (N is the set of all positive integers). For any two sequences $(U_n ; n \in N)$ and $(V_n ; n \in N)$ of neighborhoods in E and F, respectively, we set

$$(1) \quad \Omega((U_n),(V_n)) = \bigcup_{n=1}^{\infty} (U_1 \otimes y_1 + x_1 \otimes V_1 + \dots$$
$$\dots + U_n \otimes y + x_n \otimes V_n) \ .$$

If U_n and V_n are variables, then the family of all such subsets determines a base of a filter in $E \otimes F$. If U'_n and V'_n, $n \in N$ are neighborhoods in E and F, respectively, with $U'_n + U'_n \subseteq U_n$, $V'_n + V'_n \subseteq V_n$, then

$$\Omega((U'_n),(V'_n)) + \Omega((U'_n),(V'_n)) \subseteq \Omega((U_n),(V_n)) \ .$$

Hence, the family of all $\Omega((U_n),(V_n))$ defines on $E \otimes F$ a group topology (depending on ν and μ); it will be denoted by $T(\nu, \mu)$.

Theorem 2. The topology $T(\nu, \mu)$ is the unique group topology on $E \otimes F$ with the properties:

1^o The mappings $x \longrightarrow x \otimes y_n$, $y \longrightarrow x_n \otimes y$ are continuous for each $n \in N$.

2^o If u is a Z-bilinear separately continuous mapping with respect to $y = y_n$ and $x = x_n$, $n \in N$, of $E \times F$ into a topological group G , then $\tilde{u}$ is continuous on $E \otimes F$ under $T(\nu, \mu)$.

Proof. The property 1^o and the uniqueness is clear. We prove the property 2^o. Let u be a Z-bilinear separately continuous mapping with respect to $x = x_n$, $y = y_n$, $n \in N$, of $E \times F$ into G. If W is a neighborhood in G , then we take a sequence $(W_i ; i \in N)$ of neighborhoods in G with $W_1 + W_1 \subseteq W, \dots, W_{n+1} + W_{n+1} \subseteq W_n, \dots$. If $u(U_i, y_i) \subseteq W_{i+1}$, $u(x_i, V_i) \subseteq W_{i+1}$, then evidently

$$\tilde{u}[\Omega((U_i),(V_i))] \subseteq W \;,$$

hence $\tilde{u}$ is continuous on $E \otimes F$ under the topology $T(\nu, \mu)$.

The topology $T(\nu,\mu)$ is projectively generated, according to Theorem 2, by a more extensive family of Z-bilinear mappings than $\mathcal{Y}$, whence $T(\nu,\mu) \geq T(\mathcal{Y})$. One might, therefore, expect that it will yield a certain approximation for the inductive topology. Indeed, it holds:

<u>Theorem 3</u>. The inductive tensor topology $T(\mathcal{Y})$ is the greatest lower bound of all group topologies $T(\nu, \mu)$, where ν and μ run over the collection of all sequences in E and F.

<u>Proof</u>. Denote by T' the greatest lower bound of the system $(T(\nu,\mu);\ \nu,\ \mu)$. It suffices to prove that $T' \leq T(\mathcal{Y})$. Let x_0 and y_0 be arbitrary points of E and F. Taking ν_0 and μ_0 such that $x_0 \in \nu_0$, $y_0 \in \mu_0$, we see that the canonical mappings $x \to x \otimes y_0$, $y \to x_0 \otimes y$ are continuous if we consider $E \otimes F$ under the topology $T(\nu_0, \mu_0)$, hence the stated mappings are continuous also from E and F, respectively, into $(E \otimes F, T')$. From Theorem 1 we may now derive that $T(\mathcal{Y}) \geq T'$.

Thus, the topology $T(\mathcal{Y})$ is the least upper bound of the topologies generated by single mappings $u \in \mathcal{Y}$ and, at the same time, the greatest lower bound of the topologies $T(\nu,\mu)$, ν and μ variables. If we intend now to describe the structure of neighborhoods of the inductive tensor topology, then the first stated property may afford scarcely a convenient approach for principal reasons to the immediate solution of the presented problem. This is why we have introduced the topologies $T(\nu, \mu)$ by means of which we shall now construct the base of neighborhoods of $T(\mathcal{Y})$.

For any sequence $\mu = (y_m ; m \in N)$ in F and for any sequence $\nu = (x_m ; m \in N)$ in E we take neighborhoods U_{i0}^{μ} and V_{i0}^{ν}, $i \in N$, the choice is otherwise arbitrary, of the zero element in E and F. Denote

$$(2) \quad \Phi_0 = \bigcup_{\nu,\mu} \bigcup_{m=1}^{\infty} (U_{10}^{\mu} \otimes y_1^{\mu} + x_1^{\nu} \otimes V_{10}^{\nu} + \dots$$
$$\dots + U_{m0}^{\mu} \otimes y_m^{\mu} + x_m^{\nu} \otimes V_{m0}^{\nu}) ,$$

where ν and μ run over the collection of all sequences in E and F. What we are going to do is, roughly speaking, to enlarge the subsets of the form (2) in such a way in order to obtain subsets in $E \otimes F$ satisfying the additive axiom of some group topology.

First, take neighborhoods U_{i1}^{μ}, V_{i1}^{ν} in the spaces E, F for any $i = 1, 2, \dots$ and for each μ, ν and such that

$$U_{i1}^{\mu} + U_{i1}^{\mu} \subseteq U_{i0}^{\mu} , \quad V_{i1}^{\nu} + V_{i1}^{\nu} \subseteq V_{i0}^{\nu} .$$

We now form the subsets $(m = 1, 2, \dots)$

(3)

$$(U_{11}^{\mu} \otimes y_1^{\mu} + U_{11}^{\mu'} \otimes y_1^{\mu'} + U_{21}^{\mu} \otimes y_2^{\mu} + U_{21}^{\mu'} \otimes y_2^{\mu'} + \dots$$
$$\dots + U_{m1}^{\mu} \otimes y_m^{\mu} + U_{m1}^{\mu'} \otimes y_m^{\mu'} + x_1^{\nu} \otimes V_{11}^{\nu} + x_1^{\nu'} \otimes V_{11}^{\nu'} + \dots$$
$$\dots + x_m^{\nu} \otimes V_{m1}^{\nu} + x_m^{\nu'} \otimes V_{m1}^{\nu'})$$

for any two sequences $\mu = (y_i^{\mu} ; i \in N)$, $\mu' = (y_i^{\mu'} ; i \in N)$ in F and $\nu = (x_i^{\nu} ; i \in N)$, $\nu' = (x_i^{\nu'} ; i \in N)$ in E. The union of all subsets given by formula (3) which are constructed to all couples of sequences (μ, μ') and (ν, ν'), will be denoted by Φ_1'. Define $\Phi_1 = \Phi_0 \cup \Phi_1'$. Rearranging the systems of sequences $\{\mu\}, \{\nu\}, \{(\mu, \mu')\}$ and $\{(\nu, \nu')\}$, we repeat the same procedure with Φ_1 (with respect to the new indices) and we obtain the subset Φ_2.

Suppose that $\Phi_0 \subseteq \Phi_1 \subseteq \ldots \subseteq \Phi_k$ have been determined; let the defining subsets for Φ_k be of the form $\mathcal{U}_{ik}^{\mu} \otimes y_i^{\mu}$, $x_i^{\nu} \otimes V_{ik}^{\nu}$, where μ and ν stand for indices corresponding to the family Φ_k. We now select neighborhoods $\mathcal{U}_{i,k+1}^{\mu}$, $V_{i,k+1}^{\nu}$ with

$$\mathcal{U}_{i,k+1}^{\mu} + \mathcal{U}_{i,k+1}^{\mu} \subseteq \mathcal{U}_{ik}^{\mu}, \quad V_{i,k+1}^{\nu} + V_{i,k+1}^{\nu} \subseteq V_{ik}^{\nu} .$$

Define, for any couple $(\mu, \mu'), (\nu, \nu')$ of indices, the subsets of the form (3), where $\mathcal{U}_{i1}^{\mu}$, V_{i1}^{ν} and $\mathcal{U}_{i1}^{\mu'}$, $V_{i1}^{\nu'}$ are replaced by $\mathcal{U}_{i,k+1}^{\mu}$, $V_{i,k+1}^{\nu}$ and by $\mathcal{U}_{i,k+1}^{\mu'}$, $V_{i,k+1}^{\nu'}$. The union of all such subsets with respect to the couples (μ, μ'), (ν, ν') determines now Φ'_{k+1}.
Putting $\Phi_{k+1} = \Phi_k \cup \Phi'_{k+1}$, we obtain an increasing sequence $(\Phi_n ; n \in N)$ and, finally, we define

$$(4) \qquad \Phi_w = \bigcup_{n=1}^{\infty} \Phi_n .$$

Hence, we construct to any subset Φ_0 of the form (2) the set Φ_w (the last one may be called the additive envelope of Φ_0 in spite of a certain indeterminity consisting in the liberty of choices of $\mathcal{U}_{ik}^{\mu}$ and V_{ik}^{ν}). The system of all such Φ_w forms a base of neighborhoods of a group topology in $E \otimes F$. To prove this statement, it suffices to determine a subset Ψ_w of the form (4) satisfying, moreover, the inclusion $\Psi_w + \Psi_w \subseteq \Psi_w$ for any Φ_w. Namely, we may define the subset Ψ_0 by

$$\Psi_0 = \bigcup_{\mu,\nu} \bigcup_{n=1}^{\infty} (\mathcal{U}_{11}^{\mu} \otimes y_1^{\mu} + x_{11}^{\nu} \otimes V_{11}^{\nu} + \ldots + \mathcal{U}_{n1}^{\mu} \otimes y_{n1}^{\mu} + x_{n1}^{\nu} \otimes V_{n1}^{\nu})$$

and proceed in the same way as before (i.e., Ψ_k depends on the neighborhoods $\mathcal{U}^{\mu}_{i,k+1}$, $V^{\nu}_{i,k+1}$). Evidently, $\Psi_w = \bigcup \Psi_k$ satisfies the requested relation.

Theorem 4. The inductive tensor topology $T(\mathcal{Y})$ on $E \otimes F$ coincides with the group topology which has the subsets of the form (4) as a base of neighborhoods.

Proof. It suffices to prove that the group topology T defined above satisfies the properties 1^0 and 2^0 of Theorem 1. But 1^0 is clear and the proof of 2^0 may be carried out as in Theorem 2.

Remark 2. (a) Suppose that u_i $(i = 1, 2)$ in Remark 1 (a), are both open. Then $u_1 \otimes u_2$ is an open mapping from $E_1 \otimes F_1$ onto $E_2 \otimes F_2$ under the inductive tensor topologies. The proof follows from the construction of the neighborhoods Φ_w .

(b) Let E and F be two topological groups, M a subgroup in E , N a subgroup in F . Then the quotient group $E \otimes F / \Gamma(M, N)$ is topologically isomorphic with $(E/M) \otimes (F/N)$, where $\Gamma(M, N)$ stands for the subgroup in $E \otimes F$ generated by all $x \otimes y$, $x \in M$ or $y \in N$.

Remark 3. Suppose that $\{C_q(K), \partial_q\}$ is a chain complex, $C_q(K)$ a topological abelian group, $\partial_q : C_q(K) \longrightarrow$ $\longrightarrow C_{q-1}(K)$ a continuous boundary homomorphism. The category of all such chains complexes K will be denoted by $\partial \mathcal{G}$. A pair of mappings (φ, ψ) is termed to form a direct couple if the sequence

$$0 \longrightarrow K \xrightarrow{\varphi} K' \xrightarrow{\psi} K'' \longrightarrow 0$$

is exact, φ and ψ are open homomorphisms and $\varphi_n(C_n(K))$

possess topological complements in $C_m(K')$, $m \in N$, where K, K' and K'' are chain complexes.

Any such couple defines, in the same way as in [2] (see pp.126-127), a continuous boundary homomorphism ∂_* : $: H_m(K'') \longrightarrow H_{m-1}(K)$. Similarly as in Theorem 4.9 of [2] (cf.p.130), we obtain that the system $(H_q(K), f_*, \partial_*)$ is a homology theory on the $\mathcal{h}$-category $\partial\mathcal{G}$ of chain complexes with couples (with direct couples) into the category $\mathcal{G}$. Let $\otimes G$ be the functor on $\partial\mathcal{G}$ with direct couples defined by

$\otimes G: K \longrightarrow K \otimes G$ (with W or with $T(\mathcal{Y})$ -topology; cf. [11]),

$\otimes G: f \longrightarrow f \otimes 1_G$, where 1_G is the identical mapping of G. Then $\otimes G$ is (compare with Theorem 11.2 of [2],p. 150) a covariant $\mathcal{h}$-functor on $\partial\mathcal{G}$. Hence, $\otimes G$ defines a homology theory with the coefficient group G on $\partial\mathcal{G}$ (with the homology theory on $\partial\mathcal{G}$).

2. Inductive tensor topologies and vector spaces

In further discussion, we consider vector spaces over the same and usual field of scalars.

Let E and F be two fixed TVS´s. The class of all TVS´s will be denoted by $\mathcal{L}$. For any $G \in \mathcal{L}$ we mean by $\mathcal{S}(G)$ the family of all separately continuous bilinear mappings of $E \times F$ into G; $\mathcal{S}$ stands for the union of all systems $\mathcal{S}(G)$, $G \in \mathcal{L}$. The vector topology $T(\mathcal{S})$ on $E \otimes F$ generated by the family $\mathcal{S}$ will be termed the <u>inductive tensor topology</u> on $E \otimes F$

Theorem 5. The topology $T(\mathcal{S})$ is the unique vector topology on $E \otimes F$ with the properties:

1^0 The canonical bilinear mapping $\varphi : E \times F \longrightarrow E \otimes F$ is separately continuous.

2^0 For any separately continuous bilinear mapping u of $E \times F$ into a TVS G, its linear extension $\tilde{u}$ is continuous on $(E \otimes F, T(\mathcal{S}))$.

The proof is analogous to that of Theorem 1.

If $\mathcal{L}'$ is the class of all locally convex spaces, then $\mathcal{S}'$ means the union of all systems $\mathcal{S}(G)$, $G \in \mathcal{L}'$. It is immediate that the projectively generated topology $T(\mathcal{S}')$ by the system $\mathcal{S}'$ is locally convex and that $T(\mathcal{S}') \leq T(\mathcal{S})$ on $E \otimes F$.

The topology $T(\mathcal{S}')$ will be called the inductive tensor ℓ-topology on $E \otimes F$.

Theorem 6. (a) The inductive tensor ℓ-topology $T(\mathcal{S}')$ is the uniquely determined locally convex topology on $E \otimes F$ satisfying the property 1^0 of Theorem 5 and:

$2^{0\prime}$ For any $G \in \mathcal{L}'$ and for each $u \in \mathcal{S}'(G)$ the linear extension $\tilde{u}$ is continuous on $(E \otimes F, T(\mathcal{S}'))$.

(b) The topology $T(\mathcal{S}')$ is the finest locally convex topology t on $E \otimes F$ with $t \leq T(\mathcal{S})$.

(c) If E and F are locally convex, then $T(\mathcal{S}')$ coincides with the usual inductive tensor topology (in the sense of A. Grothendieck; cf.[3]) on $E \otimes F$.

(d) Let (E, t_1) and (F, t_2) be two TVS´s. Denote by t_1' and t_2' the finest locally convex topologies on E and F, respectively, satisfying $t_1' \leq t_1$ and $t_2' \leq t_2$.

If $E' = (E, t'_1)$, $F' = (F, t'_2)$, then the spaces $E' \otimes F'$ and $E \otimes F$ are topologically isomorphic under the inductive ℓ-topologies.

The proof is clear.

Remark 4. The tensor products $(E \otimes F, T(\mathcal{S}))$ and $(E \otimes F, T(\mathcal{S}'))$ have the same topological dual, but, in general, they need not be topologically isomorphic. It is easy to verify that, e.g., on the metrizable space of all measurable functions on a measure space (X, S, μ) (cf.[5], § 42), the topology $T(\mathcal{S}')$ is actually strictly coarser than $T(\mathcal{S})$.

Theorem 7. (a) For any TVS's E and F the topology $T(\mathcal{S})$ is the greatest lower bound of the group topologies $T(\nu, \mu)$ determined by the neighborhoods of the form (1).

(b) The topology $T(\mathcal{S})$ is defined as the vector topology on $E \otimes F$, E and F TVS's, having the subsets of the form (4) as a base of neighborhoods.

(c) The convex envelopes of all Φ_0 (see formula (2)) form the base of neighborhoods of the inductive tensor ℓ-topology $T(\mathcal{S}')$.

Proof. The subset Φ_0 (see formula (2)) is evidently absorbent and balanced whenever all U^{μ}_{i0}, V^{ν}_{i0} are balanced neighborhoods. The additive axiom is clear. This implies (c). The statement (b) follows from Theorem 4, because Φ_w is absorbent and balanced whenever all U^{μ}_{ik}, $V^{\nu}_{i,k}$ are of this kind. The statement (a) is clear.

Remark 5. If E and F are two TVS's, then the sequences μ and ν in (2), Section 1, may be selected from algebraic bases of the vector spaces E and F. In fact, any subset of the form (4) is absorbent in $E \otimes F$ in this case.

Theorem 8. If E and F have countable algebraic bases $(x_n ; n \in N)$ and $(y_n ; n \in N)$, then the subsets

$$(5) \quad \Omega((\mathcal{U}_n),(V_n)) = \bigcup_{n=1}^{\infty} (\mathcal{U}_1 \otimes y_1 + x_1 \otimes V_1 + \ldots$$
$$\ldots + \mathcal{U}_n \otimes y_n + x_n \otimes V_n) \;,$$

$\mathcal{U}_n$ neighborhoods in E and V_n neighborhoods in F, determine the inductive tensor topology $T(\mathcal{S})$.

The proof is elementary.

Corollary. Suppose that E and F are two locally convex spaces with countable algebraic bases. Then the inductive topology $T(\mathcal{S})$ coincides with the inductive $\mathcal{L}$-topology $T(\mathcal{S}')$.

Remark 6. We could also consider the tensor product $E \otimes F$ under one-sided topologies for any pair of TVS's. For symmetrical reasons we shall illustrate this idea only for the one-sided topology generated by the space E . Suppose further, without loss of generality and for simplification, that $(y_n ; n \in N)$ is a countable algebraic basis of F . Define a topology $T(E)$ by a base of neighborhoods

$$\Omega(\mathcal{U}_n) = \bigcup_{n=1}^{\infty} (\mathcal{U}_1 \otimes y_1 + \ldots + \mathcal{U}_n \otimes y_n) \;.$$

The following properties are now immediate:

(a) $T(E)$ is a vector topology on $E \otimes F$.

(b) $T(E)$ is separated whenever E is separated.

(c) If E' is a subspace of E , then $E' \otimes F$ is a topological vector subspace of $E \otimes F$.

(d) The topology $T(E)$ may be characterized by analogous properties as in Theorem 5.

3. Further properties of tensor topologies

The W -topology $T(\mathcal{B})$ (cf.[11]) on $E \otimes F$ (denoted also by $E \otimes_W F$), projectively generated by the system $\mathcal{B}$ of all continuous bilinear mappings from $E \times F$ into G, $G \in \mathcal{E}$, is, in general, coarser than $T(\mathcal{S})$ (denoted also by $E \otimes_s F$). It is not hard to show examples when $T(\mathcal{S})$ is strictly finer than $T(\mathcal{B})$. Take, e.g., a locally convex non-normable space E, F means the topological dual of E. Since the separately continuous bilinear function $(u, x) \longrightarrow \langle u, x \rangle$ is not continuous on $E \times F$, we may conclude (similarly as in [3]) that the tensor topologies in question are different on $E \otimes F$. On the other hand, if E and F are both metrizable, E M -barrelled, then any separately continuous bilinear mapping into an arbitrary TVS is continuous on $E \times F$ (cf.[12]). This implies:

Theorem 9. Let E and F be two metrizable TVS's, E M -barrelled. Then $T(\mathcal{B})$ and $T(\mathcal{S})$ coincide on $E \otimes F$.

Consider now two spectra $(E_n ; n \in N)$ and $(F_n ; n \in N)$ of TVS's with continuous injections i_n and j_n, respectively. Denote $E = \lim\ ind\ E_n$ and $F = \lim\ ind\ F_n$. Then it holds:

Theorem 10. Under the stated oonditions the inductive limit $\lim\ ind (E_n \otimes_s F_n)$ is topologically isomorphic with $E \otimes_s F$. If, in addition, all E_n and F_n are F -spaces, then $\lim\ ind (E_n \otimes_W F_n)$ is topologically isomorphic with $E \otimes_s F$.

Proof. Denote by I the identical mapping from $\lim\ \mathrm{ind}\,(E_n \otimes_s F_n)$ onto $E_n \otimes_s F$. Since the restriction of I to each $E_n \otimes_s F_n$ (equal to $i_n \otimes j_n$) is continuous, we derive from the definition of the inductive limit topology the continuity of I.

On the other hand, it suffices to prove the separate continuity of the bilinear mapping $(x, y) \longrightarrow x \otimes y$ from $E \times F$ into $\lim\ \mathrm{ind}\,(E_n \otimes_s F_n)$. Take therefore some $y \in F$, for a certain m we obtain $y \in F_m$. The restriction u_k of the linear mapping $u : x \longrightarrow x \otimes y$ to each E_k, $k \geq m$, is evidently continuous. But the same statement is true for the restriction of u to E_k, $1 \leq k < m$. This implies the continuity of u on E. Similarly, one proves the continuity of $v : y \longrightarrow x \otimes y$.

Remark 7. From the precedent theorem we can derive an example when the statement of Remark 1 (b) (for TVS's) need not be true for infinite topological products. Consider now the spaces over the field of real numbers and take for any $i \in N$ the real line $R_i = R$. Denoting by E the topological product $\prod_{i \in N} R_i$ and by F the topological direct sum $\sum_{i \in N} R_i$, we obtain, as in [3], from Theorem 10, since $\sum_{i \in N} R_i = \lim_n \mathrm{ind}\,(\sum_{i=1}^{n} R_i)$ that

$$E \otimes_s F = \lim_n \mathrm{ind}\,(E \otimes \sum_{i=1}^{n} R_i) = \lim_n \mathrm{ind}\,[\sum_{i=1}^{n} (E \otimes R_i)] = \sum_{i \in N} E_i ,$$

where E_i is topologically isomorphic with E for each $i \in N$. On the other hand, it holds $\prod_{i=1}^{\infty} (R_i \otimes F) = \prod_{i=1}^{\infty} F_i$, where F_i is topologically isomorphic with F for any $i \in N$.

One observes that $T(\mathcal{Y})$ is the finest vector topology on $E \otimes F$ for which $(x, y) \longrightarrow x \otimes y$ is separately continuous. Similarly, $T(\mathcal{B})$ is the maximal vector topology on $E \otimes F$ under which the canonical bilinear mapping $(x, y) \longrightarrow x \otimes y$ is continous. These properties suggest also the possibility of extending the theory of completions of tensor products to a particular case when the spaces in question are non-metrizable. Denote, therefore, by $\tau_w(E)$ (by $\tau'_w(E)$) the finest vector (the finest locally convex) topology on a vector space E. Putting $E_w = (E, \tau_w(E))$, $E'_w = (E, \tau'_w(E))$, we may now formulate in these terms:

<u>Theorem 11</u>. Let E and F be two vector spaces.

(a) For the topology $T(\mathcal{S})$ of $E_w \otimes F_w$ it holds

$$T(\mathcal{S}) = \tau_w(E \otimes F) .$$

(b) The space $E'_w \otimes F'_w$ under the topology $T(\mathcal{S}')$ is topologically isomorphic with $E \otimes F$ under $\tau'_w(E \otimes F)$.

(c) If, in particular, E and F possess countable algebraic bases, then the space $E_w \otimes F_w$ under $T(\mathcal{S})$ is topologically isomorphic with $E'_w \otimes F'_w$ under $T(\mathcal{S}')$.

(d) The space $E_w \otimes F_w$ under $T(\mathcal{S})$ is complete. Similarly, $E'_w \otimes F'_w$ under $T(\mathcal{S}')$ is complete.

<u>Proof</u>. For any $y \in F$ the inverse image of the topology $\tau_w(E \otimes F)$ defined by $u: x \longrightarrow x \otimes y$ is a vector topology on E coarser than $\tau_w(E)$ for which u is continuous. Hence u is continuous on E_w. For symmetrical reasons $y \longrightarrow x \otimes y$ is continuous on F_w for any $x \in E$. Thus $T(\mathcal{S}) = \tau_w(E \otimes F)$. Similarly, one proves (b).

If E and F are, moreover, with countable algebraic bases, then $E \otimes F$ possess also a countable algebraic basis, hence $\tau'_w(E \otimes F) = \tau_w(E \otimes F)$. From (a) and (b) we may now derive (c).

The finest vector topology of a vector space G coincides with the direct sum topology of a system of finite-dimensional F-spaces, hence G_w is complete (cf.[14]). This implies (d) by taking $G = E \otimes F$. The same arguments may be used in proving the second statement of (d).

Since the topology $T(\mathcal{Y})$ (and consequently, $T(\mathcal{Y}')$) is identical under the conditions of Theorem 11 (c) with the finest vector topology $\tau_w(E \otimes F)$, one is tempted to ask whether in this case also the locally convex projective topology $T(\mathcal{B}')$ (in the sense of A. Grothendieck) does not coincide with $\tau'_w(E \otimes F)$. If it were so, then we could obtain, in view of the above stated equality, that $E_w \otimes F_w$ is complete under $T(\mathcal{B}) = \tau_w(E \otimes F)$ (hence under $T(\mathcal{B}') = \tau'_w(E \otimes F)$). To prove our conjecture, it suffices to establish that, under the conditions of Theorem 11 (c), any separately continuous bilinear mapping of $E'_w \times F'_w$ into a locally convex space G is continuous.

Suppose therefore that u is a separately continuous mapping from $E'_w \times F'_w$ into a locally convex space G. If $y_i \in F$, $1 \leq i \leq n$, then for any neighborhood $\mathcal{U}$ in G there is a neighborhood V in E such that $u(x, y_i) \in \mathcal{U}$ for all $x \in V$, $1 \leq i \leq n$. This implies for a finite convex combination $\Sigma \lambda_i = 1$, $\lambda_i \geq 0$, the inclusion $u(x, \Sigma \lambda_i y_i) = \Sigma \lambda_i \cdot u(x, y_i) \in \mathcal{U}$.

for all $x \in V$. But, taking for y_i linearly independent points of $E_n = \sum_{i=1}^{n} R_i$ (the notation is the same as in Remark 7) we may observe that u is hypoequicontinuous in the sense of A. Grothendieck, i.e., with respect to the family of all bounded subsets in E and F. But E'_w and F'_w are DF-spaces. From the well-known theorem of A. Grothendieck (cf.[4]) we conclude that u is continuous on $E \times F$. Thus we proved:

Theorem 12. Let E and F be two vector spaces with countable algebraic bases. Then on $E_w \otimes F_w$ it holds

$$T(\mathcal{B}) = T(\mathcal{B}') = \tau'_w(E \otimes F) = \tau_w(E \otimes F) .$$

Especially, the space $E_w \otimes F_w$ under $T(\mathcal{B})$ (under $T(\mathcal{B}')$, respectively) is complete.

An open question concerning the topologies $T(\mathcal{S})$ and $T(\mathcal{B})$ is the separatedness of $E \otimes F$. Our hypothesis is that $E \otimes F$ under $T(\mathcal{B})$ (hence under $T(\mathcal{S})$) is separated whenever E and F are separated. In any case, if there are separated locally convex topologies coarser than the original on E and F, then $T(\mathcal{B})$ is separated on $E \otimes F$. Such situation occurs, e.g., for the spaces ℓ^p, $0 < p \leq 1$.

We shall now note an example which may serve as a prototype for the decision of the separatedness of $E \otimes F$ in certain cases.

Example. Suppose that E is separated and that $\mathcal{F} = \mathcal{F}(X)$ is a space of bounded scalar-valued functions on X with the topology of uniform convergence on a system $\mathcal{T} = \{B\}$ of subsets in X, $\bigcup B = X$. Denote further, by $\mathcal{F}_E(X)$, a vector space of vector-valued functions on X with values in E, bounded on each $B \in \mathcal{T}$ and such that $a \cdot f \in \mathcal{F}_E(X)$ for all $f \in \mathcal{F}(X)$, $a \in E$. The topology of $\mathcal{F}_E(X)$ is that of uniform convergence on $\mathcal{T}$. The bilinear mapping $u:(a, f) \to$ $\to a \cdot f$ of $E \times \mathcal{F}$ into $\mathcal{F}_E(X)$ is evidently continuous, hence it defines a linear continuous mapping $\tilde{u}$ of $E \otimes \mathcal{F}$ under $T(\mathcal{B})$ into $\mathcal{F}_E(X)$. Further, one may verify that $\tilde{u}$ is one-to-one (by Lemma 1.1 of [9]). Since $\mathcal{F}_E(X)$ is separated, the space $E \otimes \mathcal{F}$ under $T(\mathcal{B})$ is also separated.

4. The problem of topologies

As it was established in [11], the space $\mathcal{B}(E, F; G)$ of all continuous bilinear mappings from $E \times F$ into G is algebraically isomorphic with the space $\mathcal{L}(E \otimes F; G)$ of all continuous linear mappings from $E \otimes F$ into G. It is natural to ask under what conditions this algebraic isomorphism is a topological one ($\mathcal{B}(E, F; G)$ is taken with the topology of bi-bounded convergence; $\mathcal{L}(E \otimes F; G)$ is under the strong topology).

If E and F are locally convex, then any bounded set in $E \otimes_W F$ is also bounded under the projective locally convex topology. Consequently, the stated correspondence will be a topological isomorphism (G locally convex) whenever it is a topological isomorphism for the locally convex projective topology. Such situation occurs, e.g., if E and F are DF-spaces.

Recall that a TVS is said to be locally bounded if it contains a bounded neighborhood of the origin (cf.[7]). In [1] it was established that the topology of such a space is generated by a quasi-norm $x \longrightarrow \|x\|$ satisfying, especially, the Bourgin's multiplier property $\|x+y\| \leq \leq \beta(\|x\| + \|y\|)$, for a certain multiplier $\beta \geq 1$.

Theorem 13. Let E, F and G be locally bounded spaces. Then $\mathcal{B}(E, F; G)$ is topologically isomorphic with $\mathcal{L}(E \otimes F; G)$.

Proof. If A and B are bounded in E and F, then also $A \otimes B$ is bounded in $E \otimes F$, hence the canonical isomorphism of $\mathcal{L}(E \otimes F; G)$ onto $\mathcal{B}(E, F; G)$ is continuous.

Conversely, let C be a bounded subset in $E \otimes F$ and let W be a bounded neighborhood of the origin in G. It will be supposed that W is defined by means of a quasi-norm as the subset of all x in G with $\|x\| \leq 1$.

The bounded neighborhoods in E and F are denoted by $\mathcal{U}_0$ and V_0, respectively. Since C is bounded in $E \otimes F$, there is $\lambda > 0$ such that

$$\lambda C \subseteq \Omega = \bigcup_{k=1}^{\infty} \sum_{n=1}^{k} (\mathcal{U}_n \otimes V_n),$$

where $\mathcal{U}_n = \frac{1}{2^n} \cdot \mathcal{U}_0$, $V_n = \frac{1}{2^n} \cdot V_0$, $n \in N$ (cf.[11]). Further, we take a number m such that for the multiplier β of the quasi-norm $x \longrightarrow \|x\|$ it holds $\beta \leq 2^m$. Put now

$$W^* = W + \frac{1}{2^2} \cdot W + \frac{1}{2^4} \cdot W + \ldots + \frac{1}{2^{2m-2}} \cdot W .$$

Since W^* is bounded in G, we obtain $W^* \subseteq \alpha \cdot W$ for a fixed $\alpha \geq 1$. Consider now the set

$$W^{**} = \bigcup_{n=1}^{\infty} \sum_{k=0}^{n} \frac{1}{2^{2km}} \cdot W^{*} \quad .$$

Evidently, $u(\Omega) \subseteq W^{**}$ whenever the bilinear mapping u satisfies $u(U_0, V_0) \subseteq W$. By an elementary and immediate calculation one verifies that if $z_i \in W^*$, $1 \le i \le k$, k arbitrary, then

$$\left\| \frac{1}{2^{2m}} \cdot z_1 + \ldots + \frac{1}{2^{2km}} z_k \right\| \le \alpha \left(\frac{1}{2^m} + \frac{1}{2^{2m}} + \ldots + \frac{1}{2^{2(k-1)m}} \right) .$$

Hence, if $z \in W^{**}$, then $z = x + y$, $x \in W^*$, $y \in \sum_{i=1}^{k} \frac{1}{2^{2im}} \cdot W^*$ for some $k \in N$, consequently $\|z\| \le \beta(\|x\| + \|y\|) \le 2\alpha\beta$, i.e., $W^{**} \subseteq 2\alpha\beta \cdot W$.

Thus we have proved: To any $u \in \mathcal{B}(E, F; G)$ with $u(U_0, V_0) \subseteq \frac{\lambda}{2\alpha\beta} W$ the corresponding linear extension $\tilde{u}$ satisfies $\tilde{u}(C) \subseteq W$. This property ensures the continuity of the canonical mapping from $\mathcal{B}(E, F; G)$ into $\mathcal{L}(E \otimes F; G)$.

By carrying through the same type of arguments used in proving Theorem 13 and taking into account, for essential simplification of the procedure, that $W^{**} \subseteq 2 \cdot W$ whenever W is locally convex, one may prove the following statement:

<u>Theorem 14</u>. Let E and F be two locally bounded spaces, G locally convex. Then $\mathcal{B}(E, F; G)$ is topologically isomorphic with $\mathcal{L}(E \otimes F; G)$.

<u>Remark 8</u>. Consider now locally convex spaces and denote by $E \otimes_G F$ the tensor product $E \otimes F$ under the projective tensor topology in the sense of A. Grothendieck. The identical mapping of $\mathcal{L}(E \otimes_G F; G)$ onto

$\mathcal{L}(E \otimes_W F ; G)$ is evidently continuous. The space $\mathcal{L}(E \otimes_G F ; G)$ being isomorphic with $\mathcal{B}(E, F; G)$, is also topologically isomorphic with $\mathcal{L}(E \otimes_W F ; G)$ in the class of DF-spaces and in other special cases (e.g., it holds for $L^1(\mu) \otimes E$, E an F-space; cf.[3].

Hence, the following question is of some interest: Under what conditions $\mathcal{L}(E \otimes_W F ; G)$ and $\mathcal{L}(E \otimes_G F ; G)$ are topologically isomorphic?

5. Embedding theorems and concluding remarks

In conclusion, we note certain embedding theorems of tensor products into spaces of linear mappings and bilinear functions.

First, consider two TVS's E and F. Under E^* and F^* we shall understand the strong topological duals of E and F. If x is fixed in E, y in F, then

$$u_{x,y} : (f, g) \longrightarrow \langle f, x \rangle \cdot \langle g, y \rangle$$

is a continuous bilinear function on $E^* \times F^*$, hence an element of $\mathcal{B}(E^*, F^*)$; it will be assumed $\mathcal{B}(E^*, F^*)$ to be under the bi-bounded topology. One verifies immediately

Theorem 15. Suppose E and F to be locally bounded TVS's.

(a) The bilinear mapping $(x, y) \longrightarrow u_{x,y}$ induces a continuous linear mapping u of $(E \otimes F, T(\mathcal{B}))$ into $\mathcal{B}(E^*, F^*)$.
In particular, if (E, E^*) and (F, F^*) are dual pairs, then u is, moreover, an algebraic isomorphism.

(b) The spaces E^*, F^* are locally bounded and $\mathcal{B}(E^*, F^*)$ is topologically isomorphic with $(E^* \otimes_W F^*)^*$.

Remark 9. If we eliminate the hypothesis that E and F are locally bounded, then, similarly as in Theorem 15, we could embed $E \otimes F$ into the space $\mathcal{S}(E^*, F^*)$ of all separately continuous bilinear forms on $E^* \times F^*$. If $\mathcal{S}(E^*, F^*)$ is under the bi-equicontinuous topology (cf.[3]), then it is not hard to observe that the mapping $(x, y) \longrightarrow u_{x,y}$ is separately continuous, hence that its linear extension is also continuous on $E \otimes F$ under the topology $T(\mathcal{S})$.

Each $x \otimes y \in E \otimes F$ induces also a continuous linear mapping from E^* into F by formula

$$v_{x,y} : f \longrightarrow \langle x, f \rangle \cdot y \quad .$$

Similarly, to any $f \otimes y \in E^* \otimes F$ there corresponds a continuous linear mapping $w_{f,y} : x \longrightarrow \langle x , f \rangle \cdot y$ from E into F .

In these terms it is easy to establish:

Theorem 16. If E and F are locally bounded, then:

(a) the mapping $(x, y) \longrightarrow v_{x,y}$ induces a linear continuous mapping v from $E \otimes_W F$ into $\mathcal{L}(E^*, F)$;

(b) the mapping $(f, y) \longrightarrow w_{f,y}$ induces a linear continuous mapping w from $E^* \otimes_W F$ into $\mathcal{L}(E; F)$;

(c) if (E, E^*) is, moreover, a dual pair, then v and w are one-to-one.

Remark 10. Assuming $E^* \otimes_W F$ to be separated, we may form the completion $E^* \hat{\otimes}_W F$. If $\mathcal{L}(E; F)$ is now complete (E and F being locally bounded and separated), then there exists a continuous linear transformation of the nuclear type $\tilde{w}$ from $E^* \hat{\otimes}_w F$ into $\mathcal{L}(E;F)$. Note that $\mathcal{L}(E;F)$ is complete in this discussed case whenever F is complete (cf. [13]).

Remark 11. There are still other possibilities to topologize the tensor product $E \otimes F$; we could determine (as in locally convex spaces) topologies on $E \otimes F$ lying between $T(\mathcal{B})$ and $T(\mathcal{S})$. Such a topology may be defined, e.g., as that projectively generated by the system of all hypo-equicontinuous bilinear mappings on $E \times F$.

Remark 12. If E and F are both locally bounded, then it is easy to prove that $E \otimes F$ under $T(\mathcal{B})$, or under $T(\mathcal{S})$, is M-bornological. The author does not know whether or not the taking of such tensor products preserves the class of M-barrelled, or of M-bornological spaces.

References

[1] D.F. BOURGIN: Linear topological spaces. Amer.J.Math. 65(1943),637-659.

[2] S. EILENBERG, N. STEENROD: Foundations of Algebraic Topology. Princeton 1952.

[3] A. GROTHENDIECK: Produits tensoriels topologiques et espaces nucléaires. Mem.Amer.Math.Soc.16 (1955)

[4] A. GROTHENDIECK: Sur les espaces F et DF . Summa Bras. Math.3(1954),57-123.

[5] P.R. HALMOS: Measure Theory. Princeton 1950.

[6] K.H. HOFMANN: Tensorprodukte lokal kompakter abelscher Gruppen. J.reine angew.Math.216(1964), 134-149.

[7] D.H. HYERS: Locally bounded linear topological spaces. Revista de Ciencias 41(1939),555-574.

[8] G. KÖTHE: Topologische lineare Räume.I. Springer,Berlin 1960.

[9] R. SCHATTEN: A Theory of Croos-Spaces. Princeton 1950.

[10] S. TOMÁŠEK: On tensor products of Abelian groups. Comm. Math.Univ.Carolinae 6(1965),73-83.

[11] S. TOMÁŠEK: Some remarks on tensor products. Comment. Math.Univ.Carolinae 6(1965), 85-96.

[12] S. TOMÁŠEK: M-barrelled spaces (submitted to Comm.Math. Univ.Carolinae).

[13] S. TOMÁŠEK: M-bornological spaces (ibid.).

[14] S. TOMÁŠEK: Spectra of topological vector spaces. I. (To be published later.)

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