Commentationes Mathematicae Universitatis Carolinae

Karel Najzar

Error-estimates for the Ritz's method of finding eigenvalues and eigenfunctions

Commentationes Mathematicae Universitatis Carolinae, Vol. 12 (1971), No. 3, 485--501

Persistent URL: http://dml.cz/dmlcz/105360

Commentationes Mathematicae Universitatis Carolinae

12,3 (1971)

ERROR - ESTIMATES FOR THE RITZ' S METHOD OF FINDING EIGENVALUES AND EIGENFUNCTIONS

K. NAJZAR, Praha

In [1] - [3] , we studied the method of least squares for approximating the eigenvalues and the eigenfunctions of a DS-operator. In this paper, we present a priori error-estimates for the Ritz's method for eigenvalue problems. Upper and lower error bounds are found.

We assume throughout this paper that A is a DS-operator whose domain $\mathcal{D}(A)$ is dense in a separable Hilbert space H, i.e., A is a symmetric operator in H such that the set of its eigenvalues is of the first category on the real axis and the spectrum $\sigma(A)$ is the closure of this set.

Suppose A is bounded below and such that the eigenvalues $\{\lambda_i\}$ of A satisfy the relations

(1) $$\lambda_1 < \lambda_2 < \dots < \lambda_j < \dots \quad .$$

Let H_i be the closure of linear manifold generated by the eigenfunctions of A associated with the eigenva-

---

AMS classification: Ref.Ž. 7.978.8
Primary 49G05, 49G10 7.962.5
Secondary 65J05, 65L15

lue λ_i. The symbol $P_i u$ will be used to denote the orthogonal projection u on H_i. We introduce the operator $B = (A - \mu I)^{\frac{1}{2}}$, where I denotes the identity operator in H and μ is a real number such that $\mu < \lambda_1$. It is evident that $\mu_i = \sqrt{\lambda_i - \mu}$, $i = 1, 2, \dots$ are the eigenvalues of B and H_i is the closure of the linear manifold generated by the eigenfunctions of B associated with the eigenvalue μ_i. Therefore B is DS-operator. We remark that

$$\mathcal{D}(B) = \{ u \in H \,/ \sum_{i=1}^{\infty} \mu_i^2 \, \| P_i u \|^2 < \infty \} \supset \mathcal{D}(A)$$

and $Bu = \sum_{i=1}^{\infty} \mu_i P_i u$ for each $u \in \mathcal{D}(B)$.

Let $\{\Psi_i\}_{i=1}^{\infty}$ be a B-complete system (cf.[6],[1]) and let R_m and $\mathcal{R}_m$ be subspaces of H determined by functions $\{\Psi_i\}_{i=1}^{m}$ and $\{B\Psi_i\}_{i=1}^{m}$, respectively. Denote

$$Q_m^{(m)} = \inf_{\substack{u \in R_m \cap H_m^{(2)}, \\ \|u\| = 1}} (Au - \mu u, u),$$

where $H_m^{(1)} = \sum_{i=1}^{m-1} \oplus H_i$, $H_m^{(2)} = H \ominus H_m^{(1)}$ for $m > 1$ and $H_1^{(1)} = \{\theta\}$, $H_1^{(2)} = H$.

Then the sequence $\{Q_m^{(m)}\}_{m=1}^{\infty}$ is monotonically decreasing and converging to $\lambda_m - \mu$ (cf. Theorem 5 of [1]).

Let φ_1 be a normalized eigenfunction corresponding to μ_1. Let us construct the sequence of numbers $\{q_m\}_{m=1}^{\infty}$ such that $q_m = \min_{\substack{u \in R_m \\ \|u\| = 1}} \|Bu\|$. By Lemma 3 and Remark 3 of [3] we have for $m \geq m_0$

(2) $$q_m^2 - \mu_1^2 \leq C_1 \cdot \| \varphi_1 - {}^{(n)}\varphi_1 \|^2 ,$$

where $C_1 = (\lambda_1 - \mu) \cdot (1 - \| \varphi_1 - {}^{(n)}\varphi_1 \|)^{-2}$,

${}^{(n)}\varphi_1$ is the orthogonal projection of φ_1 on $\mathcal{L}\{B\Psi_i\}_{i=1}^n$ and n_0 is a positive integer such that ${}^{(n_0)}\varphi_1 \neq 0$ and ${}^{(n_0-1)}\varphi_1 = 0$. If μ_1 is a simple eigenvalue of B , then

(3) $$q_m^2 - \mu_1^2 \geq C_2 \cdot \| \varphi_1 - {}^{(n)}\varphi_1 \|^2 ,$$
$$C_2 = \mu_1^2 \cdot (\mu_2^2 - \mu_1^2) \cdot (\mu_2^2 + \mu_1^2)^{-1}$$

and there exists $\{u_m\}_{m=1}^\infty$ such that the following conditions are satisfied:

(4)
1) $u_m \in R_m$, $\| u_m \| = 1$,
2) $\| B u_m \| = q_m$,
3) $\lim\limits_{m \to \infty} u_m = \varphi_1$,
4) $(u_m, \varphi_1) \geq 0$ for $m = 1, 2, \dots$.

The proof is similar to that of Theorem 1 in [3] and Theorem 3 in [2]. Further there exist constants C_3 and C_4 such that for $n \geq n_0$ we have

(5)
$$\mu_1 \cdot \| \varphi_1 - {}^{(n)}\varphi_1 \| \leq \| B u_m - B \varphi_1 \| \leq C_3 \cdot \| \varphi_1 - {}^{(n)}\varphi_1 \|$$
$$\| \varphi_1 - \varphi_1^{(n)} \| \leq \| u_m - \varphi_1 \| \leq C_4 \cdot \| \varphi_1 - {}^{(n)}\varphi_1 \| ,$$

where $\varphi_1^{(n)}$ is the orthogonal projection of φ_1 on R_m (cf. Theorem 2 of [3]).

1. In this section, we give upper and lower bounds for $Q_m^{(1)} - \lambda_1$ and for $\| u_m - \varphi_1 \|_1$, where $\| u \|_1^2 = (Au - \mu \cdot u, u)$, respectively.

Since μ is smaller than λ_1, the bilinear form defined by $(u, v)_1 = (Au - \mu u, v)$, u, v in $\mathcal{D}(A)$, is a scalar product. Denote by $\mathcal{H}_1$ the complete hull of $\mathcal{D}(A)$ with the norm $\| u \|_1 = \sqrt{(u, u)_1}$. Let $\{\Psi_i\}_{i=1}^{\infty}$ be a complete system in $\mathcal{H}_1$ and let $\lambda_1^{(n)}$ be an approximation to λ_1 obtained by applying the Ritz's method to the subspace $R_m = \mathcal{L}\{\Psi_i\}_{i=1}^{n}$ of $\mathcal{H}_1$. It follows from the definition of B, that $\lambda_1^{(n)} = Q_n^{(1)} + \mu = Q_m^2 + \mu$. If, in addition, λ_1 is a simple eigenvalue of A, then $\{u_m\}_{m=1}^{\infty}$ has the following properties

$$
\begin{aligned}
& 1) \quad u_m \in R_m, \quad \| u_m \| = 1 , \\
(6) \qquad & 2) \quad \| u_m \|_1^2 = \lambda_1^{(n)} - \mu , \\
& 3) \quad (u_m, u_{m+1}) \geq 0 .
\end{aligned}
$$

Therefore u_m is an approximation to φ_1 obtained by applying the Ritz's method to R_m.

Since

$$
(7) \quad \| \varphi_1 - {}^{(n)}\varphi_1 \|^2 = \inf_{u \in R_m} \| \varphi_1 - B u \|^2 = \frac{1}{\lambda_1 - \mu} \cdot \inf_{v \in R_m} \| \varphi_1 - v \|_1^2
$$

and $\| B u_m - B \varphi_1 \| = \| u_m - \varphi_1 \|_1$,

the following theorem is a direct consequence of (2) - (7).

<u>Theorem 1</u>. Let A be a DS-operator which is bounded below. Let $\lambda_1 < \lambda_2 < \ldots$ be an enumeration of its dis-

tinct eigenvalues increasing order of values and let μ be such a number that $\mu < \lambda_1$. Let $\{\Psi_i\}_{i=1}^{\infty}$ be a complete system in $\mathcal{H}_1$. Denote by $\lambda_1^{(n)}$ an approximation to λ_1 obtained by applying the Ritz's method to subspace $R_m = \mathcal{L}\{\Psi_i\}_{i=1}^{m}$ of $\mathcal{H}_1$. Then there exists a positive number D which does not depend on m such that for $n \geq n_0$

$$\lambda_1^{(n)} - \lambda_1 \leq D \cdot \inf_{\substack{\varphi \in H_1 \\ \|\varphi\| = 1}} \left(\inf_{u \in R_m} \| \varphi - u \|_1^2 \right) ,$$

where H_1 is the closure of linear manifold generated by the eigenfunctions of A associated with the eigenvalue λ_1 and

$$n_0 = \inf_{u \in H_1} \max \{ n \mid (u, v)_1 = 0,\ v \in R_m \} .$$

If, in addition, λ_1 is a simple eigenvalue of A, then there exist constants $D_1, D_2 \neq 0, D_3, D_4$ which do not depend on m such that

$$D_2 \cdot (E_1^{(n)})^2 \leq \lambda_1^{(n)} - \lambda_1 \leq D_1 \cdot (E_1^{(n)})^2 ,$$

$$E_1^{(n)} \leq \| u_m - \varphi_1 \|_1 \leq D_3 \cdot E_1^{(n)} ,$$

$$\| u_m - \varphi_1 \| \leq D_4 \cdot E_1^{(n)} ,$$

where u_m is the Ritz's approximation to a normalized eigenfunction φ_1 with the properties (6) and $E_1^{(n)}$ is the error of the best approximation to φ_1 by functions of R_m in the norm $\| u \|_1$, i.e. $E_1^{(n)} = \inf_{v \in R_m} \| \varphi_1 - v \|_1$.

<u>Remark 1</u>. It follows from Lemma 3, Theorems 1 and 2 of [3] and from (7) that Theorem 1 is also valid when we

replace D_i by D_i^* for $i = 1,2,3,4$, where

$$D_1^* = (1 - v_n)^{-2}, \quad v_n = (\lambda_1 - \mu)^{-\frac{1}{2}} \cdot E_1^{(n)} ,$$

$$D_2^* = \left(1 + 2 \cdot \frac{\lambda_1 - \mu}{\lambda_2 - \lambda_1}\right)^{-1} ,$$

$$D_3^* = (D_1^* / D_2^*)^{\frac{1}{2}} ,$$

$$D_4^* = [2 D_1^* / (\lambda_2 - \lambda_1)]^{\frac{1}{2}} .$$

Consequently $\lim_{n \to \infty} D_1^* = 1$.

2. In this section, we derive upper and lower bounds for the errors of Ritz's approximation to λ_j and φ_j , $j > 1$. For simplicity we assume that

(8) $\lambda_1 < \lambda_2 < \ldots < \lambda_j < \lambda_{j+1}$ and $\lambda_i, i = 1, \ldots, j$ are simple eigenvalues of A .

Let φ_i be a normalized eigenfunction of A corresponding to the eigenvalue λ_i for $i = 1, \ldots, j$.

We now present a number of results which is useful to have on record for later use. First of all we consider the problem of approximating the eigenfunctions of B .

Lemma 1. With assumption (8), let $\{v_n\}_{n=1}^{\infty}$ be a sequence of normalized functions belonging to $\mathfrak{D}(B)$ such that $\lim_{n \to \infty} \| B v_n \| = \mu_j$ and $\lim_{n \to \infty} (v_n, \varphi_i) = 0$ for $i = 1, \ldots, (j-1)$. Then there exists a convergent subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ such that its limit is an eigenfunction of B belonging to μ_j .

Proof: By direct computation it follows that

$$\|Bv_n\|^2 - \mu_j^2 \geq (\mu_{j+1}^2 - \mu_j^2) \cdot \sum_{i=j+1}^{\infty} \|P_i v_n\|^2 + \sum_{i=1}^{j-1} (\mu_i^2 - \mu_j^2) \cdot \|P_i v_n\|^2 .$$

Since $\lim_{n\to\infty} \|P_i v_n\| = \lim_{n\to\infty} |(v_n, \varphi_i)| = 0$ for $i = 1, \dots, (j-1)$ and $\lim_{n\to\infty} \|Bv_n\| = \mu_j$ we have $\lim_{n\to\infty} |(v_n, \varphi_j)| = 1$ and the remainder of this lemma may be proved in the same way as Lemma 2 in [2].

The following theorem is a generalization of Theorem 3 from [2] and gives information on the construction of the approximation to φ_j .

Theorem 2. With the assumptions of Theorem 1 and (8) construct the sequences $\{u_n^{(i)}\}_{n=1}^{\infty}$, $i = 1, \dots, j$ with the following properties

1) $$q_n^{(i)} = \|Bu_n^{(i)}\| = \min_{\substack{u \in R_n \\ (u, u_n^{(k)}) = 0,\ k = 1, \dots, (i-1) \\ \|u\| = 1}}$$

for $i = 2, \dots, j$; $n \geq i$;

$$q_n^{(1)} = \|Bu_n^{(1)}\| = \min_{u \in R_n, \|u\|=1} \|Bu\| ,$$

2) $\|u_n^{(i)}\| = 1$, $i = 1, \dots, j$; $n \geq i$,

3) $(u_n^{(i)}, u_{n+1}^{(i)}) \geq 0$, $i = 1, \dots, j$; $n \geq i$.

Then

a) The sequence $\{u_n^{(i)}\}_{n=1}^{\infty}$ converges to a normalized eigenfunction φ_i of B associated with the eigenvalue $\mu_i = \sqrt{\lambda_i - \mu}$ for $i = 1, \dots, j$,

b) $(Bu_n^{(i)}, Bu_n^{(k)}) = 0$ for $i \neq k$; $i, k = 1, \dots, j$.

__Proof__: In case $j = 1$ the proof follows from Theorem 3 and Lemma 3 of [2]. We now proceed inductively. Let the theorem be true for $i = 1, \dots, (j-1)$. It then follows, since $\lim_{n \to \infty} \| u_n^{(i)} - \varphi_i \| = 0$ for $i = 1, \dots, (j-1)$, that $\lim_{n \to \infty} \alpha_n^{(i)} = 0$ for $i = 1, \dots, (j-1)$, where $\alpha_n^{(i)} = (u_n^{(j)}, \varphi_i) = (u_n^{(j)}, \varphi_i - u_n^{(i)})$.
Now, if $j^{-1} > \varepsilon > 0$, there exists a positive integer n_0 such that for $n \geq n_0$ $\| B y \| \leq q_n^{(j)} + \varepsilon$, where

$$y = \frac{u_n^{(j)} - \sum_{k=1}^{j-1} \alpha_n^{(k)} \varphi_k}{\| u_n^{(j)} - \sum_{k=1}^{j-1} \alpha_n^{(k)} \varphi_k \|} .$$

Since $y \perp \mathcal{L} \{ \varphi_i \}_{i=1}^{j-1}$, we have $\| B y \| \geq (Q_n^{(j)})^{\frac{1}{2}}$.
Therefore

$$(9) \qquad \lim_{n \to \infty} q_n^{(j)} \geq \lim_{n \to \infty} (Q_n^{(j)})^{\frac{1}{2}} = \mu_j .$$

On the other hand, let us construct normalized functions $v_n \in R_n \cap H_j^{(2)}$, $n \geq j$ so that $\| B v_n \| = (Q_n^{(j)})^{\frac{1}{2}}$.
Let $\beta_n^{(i)} = (v_n, u_n^{(i)}) = (v_n, u_n^{(i)} - \varphi_i)$.
Since $\lim_{n \to \infty} \beta_n^{(i)} = 0$ for $i = 1, \dots, (j-1)$, there exists a positive integer n_1 such that for $j^{-1} > \varepsilon > 0$

$$\| B w \| \leq (Q_n^{(j)})^{\frac{1}{2}} + \varepsilon \qquad \text{for } n \geq n_1 ,$$

where $w = v_n - \sum_{i=1}^{j-1} \beta_n^{(i)} u_n^{(i)}$.

Since $w \perp \mathcal{L} \{ u_n^{(i)} \}_{i=1}^{(j-1)}$, it follows that $q_n^{(j)} \leq$ $\leq \| B w \|$ and hence

(10) $$\lim_{n\to\infty} q_n^{(j)} \leq \lim_{n\to\infty} (Q_n^{(j)})^{\frac{1}{2}} = \mu_j .$$

Therefore, we see by (9) that $\lim_{n\to\infty} q_n^{(j)} = \mu_j$. Since $\lim_{n\to\infty} (u_n^{(j)}, \varphi_i) = 0$ for $i = 1, \dots, (j-1)$, we know by Lemma 1 that the sequence $\{u_n^{(j)}\}_{n=1}^{\infty}$ contains a convergent subsequence and that every its convergent subsequence converges to a limit which is a normalized eigenfunction of B associated with the eigenvalue μ_j. It follows that the sequence $\{u_n^{(j)}\}_{n=1}^{\infty}$ contains at most two accumulation points. These points are φ_j and $(-\varphi_j)$, where φ_j is a normalized eigenfunction of B associated with the eigenvalue μ_j. The remainder of the proof is similar to that in the proof of Theorem 3 in [2].

The assumption 3) implies that $\{u_n^{(j)}\}_{n=1}^{\infty}$ has one accumulation point. It then follows, by virtue of Lemma 1, that the sequence $\{u_n^{(j)}\}_{n=1}^{\infty}$ is convergent and so the first part of this theorem is proved.

To prove b), let $\{x_k\}_{k=1}^{n+1-i}$ be an orthonormal basis of the space $X = \{u \,/\, u \in R_n, (u, u_n^{(l)}) = 0$ for $l = 1, \dots, (i-1)\}$. Let $u_n^{(i)} = \sum_{k=1}^{n+1-i} \alpha_k x_k$.

Using the definition of $u_n^{(i)}$, we have

$$\sum_{l=1}^{n+1-i} \alpha_l \,[\,(Bx_k, Bx_l) - \delta_{lk} \cdot (q_n^{(i)})^2\,] = 0$$

for $k = 1, \dots, (n+1-i)$

and hence

$$(Bu_n^{(i)}, Bx_k) = (q_n^{(i)})^2 \cdot \alpha_k , \quad k = 1, \dots, (n+1-i).$$

By direct calculation we see that

$$(Bu_n^{(i)}, Bv) = (q_n^{(i)})^2 \cdot (u_n^{(i)}, v) \quad \text{for any } v \in X$$

and this completes the proof of b). Theorem 2 is completely proved.

The following theorem states how accurately the eigenfunction φ_j and the eigenvalue μ_j of B can be approximated by functions of $\mathcal{L}\{\Psi_i\}_{i=1}^{n}$ for $n \geq j$.

<u>Theorem 3.</u> Under the same hypotheses as in Theorem 2 there exist a positive integer n_0 and the constants $C_1, C_2 \neq 0, C_3, C_4$ which are independent of n such that for $n \geq n_0$

(a) $C_2 \cdot \|\varphi_j - {}^{(n)}\varphi_j\|^2 \leq Q_n^{(j)} - \mu_j \leq C_1 \cdot \max_{i=1,\dots,j} \|\varphi_i - {}^{(n)}\varphi_i\|^2$,

(b) $\mu_j \cdot \|\varphi_j - {}^{(n)}\varphi_j\| \leq \|Bu_n^{(j)} - B\varphi_j\| \leq C_3 \cdot \max_{i=1,\dots,j} \|\varphi_i - {}^{(n)}\varphi_i\|$,

(c) $\|u_n^{(j)} - \varphi_j\| \leq C_4 \cdot \max_{i=1,\dots,j} \|\varphi_i - {}^{(n)}\varphi_i\|$,

where ${}^{(n)}\varphi_i$ is the orthogonal projection of φ_i on

$\mathcal{R}_n = \mathcal{L}\{B\Psi_k\}_{k=1}^{n}$ for $i = 1, \dots, j$.

<u>Proof</u>: We proceed by induction. For $j = 1$ the statement follows from Theorems 1 and 2 of [3]. We now define T as the restriction of B to $R_n = \mathcal{L}\{\Psi_k\}_{k=1}^{n}$. Since $0 \bar{\in} \sigma(B)$, it follows that T and T^{-1} are continuous linear operators on R_n and $\mathcal{R}_n = \mathcal{L}\{B\Psi_k\}_{k=1}^{n}$, respectively. In a similar way, by methods analogous to those employed in the proof of Lemma 1 from [3], we can obtain

$$\|{}^{(n)}\varphi_j\|^2 - \mu_j^2 \cdot \|T^{-1}\,{}^{(n)}\varphi_j\|^2 \leq \|\varphi_j - {}^{(n)}\varphi_j\|^2$$

for each positive integer n .

Let

$$u = T^{-1}\,{}^{(n)}\varphi_j + \sum_{i=1}^{j-1} c_i u_n^{(i)} ,$$

where

$$c_i = -(T^{-1}\,{}^{(n)}\varphi_j , u_n^{(i)}) .$$

Then $(u, u_n^{(i)}) = 0$ for $i = 1, \ldots, (j-1)$ and from Theorem 2 it follows that

$$\| u \|^2 = \| T^{-1}\,{}^{(n)}\varphi_j \|^2 - \sum_{i=1}^{j-1} c_i^2 .$$

It is easy to see that

$$c_i = -(T^{-1}\,{}^{(n)}\varphi_j , u_n^{(i)} - \varphi_i) - \frac{1}{\mu_i}({}^{(n)}\varphi_j , \varphi_i), \; i = 1, \ldots, (j-1).$$

Since

$$\| T^{-1}\,{}^{(n)}\varphi_j \| \leq \frac{1}{\mu_1} \cdot \| {}^{(n)}\varphi_j \| \leq \frac{1}{\mu_1}$$

and $|({}^{(n)}\varphi_j , \varphi_i)| \leq \| \varphi_j - {}^{(n)}\varphi_j \|$ for $i = 1, \ldots, (j-1)$,

we conclude by induction that

(11) $$|c_i| \leq C \cdot \ell_j \quad \text{for } i = 1, \ldots, (j-1),$$

where $\ell_j = \max_{i=1,\ldots,j} \| \varphi_i - {}^{(n)}\varphi_i \|$ and C is a constant which does not depend on n .

But $\lim_{n\to\infty} \ell_j = 0$ and hence there exists for $0 < \varepsilon < \mu_j^{-1}$ a positive integer n_0 such that $\| u \| \geq \frac{1}{\mu_j} - \varepsilon > 0$.

By the definition of $q_n^{(j)}$, we have

(12) $$q_n^{(j)} - \mu_j \leq C_n \cdot (\| Bu \|^2 - \mu_j^2 \cdot \| u \|^2) ,$$

where

$$C_n = \frac{1}{\| u \| \cdot (\| Bu \| + \mu_j \cdot \| u \|)}$$

By direct calculation we see that

(13) $$\lim_{m \to \infty} C_m = \frac{1}{2} \mu_j .$$

We have

(14) $$\begin{aligned} \| Bu \|^2 - \mu_j^2 \cdot \| u \|^2 = (\| {}^{(m)}\varphi_j \|^2 - \mu_j^2 \cdot \| T^{-1}\, {}^{(m)}\varphi_j \|^2) + \\ + \sum_{i=1}^{j-1} c_i^2 \cdot [(q_m^{(i)})^2 + \mu_j^2] + 2 \cdot \sum_{i=1}^{j-1} c_i \cdot ({}^{(m)}\varphi_j , B u_m^{(i)}), \end{aligned}$$

by Theorem 2.

Further, by induction

(15) $$\begin{aligned} | ({}^{(m)}\varphi_j , B u_m^{(i)}) | = | ({}^{(m)}\varphi_j , B u_m^{(i)} - B\varphi_i) | + \\ + | ({}^{(m)}\varphi_j - \varphi_j , B \varphi_i) | \leq D \cdot e_j \end{aligned}$$

for $i = 1, \ldots, (j-1)$, where D is a constant which does not depend on m. On the basis of (11) - (15)

$$q_m^{(j)} - \mu_j \leq C_1 \cdot e_j^2 \quad \text{for} \quad m \geq m_0 .$$

For proving (b) we remark that

(16) $$\| B u_m^{(j)} - B \varphi_j \|^2 = (q_m^{(j)})^2 - \mu_j^2 + 2 \mu_j^2 \cdot (1 - \alpha_j^{(j)}) ,$$

where $\alpha_j^{(j)} = (u_m^{(j)}, \varphi_j)$. By Theorem 2 it follows that $1 \geq \alpha_j^{(j)} \geq 0$, whence $1 - \alpha_j^{(j)} \leq 1 - (\alpha_j^{(j)})^2$.

Writing $u_m^{(j)} = \sum_{i=1}^{\infty} \alpha_i^{(j)} \cdot \varphi_i$, $\alpha_i^{(j)} = (u_m^{(j)}, \varphi_i)$, we have

$$(q_m^{(j)})^2 - \mu_j^2 \geq \mu_{j+1}^2 - \mu_j^2 - \sum_{i=1}^{j} (\alpha_i^{(j)})^2 \cdot (\mu_{j+1}^2 - \mu_i^2),$$

whence

(17) $$\begin{aligned} 1 - (\alpha_j^{(j)})^2 \leq \frac{1}{\mu_{j+1}^2 - \mu_j^2} \cdot ((q_m^{(j)}) - \mu_j^2) + \\ + \frac{\mu_{j+1}^2 - \mu_1^2}{\mu_{j+1}^2 - \mu_j^2} \cdot \sum_{i=1}^{j-1} (\alpha_i^{(j)})^2 . \end{aligned}$$

We now show that an estimate for $\sum_{i=1}^{j-1}(\alpha_i^{(j)})^2$ leads to an estimate for $(Q_m^{(j)})^2 - \mu_j^2$. Let us consider the matrix $\mathcal{A} = \{\alpha_k^{(i)}\}_{k,i=1,\dots,(j-1)}$. By the induction we have $\lim_{m\to\infty} \alpha_k^{(i)} = \delta_{k,i}$. From this it follows that there exists for $0 < \varepsilon < 1$ a positive integer m_1 such that for $m \geq m_1$

(18) $|\mathcal{A}^{-1}| < 1 + \varepsilon$ and $|\mathcal{A}^{-1}u| \geq (1-\varepsilon)\cdot|u|$,

where $|\cdot|$ denotes the Euclidean norm.

Define for $m \geq m_1$

(19) $$\kappa = \mathcal{A}^{-1}\cdot\alpha^{(j)} ,$$

where $\alpha^{(j)} = \{\alpha_k^{(j)}\}_{k=1}^{j-1}$ and $\kappa = \{\kappa_i\}_{i=1}^{j-1}$.

We introduce $p_j^2 = \dfrac{1}{1+|\kappa|^2}$, $p_i = -\kappa_i\cdot p_j$ for $i = 1,\dots,(j-1)$. Then

(20) $$\sum_{i=1}^{j} p_i^2 = 1 \quad \text{and} \quad \sum_{i=1}^{j-1} p_i^2 = \frac{|\kappa|^2}{1+|\kappa|^2} .$$

It follows from (18) - (20) that

(21) $$\sum_{i=1}^{j-1} p_i^2 \geq C\cdot\sum_{i=1}^{j-1}(\alpha_i^{(j)})^2 \qquad \text{for } m \geq m_0 ,$$

where
$$C = \frac{(1-\varepsilon)^2}{1+(1+\varepsilon)^2} > 0 .$$

Letting $v = \sum_{i=1}^{j} p_i u_m^{(i)}$ we find $\|v\| = 1$ and $(v, \varphi_i) = 0$ for $i = 1,\dots,(j-1)$. Hence, by the definition of $Q_m^{(j)}$,

$$\|Bv\|^2 \geq Q_m^{(j)} \geq \mu_j^2$$

and using Theorem 2 and (21) we find for $n \geq n_1$

$$(q_n^{(j)})^2 - \mu_j^2 \geq (q_n^{(j)})^2 - \| Bv \|^2 =$$

$$= \sum_{i=1}^{j-1} [(q_n^{(j)})^2 - (q_n^{(i)})^2] \cdot r_i^2 \geq D \cdot \sum_{i=1}^{j-1} (\alpha_i^{(j)})^2 , \tag{22}$$

where D is a constant which does not depend on n. It follows from (16),(17) and (22) that there exists a constant $C > 0$ such that for $n \geq n_1$

$$\| B u_n^{(j)} - B \varphi_j \|^2 \leq C \cdot [(q_n^{(j)})^2 - \mu_j^2] . \tag{23}$$

This, together with (a), leads to the first assertion of (b).

It is easy to see that

$$\| B u_n^{(j)} - B \varphi_j \|^2 = \mu_j^2 \cdot \| \frac{1}{\mu_j} \cdot B u_n^{(j)} - \varphi_j \|^2 \geq \mu_j^2 \cdot \| \varphi_j - {}^{(n)}\varphi_j \|^2 \tag{24}$$

and this completes the second assertion of (b).

Since $\| B u_n^{(j)} - B \varphi_j \| \geq \mu_1 \cdot \| u_n^{(j)} - \varphi_j \|$, the right side of (c) follows at once from (b). The left side of (a) follows from (23) and (24) and this completes the induction and the proof of Theorem 3.

As a corollary to Theorems 2 and 3, we obtain the main result of this paper.

<u>Theorem 4.</u> Let A be a DS-operator which is bounded below. Let $\lambda_1 < \lambda_2 < \dots$ be an enumeration of its distinct eigenvalues increasing order of values and let μ be such a number that $\mu < \lambda_1$. Suppose λ_i, $i = 1, \dots, j$ are simple. Denote by $(u, v)_1$ the scalar product $(Au - \mu u, v)$. Let $\mathcal{H}_1$ be the complete hull of $\mathcal{D}(A)$ with the norm $\| u \|_1$.

Let $\{\Psi_i\}_{i=1}^{\infty}$ be a complete system in $\mathcal{H}_1$. Construct the sequences $\{u_n^{(i)}\}_{n=1}^{\infty}$, $i = 1,\dots,j$ with the following properties

1) $$\min_{\substack{u \in \mathcal{L}\{\Psi_k\}_{k=1}^n \\ \|u\|=1}} (Au, u) = (Au_n^{(1)}, u_n^{(1)}) = \lambda_n^{(1)} ,$$

$$\min_{\substack{u \in \mathcal{L}\{\Psi_k\}_{k=1}^n \\ (u,u_n^{(k)}) = 0,\ k=1,\dots,(i-1) \\ \|u\|=1}} (Au, u) = (Au_n^{(i)}, u_n^{(i)}) = \lambda_n^{(i)},\ i = 2,\dots,j ,$$

2) $$\|u_n^{(i)}\| = 1$$

3) $$(u_n^{(i)}, u_{n+1}^{(i)}) \geqq 0 .$$

Then

a) The sequence $\{u_n^{(i)}\}_{n=1}^{\infty}$ converges to a normalized eigenfunction φ_i of A associated with the eigenvalue λ_i for $i = 1,\dots,j$ and

b) Denote by $E_i^{(n)}$ the error of the best approximation to φ_i by functions of $\mathcal{L}\{\Psi_i\}_{i=1}^n$ in the norm $\|\cdot\|_1$, i.e.,

$$E_i^{(n)} = \inf_{v \in \mathcal{L}\{\Psi_i\}_{i=1}^n} \|\varphi_i - v\|_1 .$$

There exist a positive integer n_0 and constants C_1, $C_2 \neq 0$, C_3, C_4 which do not depend on n such that for $n \geq n_0$

$$C_2 \cdot (E_j^{(n)})^2 \leq \lambda_n^{(j)} - \lambda_j \leq C_1 \cdot [\max_{i=1,\dots,j} E_i^{(n)}]^2 ,$$

$$E_j^{(n)} \leq \|u_n^{(j)} - \varphi_j\|_1 \leq C_3 \cdot \max_{i=1,\dots,j} E_i^{(n)} ,$$

$$\|u_n^{(j)} - \varphi_j\| \leq C_4 \cdot \max_{i=1,\dots,j} E_i^{(n)} .$$

Remark 2. The function $u_n^{(j)}$ and the number $\lambda_n^{(j)}$ in Theorem 4 are the Rayleigh-Ritz approximations to φ_j and λ_j, respectively.

Proof: Let $B = (A - \mu I)^{\frac{1}{2}}$. The proof of a) follows at once from Theorem 2. Since

$$E_i^{(n)} = \inf_{v \in \mathcal{L}\{\psi_i\}_{i=1}^{n}} \| B(\varphi_i - v) \| = \sqrt{\lambda_i - \mu} \cdot \| \varphi_i - {}^{(n)}\varphi_i \|$$

and $\lambda_n^{(j)} - \lambda_j = (q_n^{(j)})^2 - \mu_j^2$ the assertion of b) follows from Theorem 3.

R e f e r e n c e s

[1] K. NAJZAR: On the method of least squares of finding eigenvalues of some symmetric operators, Comment.Math.Univ.Carolinae 9(1968), 311-323.

[2] K. NAJZAR: On the method of least squares of finding eigenvalues and eigenfunctions of some symmetric operators, Comment. Math.Univ. Carolinae 11(1970),449-462.

[3] K. NAJZAR: Error - estimates for the method of least squares of finding eigenvalues and eigenfunctions, Comment. Math.Univ.Carolinae 11 (1970), 463-479.

[4] N.I. ACHIEZER - I.M. GLASMANN: Theorie der linearen Operatoren in Hilbert-Raum, 1960.

[5] A.E. TAYLOR: Introduction to functional analysis, 1958.

[6] S.G. MICHLIN: Prjamyje metody v matematičeskoj fizike, 1950.

Matematicko-fyzikální fakulta
Karlova universita
Malostranské nám.25, Praha 1
Československo

(Oblatum 5.4.1971)