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ABSTRACT SEMILINEAR EQUATIONS WITH SMALL NONLINEARITIES

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Let $\mathcal{U}, \mathcal{X}$ be two linear topological convex Banach spaces over the real or complex number field, θ a linear operator from $\mathcal{U}$ into $\mathcal{X}$ and F a transformation of $\mathcal{U}$ into $\mathcal{X}$.

For simplicity, we shall denote by $\mathbb{D}(\theta)$ the domain of definition of θ ; $\mathbb{R}(\theta) = \{\theta u : u \in \mathbb{D}(\theta)\}$, $\mathbb{N}(\theta) = \{u : u \in \mathbb{D}(\theta), \theta u = 0\}$. Naturally $\mathbb{D}(\theta)$, $\mathbb{N}(\theta)$ are linear subspaces of $\mathcal{U}$, $\mathbb{R}(\theta)$ of $\mathcal{X}$.

We shall consider and try to solve the equation $\theta u = \varepsilon F(u)$ where $\varepsilon \geq 0$ is a ("small") parameter.

We shall give two existence theorems for this problem with fairly detailed proofs and three applications to semilinear wave equations without detailed proofs which will be treated in another paper. Our main purpose is to solve also the so called critical case. The non-critical case $(\mathbb{N}(\theta) = \{0\}, \mathbb{R}(\theta) = \mathcal{X})$ is naturally included but is itself essentially simpler.

Our approach to the problem is geometrical but we have eliminated the complementability of the subspaces $\mathbb{N}(\theta)$,

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$\mathbb{R}(\theta)$ (i.e. the existence of corresponding projectors) by means of quotient spaces. The solvability of the so called bifurcation equation is postulated, but the necessary properties of the solution are deduced from the properties of θ and F .

Geometrical approach to problems of this type was initiated by Cesari and used by many authors (Hale, Lovicarová, de Simon-Torelli, Torelli, Bancroft-Hale-Sweet, Hall, etc. - see the bibliography at the end).

<u>Theorem 1</u>. <u>If</u>

(I) $\mathcal{U}$ <u>is a Banach space and</u> X <u>a normed space</u>,

(II) <u>for every sequence</u> $u_k \in \mathbb{D}(\theta)$ <u>such that</u> θu_k <u>is compact, there exists a compact sequence</u> $\bar{u}_k \in \mathcal{U}$ <u>such that</u> $u_k - \bar{u}_k \in \mathbb{N}(\theta)$,

(III) $\mathbb{N}(\theta)$, $\mathbb{R}(\theta)$ <u>are closed</u>,

(IV) <u>for every</u> $u \in \mathcal{U}$ <u>there exist an open subset</u> $M \subseteq \mathcal{U}$ <u>and a constant</u> m <u>such that</u> $u \in M$ <u>and for every</u> u_1 , $u_2 \in M$

$$\| F(u_1) - F(u_2) \| \leq m \| u_1 - u_2 \| ,$$

<u>then for every open subset</u> $S \subseteq \mathcal{U}$ <u>satisfying</u>

(α) <u>there exists a constant</u> $c > 0$ <u>such that for every</u> $u', u'' \in S$, $u' - u'' \in \mathbb{N}(\theta)$ <u>and for every</u> $x \in \mathbb{R}(\theta)$

$$\| F(u') - F(u'') - x \| \geq n \| u' - u'' \| ,$$

(β) <u>for every</u> $u \in S$, <u>there exists a</u> $\bar{u} \in S$ <u>such that</u> $u - \bar{u} \in \mathbb{N}(\theta)$ <u>and</u> $F(\bar{u}) \in \mathbb{R}(\theta)$,

(γ) $S \cap \mathbb{N}(\theta) \neq \emptyset$,

<u>there exist a</u> $\vartheta > 0$ <u>and a function</u> $\breve{u}$ on $\langle 0, \vartheta \rangle$ <u>into</u> S

such that

(a) $\breve{u}$ is continuous on $\langle 0, \vartheta \rangle$,

(b) $\breve{u}(\varepsilon) \in \mathbb{D}(\theta)$, $\theta \breve{u}(\varepsilon) = \varepsilon F(\breve{u}(\varepsilon))$ for every $0 < \varepsilon \in \vartheta$,

(c) there exists a constant $d > 0$ such that for every $u \in S$ and $0 \leq \varepsilon \leq \vartheta$ satisfying $\|u - \breve{u}(0)\| \leq d$, $u \in \mathbb{D}(\theta)$, $0 \leq \varepsilon \leq \vartheta$, $\theta u = \varepsilon F(u)$, there is $u = \breve{u}(\varepsilon)$.

Proof. Let us denote $W = \mathbb{N}(\theta)$, $Y = \mathbb{R}(\theta)$. As these subspaces are closed by (III), we can introduce the quotient spaces $\mathbb{V} = \mathcal{U}/W$ and $\mathbb{Z} = \mathcal{X}/Y$ with norms denoted by $|||\cdot|||$ and with zero denoted by $\mathbb{O}$.

Moreover, let Π be the canonical transformation of X onto $\mathbb{Z}$ which is defined by $\Pi x = x - Y$ for every $x \in X$.

Now, we can replace the inequality in (α) by

(1) $|||\Pi F(u_1) - \Pi F(u_2)||| \geq m \|u_1 - u_2\|$

and the inclusion $F(\bar{u}) \in \mathbb{R}(\theta)$ in (β) by

(2) $$\Pi F(\bar{u}) = \mathbb{O} .$$

Let S be a fixed open subset of $\mathcal{U}$ satisfying (α) - (γ).

Let us take $\mathbb{S} = \{v : v \in \mathbb{V},\ v \cap S \neq \emptyset\}$.

It follows from (β) and (2) that there exists a transformation J of $\mathbb{S}$ into $\mathcal{U}$ such that for every $v \in \mathbb{S}$

(3) $$J(v) \in v \cap S ,$$

(4) $$\Pi F(J(v)) = \mathbb{O} .$$

Further (γ) implies

(5) $$\mathbb{O} \in \mathbb{S} .$$

Bearing in mind the openness of S we obtain from (IV) that there exist two constants $\varkappa$ and m such that

(6) $$\varkappa > 0 ,$$

(7) $$\{u : \| u - J(\mathbb{O}) \| \leq \varkappa \} \subseteq S ,$$

(8) $$\| F(u_1) - F(u_2) \| \leq m \| u_1 - u_2 \|$$

for every $\| u_1 - J(\mathbb{O}) \| \leq \varkappa$, $\| u_2 - J(\mathbb{O}) \| \leq \varkappa$.

It is clear from (8) that

(9) $$m \geq 0 .$$

Now we shall prove that

(10) $$\| J(\mathfrak{v}) - J(\mathfrak{v}_0) \| \leq (\frac{m}{n} + 1) \| \mathfrak{v} - \mathfrak{v}_0 \|\|$$

for every $\| \mathfrak{v} \|\| \leq \frac{\varkappa}{4}$, $\| J(\mathfrak{v}_0) - J(\mathbb{O}) \| \leq \frac{\varkappa}{4}$.

To this purpose, let us first fix $\mathfrak{v}$ and $\mathfrak{v}_0$ so that

(11) $$|\!|\!| \mathfrak{v} |\!|\!| \leq \frac{\varkappa}{4} ,$$

(12) $$\| J(\mathfrak{v}_0) - J(\mathbb{O}) \| \leq \frac{\varkappa}{4} .$$

It follows from (3),(6),(7) and (11) that for every $0 < \eta \leq$ $\leq \frac{\varkappa}{4}$ there exists an element $v \in U$ satisfying

(13) $$v \in \mathfrak{v} \cap S , \ \| v - J(\mathfrak{v}_0) \| \leq |\!|\!| \mathfrak{v} - \mathfrak{v}_0 |\!|\!| + \eta .$$

On the other hand, as $J(\mathfrak{v}_0) \in \mathfrak{v}_0$ and $J(\mathbb{O}) \in \mathbb{O}$ by (3), we obtain from (12) immediately that $|\!|\!| \mathfrak{v}_0 |\!|\!| \leq \frac{\varkappa}{4}$ which implies together with (11) that

(14) $$|\!|\!| \mathfrak{v} - \mathfrak{v}_0 |\!|\!| \leq \frac{\varkappa}{2} .$$

Now, from (13) and (14) we obtain

(15) $$\| v - J(\mathbb{O}) \| \leq \| v - J(\mathfrak{v}_0) \| + \| J(\mathfrak{v}_0) - J(\mathbb{O}) \| \leq$$
$$\leq |\!|\!| \mathfrak{v} - \mathfrak{v}_0 |\!|\!| + \eta + \frac{\varkappa}{4} \leq \frac{3}{4} \varkappa + \frac{1}{4} \varkappa \leq \varkappa .$$

Using (8),(9),(13) and (15), we have

(16) $$|\!|\!| \Pi F(J(\mathfrak{v}_0)) - \Pi F(v) |\!|\!| \leq m \| J(\mathfrak{v}_0) - v \| \leq$$
$$\leq m |\!|\!| \mathfrak{v} - \mathfrak{v}_0 |\!|\!| + m \eta .$$

On the other hand, (α),(1),(3) and (13) imply

$$(17)\quad |||\Pi F(J(v)) - \Pi F(v)||| \geq n \, \| J(v) - v \| .$$

Now we see from (4),(16) and (17):

$$0 = |||\Pi F(J(v)) - \Pi F(J(v_0))||| \geq$$

$$\geq |||\Pi F(J(v)) - \Pi F(v)||| - |||\Pi F(J(v_0)) - \Pi F(v)||| \geq$$

$$\geq n \, \| J(v) - v \| - m \, ||| v - v_0 ||| - m \, \eta$$

which may be rewritten as

$$(18)\quad \| J(v) - v \| \leq \frac{m}{n} \, ||| v - v_0 ||| + \frac{m}{n} \, \eta \; .$$

Using (13) and (18), we have, finally,

$$\| J(v) - J(v_0) \| \leq \| J(v) - v \| + \| J(v_0) - v \| \leq$$

$$\leq \left(\frac{m}{n} + 1 \right) ||| v - v_0 ||| + \left(\frac{m}{n} + 1 \right) \eta$$

which implies (10) since η may be arbitrarily small.

Let us now denote $\mathcal{S}_0 = \{ v : v \in V, \, ||| v ||| \leq \frac{\varkappa}{4} (\frac{m}{n} + 1)^{-1} \}$.

It is clear that

(19) $\mathcal{S}_0$ is a closed ball in V, contained in $\mathcal{S}$.

We shall prove

$$(20)\qquad \| J(v_1) - J(v_2) \| \leq \left(\frac{m}{n} + 1 \right) ||| v_1 - v_2 |||$$

for every $v_1 , v_2 \in \mathcal{S}_0$.

To prove these, we use (10). Let us first take $v = v_2$ and $v_0 = 0$ in (10), which is evidently admissible. Then we obtain $\| J(v_2) - J(0) \| \leq (\frac{m}{n} + 1) ||| v_2 ||| < (\frac{m}{n} + 1) \frac{\varkappa}{4} (\frac{m}{n} + 1)^{-1} = \frac{\varkappa}{4}$. Consequently, we can take $v = v_1$ and $v_0 = v_2$ in (10) and obtain immediately (20).

Now we see easily from (6) and (20) that

$$(21)\quad \| J(v) - J(0) \| \leq \varkappa \qquad \text{for every } v \in \mathcal{S}_0 \; .$$

Let us now define $\widetilde{\theta}$ as an operator from V into Y by the following way: $v \in \mathbb{D}(\widetilde{\theta})$ if and only if

$v \subseteq D(\theta)$ and $\tilde{\theta} v$ is the common value of all θv for $v \in v$.

We need to verify now that

(22) $\tilde{\theta}$ is a one-to-one operator from V onto Y and $\tilde{\theta}^{-1}$ is bounded.

In fact, $\tilde{\theta}$ is a one-to-one operator from V onto Y by definition. We see almost immediately from (II) that $\tilde{\theta}^{-1}$ transforms compact sequences from Y into compact sequences from V. This implies that $\tilde{\theta}^{-1}$ is a bounded operator on Y into V. If this were not true, then there would exist a sequence $y_k \in Y$ such that $\| y_k \| \leq 1$ and $|||\tilde{\theta}^{-1} y_k||| \to \infty$. Moreover, we can suppose $|||\tilde{\theta}^{-1} y_k||| > 0$. Let us now take $\alpha_k = |||\tilde{\theta}^{-1} y_k|||^{-\frac{1}{2}}$. Then evidently $\alpha_k \to 0$ and $|||\tilde{\theta}^{-1} \alpha_k y_k||| = \alpha_k |||\tilde{\theta}^{-1} y_k||| \to \infty$. But this leads to a contradiction: $\alpha_k y_k$ is undoubtedly compact, on the other hand, $\tilde{\theta}^{-1} \alpha_k y_k$ cannot be compact.

Let us rewrite (4) as

(23) $F(J(v)) \in \imath$ for every $v \in S$.

Now, (22) and (23) enable us to define

$$\Phi(v) = \tilde{\theta}^{-1} F(J(v)), \quad v \in S .$$

We obtain from (7),(8),(20) - (22) that

(24) $|||\Phi(v_1) - \Phi(v_2)||| \leq \|\tilde{\theta}^{-1}\| m (\frac{m}{n} + 1) ||| v_1 - v_2 |||$

for every $v_1, v_2 \in S_0$.

It follows from (19) and (24) that we can find a constant ϑ such that

(25) $0 < \vartheta \leq \frac{1}{2} [\|\tilde{\theta}^{-1}\| m (\frac{m}{n} + 1) + 1]^{-1}$,

(26) $\vartheta \Phi(v) \in S_0$ for every $v \in S_0$,

(27) $||| \vartheta \Phi(v_1) - \vartheta \Phi v_2 ||| \leq \frac{1}{2} ||| v_1 - v_2 |||$

for every $v_1, v_2 \in \mathcal{S}_0$.

In virtue of (I),(19),(25) - (27), we can use the Banach fixed point theorem to the transformation $\varepsilon\Phi$ in the complete metric space $\mathcal{S}_0$ for every $0 \leq \varepsilon \leq \vartheta$ and we obtain a function w on $\langle 0, \vartheta \rangle$ into $\mathcal{S}_0$ such that for every $0 \leq \varepsilon \leq \vartheta$

(28) $$w(\varepsilon) \in \frac{\varepsilon}{\vartheta} \mathcal{S}_0 ,$$

(29) $$\varepsilon \Phi (w(\varepsilon)) = w(\varepsilon) ,$$

(30) $w(\varepsilon)$ is the unique solution in $\mathcal{S}_0$ of the equation $\varepsilon\Phi(v) = v$.

By the definition of Φ , we can rewrite (29) as

(31) $w(\varepsilon) \in \mathbf{D}(\tilde{\sigma}),\ \ \tilde{\sigma} \dot{w}(\varepsilon) = \varepsilon F(J(w(\varepsilon)))$

for every $0 < \varepsilon \leq \vartheta$.

Now we shall prove that

(32) w is a continuous function on $\langle 0, \vartheta \rangle$.

In fact, it follows from (24),(25) and (29) in view of the definition of $\mathcal{S}_0$ that

$$\begin{aligned}
&|||w(\varepsilon_1) - w(\varepsilon_2)||| = |||\varepsilon_1\Phi(w(\varepsilon_1)) - \varepsilon_2 \Phi(w(\varepsilon_2))||| \leq \\
&\leq |\varepsilon_1 - \varepsilon_2|\ |||\Phi w(\varepsilon_1)||| + \varepsilon_2 |||\Phi(w(\varepsilon_1)) - \Phi(w(\varepsilon_2))||| \leq \\
&\leq |\varepsilon_1 - \varepsilon_2| [\,|||\Phi(0)||| + \|\tilde{\sigma}^{-1}\|\, m (\frac{m}{n} + 1) \frac{\varkappa}{4} (\frac{m}{n} + 1)^{-1}\,] + \\
&+ \vartheta \|\tilde{\sigma}^{-1}\|\, m (\frac{m}{n} + 1)\ |||w(\varepsilon_1) - w(\varepsilon_2)||| = \\
&= |\varepsilon_1 - \varepsilon_2| [\,|||\Phi(0)||| + \|\tilde{\sigma}^{-1}\|\, m \frac{\varkappa}{4}] + \frac{1}{2} |||w(\varepsilon_1) - w(\varepsilon_2)|||
\end{aligned}$$

which implies

$$|||w(\varepsilon_1) - w(\varepsilon_2)||| \leq 2 [\,|||\Phi(0)||| + \|\tilde{\sigma}^{-1}\| m \frac{\varkappa}{4}]\, |\varepsilon_1 - \varepsilon_2|$$

and (32) is an immediate consequence.

Let us now take $u(\varepsilon) = J(w(\varepsilon))$ for $0 \leq \varepsilon \leq \vartheta$.

It is easily seen from (3),(28),(31) and (32) that u

satisfies (a) and (b).

Thus we have only to prove the unicity assertion (c).

It follows from (3),(4) and (α) that

(33) for every $u \in S$ such that $\Pi F(u) = \mathcal{O}$ we have $u = J(u + W)$.

Let us now take $d = \frac{\varkappa}{4}(\frac{m}{n} + 1)^{-1}$ and let u satisfy the suppositions stated in (c) for some fixed $0 < \varepsilon \leq \leq \vartheta$. Let us take $v = u + W$. According to (28), we have $\| u - \tilde{u}(0)\| = \| u - J(\mathcal{O})\| \leq d = \frac{\varkappa}{4}(\frac{m}{n} + 1)^{-1}$ which implies, in view of $J(\mathcal{O}) \in \mathcal{O}$ that $|\!|\!| v |\!|\!| \leq \frac{\varkappa}{4}(\frac{m}{n} + 1)^{-1}$, i.e.

(34) $$v \in S_0 .$$

As $u \in D(\theta)$, $\theta u = \varepsilon F(u)$, we see first that $\Pi F(u) = \mathcal{O}$ which implies in consequence of (33) that

(35) $$u = J(v) .$$

On the other hand, we see that

(36) $$v \in D(\tilde{\theta}) , \quad \tilde{\theta} v = \varepsilon F(u) .$$

Using (35) and (36), we have

(37) $$v = \varepsilon \Phi(v) .$$

Hence we obtain from (30),(34) and (37) that $v = \varphi(\varepsilon)$ and, as $\tilde{u}(\varepsilon) = J(\varphi(\varepsilon))$, we have from (35) that $u = \tilde{u}(\varepsilon)$, which was to verify.

Remark 1. Theorem 1 generalizes many results provable under different hypotheses for special types of operators θ and F. In particular, the differentiability of F plays an important role and was used in many papers - see the bibliography at the end. We shall give an abstract form of sufficient differentiability conditions in a separate note.

Proposition 1. If the condition (III) of Theorem 1 holds, then the condition (II) of the same theorem is equivalent with: (II$'$) for every sequence $u_k \in \mathbf{D}(\theta)$ such that θu_k is compact, there exists a weakly compact sequence $\bar{u}_k \in \mathcal{U}$ such that $u_k - \bar{u}_k \in \mathbf{N}(\theta)$.

Proof. We shall use the operator $\tilde{\theta}$ constructed in the preceding proof.

First, if (II) holds, then by (22) of the above proof, $\tilde{\theta}$ is bounded and consequently it transforms the compact sequences of $\mathbf{Y}$ into the compact ones, and thereby as well into the weakly compact sequences of $\mathbf{V}$. From here immediately (II$'$).

Conversely, if (II$'$) holds, then we obtain by almost the same argument as in proving (22) that $\tilde{\theta}$ is bounded and consequently it transforms the compact sequences of $\mathbf{Y}$ into the compact sequences of $\mathbf{V}$. But this implies (II) by a simple way.

Remark 2. According to the preceding Proposition 1, we can replace the hypothesis (II) of Theorem 1 by the condition (II$'$) of this proposition.

Proposition 2. The conditions (II),(III) in Theorem 1 are equivalent with

(II$''$) θ is a closed operator,

(III$''$) $\mathbf{R}(\theta)$ is a closed subspace.

Proof. Let us construct the operator $\tilde{\theta}$ as above in the proof of Theorem 1. Moreover, we shall use the notation of this proof.

We shall first verify that (II),(III) imply (II''), (III''). Obviously it sufficest to verify (II''). Using (22) from the above proof, we see that $\tilde{\theta}^{-1}$ is a bounded operator on Y into V. Consequently $\tilde{\theta}$ is a closed operator from V onto Y and it is easy to deduce from this the closedness of θ itself.

Conversely, let (II''), (III'') hold. It follows immediately from (II'') that $\mathbb{N}(\theta)$ is closed. Therefore (III) holds. Further we obtain easily from (II''), (III'') that $\tilde{\theta}$ is a one-to-one closed operator from V onto Y. Consequently, $\tilde{\theta}^{-1}$ is also closed, which implies, according to <u>Banach closed graph theorem</u> that it is continuous. Then (II) is an immediate consequence.

Cf. also T. Kato: Perturbation theory of linear operators, p. 231.

<u>Remark 3</u>. According to Proposition 2, we can replace the hypotheses (II),(III) of Theorem 1 by the conditions (II''), (III'') of this proposition.

<u>Theorem 2</u>. <u>If</u>

(I) $\mathcal{U}$, X <u>are normed spaces</u>,

(II) <u>for every sequence</u> $u_k \in \mathbb{D}(\theta)$ <u>such that</u> θu_k <u>is bounded, there exists a compact sequence</u> $\bar{u}_k \in \mathcal{U}$ <u>such that</u> $u_k - \bar{u}_k \in \mathbb{N}(\theta)$,

(III) $\mathbb{N}(\theta)$, $\mathbb{R}(\theta)$ <u>are closed</u>,

(IV) F <u>is continuous on</u> $\mathcal{U}$ <u>into</u> X , then for every open subset $S \subseteq \mathcal{U}$ satisfying

(α) for every two sequences $u'_k, u''_k \in S$ such that $u'_k - u''_k \in \mathbb{N}(\Theta)$ and that for some sequence $x_k \in \mathbb{R}(\Theta)$

$$F(u'_k) - F(u''_k) - x_k \to 0 \ ,$$

there is $u'_k - u''_k \to 0$,

(β),(γ) as in Theorem 1,

there exist a constant $\vartheta > 0$ and a function $\breve{u}$ on $\langle 0,\vartheta \rangle$ into S such that

(a) $\breve{u}(\varepsilon) \longrightarrow \breve{u}(0) \ (\varepsilon \to 0_+)$ and the set $\{ \breve{u}(\varepsilon) ; 0 \leq \varepsilon \leq \vartheta \}$ is compact,

(b) as in Theorem 1.

Proof. We shall use the first part of the proof of Theorem 1 till (5), only (1) must be replaced by

(1) for every two sequences $u'_k, u''_k \in S$ such that $u'_k - u''_k \in \mathbb{N}(\Theta)$ and that $\Pi F(u'_k) - \Pi F(u''_k) \to 0$, there is $u'_k - u''_k \to 0$.

Hence we continue the numbering by (6).

Now we shall prove that

(6) $\mathcal{J}$ is continuous on $\mathfrak{S}$ into $\mathcal{U}$.

In fact, let $\mathfrak{v} \in \mathfrak{S}$ and let $\mathfrak{v}_k , k \in \{1,2,\dots\}$ be an arbitrary sequence from $\mathfrak{S}$ such that

(7) $$\mathfrak{v}_k \longrightarrow \mathfrak{v} \ .$$

As $\mathcal{J}(\mathfrak{v}) \in \mathfrak{v}$ by (3), it follows from (7) that we can always choose a sequence $v_k , \ k \in \{1,2,\dots\}$ for which

(8) $v_k \in \mathfrak{v}_k$ for every $k \in \{1,2,\dots\}$,

(9) $$v_k \longrightarrow \mathcal{J}(\mathfrak{v}) \ .$$

Moreover, as S is open and $\mathcal{J}(\mathfrak{v}) \in S$ likewise by (3) it follows from (9) that there exists a $k_0 \in \{1, 2, \dots\}$ such that

(10) $v_k \in S$ for $k \geq k_0$.

According to (4), we have $\Pi F(J(v)) = 0$ and $\Pi F(J(v_k)) = 0$ for $k \in \{1, 2, \dots\}$ and consequently we can write

(11) $\Pi F(J(v_k)) - \Pi F(v_k) = \Pi F(J(v)) - \Pi F(v_k)$

for $k \in \{1, 2, \dots\}$.

But it follows from (IV) and (9) that $\Pi F(J(v)) - \Pi F(v_k) \to 0$ and therefore in consequence of (11)

(12) $$\Pi F(J(v_k)) - \Pi F(v_k) \to 0 \ .$$

Further, we see immediately from (3),(8) and (10) that

(13) $J(v_k) \in S$, $v_k \in S$, $J(v_k) - v_k \in W$ for $k \geq k_0$.

Thus (12) and (13) enable us to apply (1) from where we obtain

(14) $$J(v_k) - v_k \to 0 \ .$$

Finally, combining (9) and (14), we see that $J(v_k) \to J(v)$, which was our aim to prove (6).

In consequence of (IV) there exists a constant $\varkappa$ such that

(15) $$\varkappa > 0 \ ,$$

(16) $$\{u : \|u - J(0)\| \leq \varkappa\} \subseteq S \ ,$$

(17) $$\|F(u) - F(J(0))\| \leq 1 \quad \text{for } \|u - J(0)\| \leq \varkappa \ .$$

On the other hand, in consequence of (6), we can find a constant $\varkappa_0$ such that

(18) $$0 < \varkappa_0 \leq \varkappa \ ,$$

(19) $$\|J(v) - J(0)\| \leq \varkappa \quad \text{for } |||v||| \leq \varkappa_0 \ .$$

Let us now take $\mathfrak{S}_0 = \{v : |||v||| \leq \varkappa_0\}$.

It follows from (18) that

(20) $\mathcal{S}_0$ is a closed ball in V, contained in $\mathcal{S}$.

We see immediately from (17) and (19) that

(21) $\| F(J(v)) - F(J(0)) \| \leq 1$ for $v \in \mathcal{S}_0$.

Let us now define the operator $\tilde{\mathcal{O}}$ as in the proof of Theorem 1.

We prove without difficulty under use of (I),(II),(III) that

(22) $\tilde{\mathcal{O}}$ is a one-to-one operator from V onto Y such that $\tilde{\mathcal{O}}^{-1}$ is compact.

Especially, in consequence of (22), there exists a subset $K \subseteq V$ such that

(23) K is compact and convex in V.

(24) $\tilde{\mathcal{O}}^{-1} y \in K$ for every $y \in Y$,

$\| y - F(J(0)) \| \leq 1$.

Owing to (4)

(25) $F(J(v)) \in Y$ for every $v \in \mathcal{S}$.

Thus (22) and (25) enable us to define

$$\Phi(v) = \tilde{\mathcal{O}}^{-1} F(J(v)), \quad v \in \mathcal{S}.$$

Now, we shall prove that

(26) $\Phi(v) \in K \quad , v \in \mathcal{S}_0$,

(27) Φ is continuous on $\mathcal{S}_0$ into V.

In fact, (26) follows immediately from (21) and (24), and (27) from (III),(6) and (22).

Using (20),(23) and (26), we can find a $\vartheta > 0$ such that

(28) $\vartheta \Phi(v) \in \mathcal{S}_0$ for $v \in \mathcal{S}_0$.

So, (20),(26) - (28) enable us to apply the <u>Schauder fixed point theorem to the transformation</u> $\varepsilon \Phi$ in $\mathcal{S}_0$

for every $0 \leq \varepsilon \leq \vartheta$ and thus we obtain the existence of a function v on $\langle 0, \vartheta \rangle$ into $\mathcal{S}_0$ such that for every $0 \leq \varepsilon \leq \vartheta$

(29) $\qquad v(\varepsilon) \in \varepsilon \mathbf{K}$,

(30) $\qquad \varepsilon \Phi(v(\varepsilon)) = v(\varepsilon)$.

But we can rewrite (30) as

(31) $\qquad v(\varepsilon) \in \mathbf{D}(\tilde{\delta}), \ \tilde{\delta} v(\varepsilon) = \varepsilon F(J(v(\varepsilon)))$

for every $0 \leq \varepsilon \leq \vartheta$.

Finally, let us take $\acute{u}(\varepsilon) = J(v(\varepsilon))$ for $0 \leq \varepsilon \leq \vartheta$.

It is now an easy matter to deduce (a) from (6),(23) and (29) and (b) from (3) and (31).

<u>Remark 4</u>. The assertion on local unicity of the type (c) in Theorem 1 seems unprovable here, at least by the above method.

<u>Remark 5</u>. The condition (β) in Theorems 1 and 2 postulates the solvability of the so called <u>bifurcation equation</u> in a completely general form.

We give two more suggestive formulations:

(β') for every $u \in \mathcal{S}$ there exists a $w \in \mathbf{N}(\theta)$ such that

$$u + w \in \mathcal{S}, \quad F(u + w) \in \mathbf{R}(\theta) ,$$

(β'') for every $u \in \mathcal{S}$ there exists a $w \in \mathbf{N}(\theta)$ such that

$$u + w \in \mathcal{S}, \quad \Pi F(u + w) = 0$$

where Π is the canonical transformation of $\mathbf{X}$ onto

$X/_{R(\theta)}$ defined by $\Pi x = x + R(\theta)$ for $x \in X$.

In the sequel we shall describe (Examples 1 - 3) some applications of Theorems 1 and 2 to the periodic problem for semilinear wave equations, without detailed proofs. In these examples we denote by R the real number field and by C^{∞} the set of all infinitely differentiable functions φ on $\langle 0, 2\pi \rangle$ whose derivations are 2π -periodic, i.e. $\varphi^{(n)}(0) = \varphi^{(n)}(2\pi)$ for every $n \in \{0, 1, \dots\}$.

Example 1. Let f *be a real function on* $\langle 0, 2\pi \rangle \times \langle 0, \pi \rangle \times R$ *such that*

(I) $f(0, \xi, x) = f(2\pi, \xi, x)$ *for all* $0 \le \xi \le \pi$, $x \in R$,

(II) f *is continuous in all variables and*

$$|f(t,\xi,x_1) - f(t,\xi,x_2)| \le \mu(\varrho)\,|x_1 - x_2|$$

for all $0 \le t \le 2\pi$, $0 \le \xi \le \pi$, $|x_1|, |x_2| \le \varrho$,

(III) *there exist a constant* $c > 0$ *and a number* $\nu \in \{0, 1\}$ *so that*

$$(-1)^{\nu}[f(t,\xi,x_2) - f(t,\xi,x_1)] \ge c\,(x_2 - x_1)$$

for all $0 \le t \le 2\pi$, $0 \le \xi \le \pi$, $x_1 \le x_2$.

Then there exist a constant $\vartheta > 0$ *and a real function* $\check{u}$ *on* $\langle 0, 2\pi \rangle \times \langle 0, \pi \rangle \times \langle 0, \vartheta \rangle$ *such that*

(A) $\check{u}(0, \xi, \varepsilon) = \check{u}(2\pi, \xi, \varepsilon)$ *for all* $0 \le \xi \le \pi$, $0 \le \varepsilon \le \vartheta$, *and* $\check{u}$ *is continuous in all variables*,

(B) $\int_0^{2\pi} \varphi(\tau)\, \check{u}(\tau, \xi, \varepsilon)\, d\tau$ *is twice continuously derivable in* $0 \le \xi \le \vartheta$ *for every*

$\varphi \in C^\infty$, $0 < \varepsilon \le \vartheta$,

$$\int_0^{2\pi} \varphi''(\tau)\,\check{u}(\tau,\xi,\varepsilon)\,d\tau - \frac{\partial^2}{\partial\xi^2}\int_0^{2\pi}\varphi(\tau)\check{u}(\tau,\xi,\varepsilon)\,d\tau = \\ = \varepsilon\int_0^{2\pi}\varphi(\tau)\,f(\tau,\xi,\check{u}(\tau,\xi,\varepsilon))\,d\tau$$

<u>for every</u> $\varphi \in C^\infty, 0 \le \xi \le \pi, 0 < \varepsilon \le \vartheta$ <u>and</u> $\check{u}(t,0,\varepsilon) = \check{u}(t,\pi,\varepsilon) = 0$ <u>for every</u> $0 \le t \le 2\pi$, $0 < \varepsilon \le \vartheta$,

(C) <u>there exists a constant</u> $d > 0$ <u>such that for any real function</u> u <u>on</u> $\langle 0, 2\pi\rangle \times \langle 0,\pi\rangle$ <u>and for</u> $0 < \varepsilon \le \vartheta$ <u>satisfying</u> $|u(t,\xi) - \check{u}(t,\xi,0)| \le d$ <u>for every</u> $0 \le t \le 2\pi$, $0 \le \xi \le \pi$ <u>and the above properties (A),(B) with</u> u <u>instead of</u> $\check{u}$, <u>there is</u> $u(t,\xi) = \check{u}(t,\xi,\varepsilon)$ <u>for every</u> $0 \le t \le 2\pi$, $0 \le \xi \le \pi$.

<u>Outline of the proof</u>. We choose $\mathcal{U} = X = \mathcal{C} =$ the real Banach space of all real continuous functions on $\langle 0, 2\pi\rangle \times \langle 0,\pi\rangle$ such that $u(0,\xi) = u(2\pi,\xi)$ for $0 \le \xi \le \pi$ with the maximum norm.

Now Θ is defined as follows: $u \in \mathbb{D}(\Theta) \Longleftrightarrow$

(1) $u \in \mathcal{C}$,

(2) $u(t,0) = u(t,\pi)$ for all $0 \le t \le 2\pi$,

(3) $\int_0^{2\pi}\varphi(\tau)\,u(\tau,\cdot)\,d\tau$ is twice continuously derivable for all $\varphi \in C^\infty$,

(4) there exists a function $h \in \mathcal{C}$ such that

$$\int_0^{2\pi}\varphi''(\tau)\,u(\tau,\xi)\,d\tau - \frac{\partial^2}{\partial\xi^2}\int_0^{2\pi}\varphi(\tau)\,u(\tau,\xi)\,d\tau = \\ = \int_0^{2\pi}\varphi(\tau)\,h(\tau,\xi)\,d\tau$$

for all $\varphi \in C^\infty$ and $0 \le \xi \le \pi$.

Then $\Theta u = h$.

Further F is defined for all $u \in \mathcal{C}$ by the formula

$$F(u)(t,\xi) = f(t,\xi,u(t,\xi)) .$$

Finally we choose $\mathcal{S} = \mathcal{U}$.

Now it suffices to verify that $\Theta, F, \mathcal{S}$ satisfy the hypotheses of Theorem 1. But this is elaborate and will be described elsewhere.

<u>Example 2</u>. <u>Let</u> $f^{\cdot}$ <u>be a real function on</u> $\langle 0, 2\pi\rangle \times \times \langle 0,\pi\rangle \times R \times R \times R$ <u>such that</u>

(I) $f^{\cdot}(0,\xi,r,p,q) = f^{\cdot}(2\pi,\xi,r,p,q)$

<u>for all</u> $0 \le \xi \le \pi,\ r, p, q \in R$,

(II) $f^{\cdot}$ <u>is continuous in all variables and</u>

$$|f^{\cdot}(t,\xi,r_1,p_1,q_1) - f^{\cdot}(t,\xi,r_2,p_2,q_2)| \le$$

$$\le \mu(\varrho)(|r_1 - r_2| + |p_1 - p_2| + |q_1 - q_2|)$$

<u>for all</u> $0 \le t \le 2\pi$, $0 \le \xi \le \pi$, $|r_1|, |r_2|, |p_1|, |p_2|, |q_1|, |q_2| \le \varrho$,

(III) <u>there exist three constants</u> $a, b, c,\ a + b - 4\pi c > 0$ <u>and a number</u> $\nu \in \{0,1\}$ <u>such that</u>

$$(-1)^{\nu}[f^{\cdot}(t,\xi,r_2,p_2,q_2) - f^{\cdot}(t,\xi,r_1,p_1,q_1)] \ge$$

$$\ge a(p_2 - p_1) + b(q_2 - q_1) - c\,|r_2 - r_1|$$

<u>for all</u> $0 \le t \le 2\pi,\ 0 \le \xi \le \pi,\ p_1 \le p_2$, $q_1 \le q_2,\ r_1, r_2 \in R$.

<u>Then there exist a constant</u> $\vartheta > 0$ <u>and a real function</u> $\breve{u}$ <u>on</u> $\langle 0, 2\pi\rangle \times \langle 0,\pi\rangle \times \langle 0, \nu\rangle$ <u>such that</u>

(A) $\breve{u}(0,\xi,\varepsilon) = \breve{u}(2\pi,\xi,\varepsilon)$ <u>for all</u>

$0 \le \xi \le \pi$, $0 \le \varepsilon \le \vartheta$, $\breve{u}$ is continuous in all variables, $\breve{u}_t$, $\breve{u}_\xi$ exist everywhere and are also continuous in all variables,

(B) $\int_0^{2\pi} \varphi(\tau)\, \breve{u}(\tau, \xi, \varepsilon)\, d\tau$ is twice continuously derivable in $0 \le \xi \le \pi$ for every $\varphi \in C^\infty$, $0 < \varepsilon \le \vartheta$,

$$\int_0^{2\pi} \varphi''(\tau)\breve{u}(\tau,\xi,\varepsilon)d\tau - \frac{\partial^2}{\partial\xi^2}\int_0^{2\pi}\varphi(\tau)\breve{u}(\tau,\xi,\varepsilon)d\tau =$$
$$= \varepsilon \int_0^{2\pi}\varphi(\tau) f^*(\tau,\xi,\breve{u}(\tau,\xi,\varepsilon),\breve{u}_t(\tau,\xi,\varepsilon),\breve{u}_\xi(\tau,\xi,\varepsilon))d\tau$$

for every $\varphi \in C^\infty$, $0 \le \xi, \le \pi$, $0 < \varepsilon \le \vartheta$, $\breve{u}(t, 0, \varepsilon) = \breve{u}(t, \pi, \varepsilon) = 0$ for every $0 \le t \le 2\pi$, $0 < \varepsilon \le \vartheta$,

(C) as in Example 1.

Outline of the proof. We choose $X = \mathcal{C}$ (see Example 1) and $\mathcal{U} = \mathcal{C}^* =$ the space of all continuously differentiable functions from $\mathcal{C}$ with the norm $\| u \|_{\mathcal{C}^*} = \| u \|_{\mathcal{C}} + \| u_t \|_{\mathcal{C}} + \| u_\xi \|_{\mathcal{C}}$.

Now Θ is defined as the restriction of the Θ from Example 1 to the space $\mathcal{C}^*$ and it operates consequently from $\mathcal{U} = \mathcal{C}^*$ into $X = \mathcal{C}$.

Further F is defined for all $u \in \mathcal{C}^*$ by the formula $F(u)(t,\xi) = f^*(t,\xi,u(t,\xi), u_t(t,\xi), u_\xi(t,\xi))$.

As in Example 1 we choose $S = \mathcal{U}$.

Our example follows again from Theorem 1, but the elaborate verification of its hypotheses will be given elsewhere.

Example 3. Let f be as in Example 1 with the following properties:

(I) as in Example 1,

(II) f is continuous in all variables,

(III) there exist two constants $a > 0$, $\alpha \geq 1$ and a number $\nu \in \{0, 1\}$ such that

$$(-1)^{\nu}[f(t, \xi, \varkappa_2) - f(t, \xi, \varkappa_1)] \geq a\,[|\varkappa_2|^{\alpha-1}\varkappa_2 - |\varkappa_1|^{\alpha-1}\varkappa_1]$$

for all $0 \leq t \leq 2\pi$, $0 \leq \xi \leq \pi$, $\varkappa_1 \leq \varkappa_2$.

Then there exist a constant $\vartheta > 0$ and a real function $\check{u}$ on $\langle 0, 2\pi \rangle \times \langle 0, \pi \rangle \times \langle 0, \vartheta \rangle$ such that

(A) $\check{u}(0, \xi, \varepsilon) = \check{u}(2\pi, \xi, \varepsilon)$ for all $0 \leq \xi \leq \pi$, $0 \leq \varepsilon \leq \vartheta$, the function $\check{u}$ is uniformly bounded in all variables, $\check{u}(t, \xi, \varepsilon) \to \check{u}(t, \xi, 0)\ (\varepsilon \to 0_+)$ uniformly in $0 \leq t \leq 2\pi$, $0 \leq \xi \leq \pi$, and the functions $\check{u}(\cdot, \cdot, \varepsilon)$, $0 \leq \varepsilon \leq \vartheta$, are equicontinuous in both variables,

(B) as in Example 1.

Outline of the proof. We choose $\mathcal{U}$, X and we construct Θ, F as in the proof of Example 1.

Further we choose $S = \mathcal{U}$.

Then we can verify the hypotheses of Theorem 2, but this will be given elsewhere. Our example is an immediate consequence.

Remark 6. The assertions (B) in Examples 1 - 3 say that the function $\check{u}(\cdot, \cdot, \varepsilon)$ is a generalized solution (in the sense of Petrowsky) of the wave equation $u_{tt} - u_{\xi\xi} = \varepsilon f(t, \xi, u(t, \xi))$ (Examples 1 and 3) or $u_{tt} - u_{\xi\xi} = \varepsilon f'(t, \xi, u(t, \xi), u_t(t, \xi), u_\xi(t, \xi))$ (Example 2). This is easily verifiable by means of the integration by parts and by approximation.

Remark 7. An application of Theorem 2 to the case studied in Example 2 is not possible. In fact, for the operator θ constructed in Example 2, operating from $\mathcal{C}'$ into $\mathcal{C}$, we cannot verify (I) of Theorem 2. This problem will be solved in a subsequent paper.

Addendum. After this note was written, we have been acquainted with two preprints of W.S. Hall (The bifurcation of periodic solutions in Banach spaces I,II), where the method of quotient spaces is used, too.

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