Commentationes Mathematicae Universitatis Carolinae

Petr Hájek

Why semisets?

Commentationes Mathematicae Universitatis Carolinae, Vol. 14 (1973), No. 3, 397--420

Persistent URL: `http://dml.cz/dmlcz/105499`

Commentationes Mathematicae Universitatis Carolinae

14,3 (1973)

WHY SEMISETS ?

Petr HÁJEK, Praha

DEDICATED TO PROFESSOR GERT H. MÜLLER ON OCCASION OF HIS 50th BIRTHDAY

Abstract: The paper contains (i) some informal considerations on the notion of a semiset (theses on semisets) and (ii) an application of the theory of semisets to a (classical) analysis of the (ultra-intuitionistic) notion of feasibility.

Key words: Semisets, comprehension, set theory, feasible natural numbers, ultra-intuitionism.

AMS, Primary: 02K10, 02K15, 02A05, 02D99
Secondary: 02E99

Ref. Ž. 2.641, 2.644

Introduction. Semisets are subcollections of sets. In the first part of this paper, I formulate some theses on the notion of a semiset that should explain and stress basic facts on this notion in view of some new results [1]. In the second part we formulate two axiom systems for semisets

---

1) Many facts stated and used here are not contained in the book [17]. This is because they were not known when the book was written. I refer to my mimeographed Warsaw Lecture notes (Logical semester 1973) for up-to-date more detailed information on the theory of semisets; but the present paper does not assume any knowledge of them.
Cf. also the bibliography.

and present an interpretation relating them and due to Balcar. We use Balcar's interpretation in the third part, where we give a new application of semisets, namely to the (classical) analysis of the (ultraintuitionistic) notion of feasibility in the style of Parikh [13]. This application seems to be of independent interest but also supports - by my opinion - the theses of the first part.

Acknowledgements. I am obliged to my dear friend Professor Gert H. Müller, to whom this paper is dedicated, for several discussions on semisets on various occasions. It was our last discussion (during his visit to Prague in Autumn 1972) that helped me to find new questions, views and formulations. This paper depends heavily on the work of my Prague colleagues; above all, the permanent influence by Vopěnka is a thing that I am hardly able to evaluate. I also still feel the influence of my meetings with Professor Kreisel in 1969. Last but not least, thanks are due to Professor A. Mostowski, B. Balcar, P. Hinman, S. Krajewski, A. Sochor and P. Vopěnka for helpful criticism of earlier versions of the first part of the paper. This paper was written during my stay at the 1973 Warsaw Logical Semester.

I. T h e s e s   o n   s e m i s e t s

1. Semisets are natural

1.1. Semisets can exist. Consider [2] (i) Cantor's definition of a set as a comprehensible collection (Vielheiten, die als Einheiten gedacht werden können). (ii) The principle of iterated power-set operation (starting from $\emptyset$); the power set $\mathcal{P}(x)$ of a set x is the set of all subsets of x , i.e. of all comprehensible subcollections of x . (iii) The comprehension principle: each definable subcollection of a set is a set. Observation: The postulate "each subcollection of a set is a set" is an additional postulate, not implied by (i),(ii),(iii); do not assume this additional postulate (and cf. 1.2).

Semisets are arbitrary subcollections of sets. If we look at sets created by the iterated power-set process we are led to the axioms of ZF [3], so for sets the axioms of ZF are true; but do not forget that the process need not grasp all subcollections of sets. If a set x contains a proper subsemiset σ (a non-set) then σ is not an element of $\mathcal{P}(x)$ and not an element of any set. This consideration leads naturally to the basic axioms of the theory of semisets [4].

----

2) Cf. [9] , pp. 210-213.

3) Cf. [9] , pp. 214-216.

4) The choice of the language of sets and classes is more or less only a technical thing, but it is very convenient. Cf. part II.

1.2. How does one find axioms for proper semisets? One has to investigae various domains of familiarity [5] for semisets and write down the axioms true in the domain which one wants to study [6].

Examples: (i) The domain of genericity. Elements of the generic extension that are subsets of (the elements of) the ground model are semisets. Historically, this is the first exploited domain [7] and has led to fruitful notions of dependence and supports; but, at any rate, this is not the only one domain.

(ii) The domain of countability and an analysis of the notion of cardinality. On the one hand, one has sets of different cardinalities but, on the other hand, one is led to the axiom (GC) of general (semiset-) countability saying that any two infinite sets are semiset-equivalent [8]. Indeed, each countable model M of ZF is the set-part of a model N of (TSS, GC) (i.e. M consists of exactly all sets of N and membership in M is the restriction of membership in N); this supports the

----

5) See [12].

6) Cf. what Mostowski says on sets [11]: "Probably we shall have in the future essentially different intuitive notions of sets just as we have different notions of space, and will base our discussions of sets on axioms that correspond to the kind of sets that we want to study." For semisets this claim is true in the present.

7) Note that [17] is devoted only to this case; the possibility of other applications is mentioned but not exploited.

8) Observe that it is possible for sets x, y that there is no set which is a 1-1 mapping of x onto y but that there is a (proper) semiset with this property.

intuition of only one (semiset-) infinity. The axiom (GC) provides an explication of the so-called Löwenheim-Skolem paradox [9].

(iii) The domain of non-standardness and criticism of the notion of natural numbers and induction. One can have infinitely large (non-standard) natural numbers; they form a non-empty semiset of natural numbers with no least element. (The bald man paradox.) In particular: non-standard analysis - monads are proper semisets. Note that each model of ZF with non-standard natural numbers is the set-part of a model of (TSS + "there is a non-empty semiset of natural numbers with no least element").

----

9) Vopěnka suggested to call (GC) the Löwenheim axiom. Consider the following consequence of Löwenheim-Skolem theorem, which is called "the Löwenheim-Skolem paradox": Each consistent extension on ZF has a model M such that, from inside, there are (obviously) infinite sets of different cardinalities but, from outside, (extensions of) any two infinite sets of M are of the same (countable) cardinality. Call such models LS -models of ZF. Take now (TSS, GC) for the informal metatheory. Then the "Löwenheim-Skolem paradox" obtains the following formulation: Each consistent extension of (TSS, GC) has a model N which is a set (i.e. also proper semisets of N are interpreted as sets) and such that the set of natural numbers of N is countable (from outside). For a moment, call such a model good (it is good that there are countably many natural numbers; and it is good that proper semisets are "visualized" by sets). LS -models of ZF do not more appear paradoxal or even unnatural since they are set-parts of good models of (TSS, GC). Recall that "relativeness of cardinals was very disturbing to Skolem and von Neumann. ... How can one, for example, trust non-denumerable cardinals when it may turn out that the structure one is speaking about is such that all sets are really finite or denumerable?" [4] An answer (alternative to those in [4]) is: use semisets.

(iv) Of course, one can postulate that there are no proper semisets; then one comes back to the classical set theory.

1.3. The theory of semisets contains Set theory. (i) Model theoretically: the set-part of each model of TSS is a model of ZF ; conversely, each model $\mathcal{M}$ of ZF can be (end-) extended to various models of TSS having $\mathcal{M}$ as its set-part. (ii) Proof-theoretically: Not only the basic system TSS (corresponding to 1.1) but also all used stronger systems (corresponding to 1.2) extend ZF conservatively.[10] (iii) Psychologically: your old set-universe is not damaged; you are only invited to widen your visual angle and see (grasp, admit) also other things than sets, namely semisets.

Compare this situation with (i) the way that real numbers are extended to complex numbers (hence imaginary numbers are recognized as actually existing) and (ii) with the non-standard analysis (infinitely small objects accepted as actually existing).

2. Semisets are not sets. This means: it is not always correct to imagine semisets as sets of a bigger universe (which is an end-extension of the smaller one).

2.1. One has a mathematical notion of extendibility of

----

10) This means that if a ZF -statement is provable in such a system TSS then it is provable in ZF ; at the same time, in TSS one can have non-trivial facts about proper semisets.

models of TSS to models of Set theory and induced syntactic extendibility notions for axiom systems (convervatively extendible, not consistently extendible) [11]. Systems corresponding to 1.2(i) are conservatively extendible (which implies that each model of such a system has an end-extension which is a model of Set theory with the same ordinals) but systems corresponding to 1.2(ii), 1.2(iii) are not consistently extendible (no model of such a system has an end-extension which is a model of Set theory with the same ordinals) [12].

2.2. Thus semisets grasped by systems corresponding to 1.2(i) are (can be thought as) sets [13]; the notion of a semiset yields here an axiomatization and simplification. But the use of semisets described by semiset theoretical axiom systems not consistently extendible to Set theory consists of a description and investigation of things (objects, situations) set-theoretically not available. (i) Vopěnka's theorem (EEI) (any two relational structures of the same

----

11) The syntactic extendibility notions are introduced in [7].

12) Some models of 1.2(ii) are extendible to models of GB^- (without the power set axiom) but not all. One can consider the non-extendibility feature more closely, e.g. one can drop the clause "with the same ordinals" and look what models are still not extendible. At any rate, a model M which contains a non-empty semiset of ordinal numbers with no least element cannot have any extension N which is a model of Set theory such that the notion of an ordinal is absolute between M and N.

13) And the best way to exploit this fact is to combine set theoretical and semiset-theoretical means.

standard type that are elementarily equivalent are semiset-isomorphic) provable in a theory combining 1.2(ii) and 1.2 (iii); consequences of (EEI) for representation of models of Peano arithmetic, embeddability of relational structures into their finite substructures, inductive logic etc. [14].
(ii) Various forms of axiomatic non-standard analysis. [15]
(iii) One can have almost consistently [16] the semiset of all feasible numbers (containing 0 , closed under successor, segment of natural numbers but not containing a "Bernays number" - a big natural number described by a simple primitive recursive term) [17].

3. Conclusion: semisets need independent investigation.

3.1. The fact that semisets naturally occur in various domains (cf. 1.2 and 2.2) shows that the notion of a semiset plays a unifying and generalizing role.

3.2. It is legitimate to look for consequences of semi-set-theoretical results in traditional domains (e.g. by

----

14) Vopěnka's theorem is at least two years old but unfortunately not yet published except in [8] where an application to inductive logic is mentioned. Note that (EEI) implies e.g. that each model of Peano (which is a set) is representable as the set of natural numbers with the usual addition and with a new multiplication that may be a proper semiset. (Vopěnka.)

15) Vopěnka; not published, but Vopěnka presented his attempts in several lectures at various occasions.

16) In the sense of Parikh. See Part III of this paper.

17) Professor Kreisel suggested in 1969 that one could use semisets in the abstract recursion theory.

considering semantical models of TSS); but it is not legitimate to replace the results by their consequences [18].

3.3. The theory of semisets is not an auxiliary technical means but it studies a notion which might prove to be one of fundamental mathematical (and metamathematical) notions.

II - Axioms and an interpretation

1. Collections of sets are classes. In particular, in Gödel-Bernays set theory GB one has classes corresponding to ZF-properties of sets; call those classes GB-classes. We think of GB as of a two-sorted theory with a sort $x, y, \dots$ for sets and a sort $X^\circ, Y^\circ, \dots$ for GB-classes. The axioms are: subordinateness $(\forall x)(\exists X^\circ)(x = X^\circ)$; definability of sets $(\exists x)(x = X^\circ) \equiv (\exists Y^\circ)(X^\circ \in Y^\circ)$; extensionality for classes, existence of classes $(\exists X^\circ)(\forall x)(x \in X^\circ \equiv \varphi(x))$ (φ normal, not containing X° , may contain parameters), for sets: pairing, sum, power, infinity, replacement (single axiom), regularity. The initial part up to pairing axiom (incl.) is the theory of classes TC. Note that TC (and GB) is finitely axiomatizable.

----

18) Compare with a "reduction" of results in the analysis in complex domain to theorems on pairs of real numbers!

2. Remarks. (i) GB extends ZF conservatively.

(ii) $GB \vdash X^\circ \subseteq x \rightarrow Set(X^\circ)$,

(iii) $(\forall x) Set(X^\circ \cap x)$.

We now add a new sort $X, Y, \dots$ for general classes and assume that GB-classes are particular general classes. It is natural to suppose that sets together with general classes satisfy TC (so that e.g. intersections, complements etc. of general classes exist). This leads to the following definition due to Balcar:

3. The weak theory of semisets TSS° has three sorts x, X°, X subordinated in this order and axioms (i) of GB for x and X° and (ii) of TC for x and X. Evidently TSS° extends GB conservatively.

4. Definition (TSS°).

$Sm(X) \equiv (\exists x)(x \subseteq X)$ (semisets) ,

$Real(X) \equiv (\forall x) Set(X \cap x)$ (real classes) .

We further introduce a new sort $\sigma, \varphi, \dots$ for semisets.

5. Lemma (TSS°) (i) $(\exists X) \neg Real(X) \equiv (\exists \sigma) \neg Set(\sigma)$,

(ii) $(\forall X^\circ) Real(X^\circ)$.

6. When looking for further axioms we make use of the following economy principle for proper classes: proper classes (non-semisets, consequently non-sets) are auxiliary (since we are interested in semisets); we want as few

proper classes as possible, not loosing any semisets.

(i) GB-classes correspond to (ZF-) properties of sets; imaginary (non-real) classes yield proper semisets. We do not want any other classes and hence formulate the following axiom:

(Re) $$(\exists X^\circ)(X = X^\circ) \equiv \mathit{Real}(X) \ .$$

(ii) Recall the following definitions from [17] (serving for a coding of "systems of semisets"; $\mathcal{D}$ is domain):

$R[x] = \{y ; \langle y, x \rangle \in R\}$; $\mathit{Reg}(R) \equiv (\forall x) \mathit{Sm}(R[x])$; $\mathit{Ncon}(R) \equiv$
$\equiv (\forall x \in \mathcal{D}(R))(x \neq y \rightarrow R[x] \neq R[y])$;

$\mathit{Exct}(R) \equiv \mathit{Reg}(R) \ \& \ \mathit{Ncon}(R)$.

(exact functor - codes a "1-1 mapping of some sets to some semisets")

$$\mathit{Econ}(R) \equiv \mathit{Reg}(R) \ \& \ (\forall x \in \mathcal{D}(R))(\mathit{Sm}\{y \in \mathcal{D}(R) ; R[y] = R[x]\})$$

(economical functor - codes a "1-1 mapping of a disjointed system of semisets onto a system of semisets"). The following is a strong form of replacement axiom for semisets:

(Repl) $$\mathit{Econ}(R) \rightarrow (\mathit{Sm}(\mathcal{D}(R)) \equiv \mathit{Sm}(\mathcal{W}(R))) \ .$$

Note that this axiom is stronger that the axiom (C2) of [17]:

$$\mathit{Exct}(R) \rightarrow (\mathit{Sm}(\mathcal{D}(R)) \equiv \mathit{Sm}(\mathcal{W}(R))) \ .$$

7. <u>The theory of semisets</u> TSS is the theory (TSS°, Re, Repl).

8. <u>Semiset formulas</u> are formulas built up from set variables, semiset variables, membership and equality predicates, connectives and quantifiers.

9. <u>Theorem</u>. TSS extends TSS° conservatively with respect to semiset formulas.

<u>Proof</u>. We prove the theorem by considering an appropriate relative interpretation of TSS in TSS°. This interpretation is due to Balcar and is implicit in [18]. Sets of the interpretation are exactly all sets, GB-classes of the interpretation are exactly all GB-classes and general classes of the interpretation are all classes of the form $R''\sigma$ where R is a GB-relation (GB-class which is a relation) and σ is a semiset. It is easy to show that all axioms of TSS° hold in the interpretation and that the notions of set, semiset, GB-class are absolute. In addition, the following axiom (W) holds in the interpretation: $(\forall X)(\exists R^{\circ})(\exists \sigma)(X = R^{\circ}{}''\sigma)$.

It remains to show that $(TSS^{\circ}, W) \vdash (Re \,\&\, Repl)$.

Let $X = R''\sigma$ where R is a GB-relation and suppose $\sigma \subseteq a$ and $\mathcal{D}(R) \subseteq a$.

(i) Let $Sm(\mathcal{D}(X)) \,\&\, Reg(X)$, $\mathcal{D}(X) = p \subseteq b$; we may suppose $\langle\langle x, y\rangle, z\rangle \in R \longrightarrow y \in b$. Put $\langle x, \langle y, z\rangle\rangle \in S \equiv \langle\langle x, y\rangle, z\rangle \in R$. Then S is a GB-class, $\mathcal{D}(S) \subseteq b \times a$ is a set and we may suppose $Reg(S)$ (we may suppose $z \in \mathcal{D}(R) \longrightarrow (\forall y)(Set(\{x ; \langle\langle x, y\rangle, z\rangle \in R\}))$ since the consequent is true for each $z \in \sigma$). Then

$\mathcal{W}(S)$ is a set since S is a GB-class and we have $\mathcal{W}(X) \subseteq \mathcal{W}(S)$. Thus $\mathcal{W}(X)$ is a semiset.

(ii) Let $Econ(X)$ & $Sm(\mathcal{W}(X))$. Let $\mathcal{W}(X) \subseteq c$ and suppose $R \subseteq (c \times V) \times a$. Put $R_y = \{\langle x, z \rangle ; \langle\langle x, y \rangle, z\rangle \in R\}$. We claim:

$(*)$ $[\mathcal{D}(R_{y_0}) \cap \sigma \neq \emptyset \rightarrow Y = \{y ; R_y = R_{y_0}\}$ is a set.$]$

(Otherwise, it is a proper GB-class and for $z \in \mathcal{D}(R_{y_0}) \cap \sigma$ one has $(\forall y \in Y)(X[y] = X[y_0])$ since $X[y] = R_y" \sigma$, which contradicts $Econ(X)$.) So we may suppose without loss of generality

$[(\forall y)(R_y \neq \emptyset \longrightarrow \{\bar{y} ; R_{\bar{y}} = R_y\}$ is a set.$]$

But each R_y is a set, $R \subseteq (c \times b) \times a$ for some b and $\mathcal{D}(X) \subseteq b$. Thus $\mathcal{D}(X)$ is a semiset.

(iii) Let $Real(X)$. We may suppose $X = F^{-1}" \hat{\sigma}$ for a GB-function F and a semiset $\hat{\sigma}$. (For $x \in \mathcal{W}(R)$, $F(x) = \{y \in a ; \langle x, y \rangle \in R\}$; $x \in R" \sigma \equiv F(x) \cap \sigma \neq \emptyset$, i.e. $X = F^{-1}" \hat{\sigma}$ for $\hat{\sigma} = \mathcal{P}(a) - \mathcal{P}(a - \sigma)$. F is a GB-class and $\mathcal{W}(F) = a$.) For $y \in a$ let d_y be the set of all elements of $F^{-1}[y]$ of least rank and put $f = F \restriction \bigcup_{y \in a} d_y$ (f is a set!.) We have $\mathcal{W}(f) = \mathcal{W}(F)$, $f^{-1}" \hat{\sigma} = (F^{-1}" \hat{\sigma}) \cap \mathcal{D}(f) = X \cap \mathcal{D}(f)$ is a set (since X is supposed to be real). Put $f^{-1}" \hat{\sigma} = q$. Then $f" q = s$ is a set, $s \subseteq a$ and $F^{-1}" \hat{\sigma} = F^{-1}" s$ since $s = \hat{\sigma} \cap \mathcal{W}(F) = \hat{\sigma} \cap \mathcal{W}(f)$. Hence X is a GB-class.

10. <u>Corollary</u> (of the proof of 9). Each semantic model $\mathcal{M}$ of TSS° contains a least submodel $\mathcal{N} \models TSS^\circ$ containing all the sets, semisets and GB-classes of $\mathcal{M}$. $\mathcal{N}$ is a model of TSS and its real classes coincide with GB-classes of $\mathcal{M}$.

11. <u>Theorem</u> (TSS). $\mathcal{O}n$ is not cofinal with any ordinal number (i.e. $\neg (\exists F)(Fnc(F) \mathbin{\&} \mathcal{D}(F) \in \mathcal{O}n \mathbin{\&}$ $\mathbin{\&} \bigcup \mathcal{W}(F) = \mathcal{O}n))$.

12. <u>Remark</u>. (i) TSS as described here but with $(Repl)$ weakened to (C2) is equivalent to the theory TSS''' of [17].

(ii) Balcar's construction shows that one indeed obtains TSS applying the economy principle for proper classes; by that construction, all classes not satisfying $(Re \mathbin{\&} Repl)$ are omitted.

III - The bald man semiset

Parikh ([13] Th.2.2) proved a theorem showing that formal systems in which "large" numbers are treated as if they were infinite give correct results for all proofs of reasonable length. (The introduction of Parikh's paper explicates the ultraintuitionistic criticism of induction recalling Bernays and Jesenin-Volpin.) The <u>bald man paradox</u> (whipping out one hair does not make one bald, so non-baldness is inductive; but whipping out 50 000 hairs does) is one of the arguments that large numbers behave as if they were infinite. (This paradox has often been stressed by Vopěnka; in fact, the result of the present part can be viewed as a possible realization of an intuitive idea of Vopěnka, using Parikh's method.) We formulate Parikh's theorem (in a form slightly differing from the original but convenient for our purpose) and show how Parikh's result can be strengthened in TSS .

1. T denotes a primitively recursively axiomatized theory in which Peano arithmetic is relatively interpretable; in this part we keep our theories one-sorted. (For TSS° use a new unary predicate $\mathcal{G}\ell$ for GB-classes.) So we have $\overline{0}, S, +, \cdot$ in T ; suppose the operations $S, +, \cdot$ be defined $\overline{0}$ if one of the arguments is not a natural number. (Similarly for other number functions.) Define PR-extensions (following Feferman) as follows:

(i) T is its PR-extension; its PR-symbols are $\overline{0}, S, +, \cdot$.

(ii) If T^+ is a PR-extension of T , if F is a function symbol not in $L(T^+)$ (the language of T^+) and if $t(x)$ is a term composed from PR-symbols of T^+ (a PR-term) then $(T^+, F(\underline{x}) = t(\underline{x}))$ is a PR-extension of T^+ ; its PR-symbols are those of T^+ and F .

(iii) If T^+ is a PR-extension of T , if F is not in $L(T^+)$ and if $t(\underline{x})$, $S(\underline{x}, y, x)$ are PR-terms of T^+ then

$$(T^+, F(\overline{0}, \underline{x}) = t(\underline{x}), F(y+1, \underline{x}) = S(\underline{x}, y, F(y, \underline{x})))$$

is a PR-extension of T^+ and its PR-symbols are those of T^+ and F .

In the sequel, let T^+ denote a fixed primitively recursively axiomatized PR-extension of T in which one has definitions for all primitive recursive functions. Note that T^+ is a conservative extension of T. If t is a closed PR-term of T^+ then $|t|$ is its <u>value</u>; if H is a PR-symbol then $|H|$ is the primitive recursive function defined by $|H|(n_1, \dots) = |H(\bar{n}_1, \dots)|$.

2. Each symbol of T^+ is identified with a natural number in a reasonable way; formulas may be identified with their Gödel numbers, but we define the <u>complexity</u> of a formula (or of a sequence of symbols) as the sum of the sequence. So there are only finitely many formulas of a fixed complexity. Note that $Max(n \,;\, \text{complexity } (n) \leq m)$ is a primitive recursive function.

3. If θ is a closed PR-term and if Fe is a unary predicate not in $L(T^+)$ then $(T^+, \mathcal{P}ar_\theta)$ is the following theory:

$$(T^+, Fe(\bar{0}), Fe(x) \rightarrow Fe(x+1), (y < x \,\&\, Fe(x)) \rightarrow Fe(y), \neg Fe(\theta))$$

(read Fe as "feasible"). Evidently, $(T^+, \mathcal{P}ar_\theta)$ is inconsistent, but one has the following

<u>Theorem</u> (Parikh). There is a PR-term $H(x)$ quickly definable (i.e. the complexity of a sequence of definitions leading to H is low) such that, for each n , if θ is a closed PR-term and if $|\theta| \geq |H|(n)$ then $(T^+, \mathcal{P}ar_\theta)$ is an n-almost conservative extension of

T^+ , i.e. each T^+ -formula which has a $(T^+, \mathcal{P}ar_\theta)$ - proof of complexity $< n$ is T^+-provable. (See 6 below for a sketch of a proof.)

4. Remarks. (1) If the complexity of the sequence of definitions leading to H is k and one takes θ e.g. $H(\overline{10}^{\overline{10}^{\overline{k}}})$ then in proofs of complexity $< 10^{10^k}$ one may really use all axioms of $\mathcal{P}ar_\theta$ and many axioms of T^+ (but, by the theorem, not enough many times to derive a contradiction).

(2) Take T to be GB (or TSS). Certainly, one cannot add the assumption that the collection of all feasible numbers is a set; then the proof of contradiction would have a very low complexity (since one has induction for sets). Our question is: may we assume (almost consistently) that all feasible numbers form a semiset? We show that the answer is positive. The meaning of this result is that the collection of all feasible numbers can be thought as an object, moreover, as an object of a theory with known properties. The precise formulation follows.

5. Theorem (on the bald man semiset). Denote by $(TSS^+, \mathcal{P}ar_{\theta,\sigma})$ the theory $(TSS^+, \mathcal{P}ar_\theta, x \in \sigma \equiv Fe(x))$ (σ a constant). There is a PR-term $H(x)$ quickly definable and such that, for each n , if θ is a closed PR-term and if $|\theta| \geq |H|(n)$ then $(TSS^+, \mathcal{P}ar_{\theta,\sigma})$ is an n-almost conservative extension of TSS^+ (and therefore of TSS and of ZF via the obvious interpretation).

6. Remark on the proof of 3 . Verify that there is a primitive recursive function $|H_0|$ assigning to each

$(T^+, Fe(x) \equiv Fe(x))$ - proof d of a formula φ a disjunction D of instances of the matrix of the Herbrand variant φ_{Herb} of (a prenex form of) φ ; let, for each m, $|H|(m)$ be bigger than the maximal number of members of D for any φ with complexity $< m$. Then H satisfies the assertion. Observe:

(1) $T^+ \vdash (\forall x)\mathcal{P}ar_\theta \rightarrow \varphi$, (4) $\vdash \bigwedge_i \mathcal{P}ar_\theta(t_i) \rightarrow \bigvee_i \Phi_0(t_i, S_i)$,

(2) $\vdash (\forall x)\mathcal{P}ar_\theta \rightarrow (T_0^+ \rightarrow \varphi)$, (5) $\vdash \bigwedge_i \mathcal{A}ux(t_i) \rightarrow \bigvee_i \Phi_0(t_i, S_i)$,

(3) $\vdash (\exists x)(\exists \ldots)(\mathcal{P}ar_\theta \rightarrow \Phi_0)$, (6) $T^+ \vdash (\forall x)\mathcal{A}ux(x) \rightarrow \varphi$,

where T_0^+ is a finite subset of T^+, Φ_0 results by making the Herbrand form and $\mathcal{A}ux(x)$ is an auxiliary formula not containing Fe and provable in T^+. ($\mathcal{A}ux(x)$ is constructed using the fact that $T^+ \vdash x \neq \overline{k} \rightarrow \mathcal{P}ar_\theta(Fe(x)/x \leq \overline{k})$. One can take

$$\mathcal{A}ux(x) = [\bigwedge_{k<|\theta|}(x \neq \overline{k} \rightarrow \mathcal{P}ar_\theta(Fe(x)/x \leq k)) \& \bigwedge_{\substack{k \neq l \\ k, l < |\theta|}} \overline{k} \neq \overline{l}]$$

and show that $\bigwedge_i \mathcal{A}ux(t_i) \rightarrow \bigvee_i \Phi_0(t_i, S_i)$ is a quasi-tautology. Cf. [14].

7. We shall now present a detailed plan of the proof of the Skolem theorem on the bald man semiset. We denote by $sk(T)$ the Skolem equivalent of T (open conservative extension of T). Let φ be an arbitrary closed formula with all quantifiers restricted to be sets and containing only the predicates $\in$, $=$ (a ZF-formula). Then there is a closed universal $sk(GB)$ - formula $(\forall X, \ldots)\varphi_0(X, \ldots)$

equivalent to φ both in $sk(GB)$ and in $sk(TSS^{\circ})$. ($\varphi_0(X,\dots)$ begins with $(Set(X) \& \dots) \rightarrow$.) We shall show some transformations of proofs into proofs and we observe that these transformations are primitive recursive. Let ψ be $\neg Fe(\theta) \rightarrow \varphi$ and let d_0 be a proof of ψ in $(TSS^{+}, \mathcal{P}ar^{\circ}_{\sigma})$. ($(TSS^{+}, \mathcal{P}ar^{\circ}_{\sigma}$ is $(TSS^{+}, \mathcal{P}ar_{\theta,\sigma})$ without $\neg Fe(\theta)$.)

(1) $(TSS^{+}, \mathcal{P}ar^{\circ}_{\sigma}) \vdash \psi$ (proof d_0).

By Balcar's interpretation,

(2) $(TSS^{\circ +}, \mathcal{P}ar^{\circ}_{\sigma}) \vdash \psi$ (proof $d_1 = K_1(d_0)$,

(3) $sk(TSS^{\circ +}, \mathcal{P}ar^{\circ}_{\sigma}) \vdash \psi_0$ (proof $d_2 = K_2(d_1)$,

where ψ_0 is $\neg Fe(\theta) \rightarrow \varphi_0$, open. By Herbrand's theorem,

(4) $\bigwedge_{i \in I} inst_i(sk(TSS^{\circ +}, \mathcal{P}ar^{\circ}_{\sigma})) \rightarrow \psi_0(\underline{a})$ (proof $d_3 = K_3(d_2)$,

where $\{inst_i(\dots), i \in I\}$ is a finite set of closed instances of axioms of $sk(TSS^{\circ +}, \mathcal{P}ar^{\circ}_{\sigma})$ and $\underline{a}$ are constants. The crucial point comes now; the idea is borrowed from a construction due to Sochor (cf. [15]). One constructs a relative interpretation of $\bigwedge_{i \in I} inst_i(\dots)$ in (a conservative extension of) $sk(GB^{+}, \mathcal{P}ar^{\circ})$ (i.e. in $sk(GB^{+}, \mathcal{P}ar_{\theta})$ without $\neg Fe(\theta)$) in such a way that set-notions are absolute. The extension is denoted by $sk(GB^{+}, \mathcal{P}ar_0) + fix$. So

(5) $\vdash sk(GB^{+}, \mathcal{P}ar^{\circ}) + fix \rightarrow [(\bigwedge_i inst_i)^{*} \&$

$\& (\psi_0(\underline{a}) \equiv \psi_0^{*}(\underline{a}^{*})]$ (proof $d_4 = K_4(d_3)$

and hence

(6) $\vdash sk(GB^+, \mathcal{P}ar^o) + fix \rightarrow \psi_0(\underline{a})$ (proof $d_5 = K_5(d_4)$).

Since $\psi_0(\underline{a})$ does not contain added symbols, we have

(7) $\vdash sk(GB^+, \mathcal{P}ar^o) \rightarrow \psi_0(a)$ (proof $d_6 = K_6(d_5)$),

(8) $\vdash sk(GB^+, \mathcal{P}ar_\theta) \rightarrow \varphi_0(a)$ (proof $d_7 = K_7(d_6)$)

and hence if H_1 is as in Parikh's theorem (3 above) with respect to $sk(GB^+, \mathcal{P}ar_\theta)$ and if $|H_1|(d_7) \leq |\theta|$ then

(9) $sk(GB^+) \vdash \varphi_0(\underline{a})$

and hence, by conservativity,

(10) $GB^+ \vdash \varphi$.

So we put $H(x) = H_1(K_7(K_6(K_5(K_4(K_3(K_2(K_1(x)))))))))$.

9. It remains to describe the interpretation giving (5). We begin with a semantic motivation.

Let $\mathcal{J}$ be a finite set of closed instances of axioms of $sk(TSS^{o+}, \mathcal{P}ar^o_\sigma)$ and let $\mathcal{M}$ be a transitive model of GB. Then $\mathcal{M}$ can be (end-) extended by adding finitely many new objects to a model $\mathcal{N} \models \mathcal{J}$ such that sets and GB-classes of $\mathcal{N}$ coincide with $\mathcal{M}$.

Sketch of proof. (i) First, some remarks on the particular form of $sk(TSS^{o+}, \mathcal{P}ar^o_\sigma)$. One has predicates $=, \in, Set, Gb, Fe$; constants $V, \bar{0}, \sigma, \ldots$,axioms $Set(X) \equiv X \in V$, $X \in Y \rightarrow X \in V$. Then extensionality axiom gives a Skolem-choice function $\mathcal{A}(X,Y)$ for symmetric difference. Pairing axiom gives the operation of the unordered pair of sets, the Gödel's class existence axioms give Gödelian operations and some other Skolem functions, e.g. the axiom of domain gives an operation d with the axiom $\langle Y, Z \rangle \in X \equiv \langle d(X,Z), Z \rangle \in X$. Those other Skolem functions and the function $\mathcal{a}$ will be called <u>proper Skolem functions</u>. Then all remaining axioms of TSS° may be written as open axioms without introducing new functions, and all variables in them are restricted to Gb or even to Set . Axioms for PR-symbols and $\mathcal{P}ar^{\circ}_{\sigma}$ are open. Observe the following facts on proper Skolem functions: Their values are provably sets, in their open defining formulas no proper Skolem functions occur.

Let now $t_1, \dots, t_n$ be a sequence of all terms occurring in $\mathcal{J}$ (together with subterms), ordered respecting the subterm-relation. We construct a sequence $N_0, \dots N_n = N$ of end-extensions of M . Put $N_0 = M$; if $N_0, \dots, N_{i-1}$ are defined and if t_i is σ then form $\hat{\sigma} = \{x \in \omega^M ; M \models Fe(x)\}$. If $\hat{\sigma} \in N_{i-1}$ then put $N_i = N_{i-1}$ and if <u>not</u> put $N_i = N_{i-1} \cup \{\hat{\sigma}\}$, and extend membership in the obvious way. If t_i is not σ or does not begin with a gödelian operation put $N_i = N_{i-1}$. If $t_i = F(t_j, t_k)$, F gödelian and if $\varphi(X, Y, Z)$ is the canonical definition of $Z \in F(X, Y)$ then form $\{x \in N_{i-1} ; N_{i-1} \models \varphi(t_j^{N_{i-1}}, t_k^{N_{i-1}}, x)\}$ and proceed as above. We put $Set^{N_i} = Set^M$, $Gb^{N_i} = |M|$, $Fe^{N_i} = Fe^M$ etc. By the properties of proper Skolem functions, we may extend them from N_{i-1} to N_i such that their defining axioms are true; the same concerns PR-functions involved. (One puts $F^{N_i}(x, y) = 0^M$ for each PR-function P and for $\neg(x, y \in M)$.) Hence only gödelian operations remain. If $N_i = N_{i-1} \cup \{a\}$ then put $F^{N_i}(a, y) = F^{N_i}(y, a) = 0^M$ for each $y \in N_i$ and each gödelian operation F . For $x, y \in N_i$ one puts $F^{N_i}(x, y) = F^{N_{i-1}}(x, y)$ <u>unless</u> $t_i = F(t_j, t_k)$ and $x = t_j^{N_{i-1}}$, $y = t_k^{N_{i-1}}$. In this case one "corrects" $F^{N_{i-1}}$ putting $F^{N_i}(x, y) = a$. Hence if $inst$ is an instance of the axiom stating the characteristic property of F and if all the terms involved in $F(t_j, t_k)$ are among $t_1, \dots, t_i$ then $N_i \models inst$.

10. <u>Remark</u>. It is easy to see that an isomorphic copy of N is <u>definable</u> in M by an uniform definition. Hence one obtains the required interpretation (inventing some "coding" but in fact nothing else) and the proof is complete. One proceeds in GB^+, $\mathcal{P}ar_0$. First, one puts $Cls^{(0)}(X) \equiv (\exists Y)(X = \{0\} \times Y)$ and defines $\in^{(0)}$, $Set^{(0)}$, $Fe^{(0)}$ etc. in the obvious way; one puts $c_i = \{\langle 1, i\rangle\}$ for $i = 1, \dots, n$. Then one defines inductively $Cls^{(i)}$ and all (i)-notions $(i = 1, \dots, n)$; e.g. for $t_i = \sigma$ one defines

$$\varphi_i \equiv [(\exists Y)(Cls^{(i-1)}(Y) \& (\forall X)(X \in^{(i-1)} Y \equiv (Fe^{(i-1)}(X) \& X \in^{(i-1)} \omega^{(i-1)})))]$$

$$Cls^{(i)}(X) \equiv [\varphi_i \& Cls^{(i-1)}(X)] \vee [\neg \varphi_i \& (Cls^{(i-1)}(X) \vee X = c_i]\ ,$$

$$X \in^{(i)} Y \equiv [\varphi_i \& X \in^{(i-1)} Y] \vee [\neg \varphi_i \& (X \in^{(i-1)} Y \vee (Y = c_i \& Fe^{(i-1)}(X))]$$

etc.

References

[1] B. BALCAR: A theorem on supports in the theory of semisets, Comment.Math.univ.Carolinae 14(1973),1-6.

[2] B.BALCAR: Teorie polomnožin (thesis).

[3] K. ČUDA: Contributions to the theory of semisets III, Zeitschr.f.Math.Log.(to appear).

[4] A.A. FRAENKEL, Y. BAR-HILLEL, A. LEVY: Foundations of set theory, North-Holland P.C.1973.

[5] P. HÁJEK: On semisets, in: Logic Colloquium ´69, North-Holland P.C.1971,p.67-76.

[6] P. HÁJEK: Contributions to the theory of semisets I, Zeitschr.f.Math.Log.18(1972),241-248.

[7] P. HÁJEK: Degrees of dependence in the theory of semisets, to appear in Fund.Math.(Mostowski´s volume).

[8] P. HÁJEK, D. HARMANCOVÁ: On generalized credence functions, Kybernetika (to appear).

[9] G. KREISEL, J.L. KRIVINE: Modelltheorie, Springer Verlag, 1972.

[10] J. MLČEK, A. SOCHOR:Contributions to the theory of semisets II, Zeitschr.f.Math.Log.18(1972), 407-427.

[11] A. MOSTOWSKI: Recent results in Set theory, in: Problems in the philosophy of mathematics (Lakatos - ed.),North-Holland P.C.1967, 82-96.

[12] G.H. MÜLLER: An old philosophical question - and the recent results in the foundations of mathematics, ibid. p.133-135.

[13] R. PARIKH: On existence and feasibility in arithmetic, Journ.Symb.Log.36(1971),494-508.

[14] J.R. SHOENFIELD: Mathematical Logic, Addison-Wesley,1967.

[15] A. SOCHOR: On semisets defined by non-normal formulas (in preparation).

[16] P. VOPĚNKA: The theory of semisets, in: Proc.Int. Congr.of Math. Nice,Gauthier-Villars 1971, p.255-260.

[17] P. VOPĚNKA, P. HÁJEK: The theory of semisets, Academia Prague and North-Holland P.C.1972.

[18] P. ŠTĚPÁNEK: Některé podmodely ultraproduktu (thesis).

Matematický ústav ČSAV

Žitná 25, Praha 1

Československo

(Oblatum 19.6.1973)