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NOTE ON NONLINEAR SPECTRAL THEORY: APPLICATION TO BOUNDARY VALUE PROBLEMS FOR ORDINARY INTEGRODIFFERENTIAL EQUATIONS[x)]

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Abstract: In this paper we prove that under some assumptions it is possible to apply the whole nonlinear spectral theory to the boundary value problem for ordinary integrodifferential equations.

Key words: Spectral analysis of nonlinear operators, Fredholm alternative for nonlinear operators, Ljusternik-Schnirelman theory, weak solution of the boundary value problem for nonlinear integrodifferential equation, regularity properties of the weak solutions.

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Introduction. Three main parts of the nonlinear spectral theory, i.e., Fredholm alternative for nonlinear operators ([2] and [7, Chapt. II]), Ljusternik-Schnirelman theory ([4] and [7, Chapt. III]) and its converse ([5] and [7, Chapt. V]) were up to this time applied to the existence of the solution of nonlinear integral equations of the Lichtenstein type ([6] and [7, Appendix II]) and to the exis-

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x) This paper is taken from a part of the thesis of the second named author which was written on Department of Mathematical Analysis, Charles University, under the supervision of the first author.

tence and multiplicity of the solutions of the boundary value problems for nonlinear ordinary or partial differential equations. Unfortunately, we do not know any example of boundary value problem for nonlinear differential equation which would satisfy at the same time all the assumptions of the nonlinear spectral theory. On the other hand, such examples can be given for integral equations.

In this paper we prove that under some assumptions it is possible to apply the whole nonlinear spectral theory to the boundary value problem for ordinary integrodifferential equations.

1. Definitions and Statement of the Main Results.

Let $2 \le p < \infty$ and let k be a positive integer. Denote by $W_p^k = W_p^k(\langle 0,1 \rangle)$ x) the Sobolev space of all absolutely continuous real functions u on the interval $\langle 0,1 \rangle$ whose derivatives up to the order $k-1$ are also absolutely continuous and whose derivative of the order k is an L_p function. Set

$$\overset{\circ}{W}{}_p^k = \{ u \in W_p^k : u(0) = \ldots = u^{(k-1)}(0) = u(1) = \ldots = u^{(k-1)}(1) = 0 \} .$$

It is easy to see that $\overset{\circ}{W}{}_p^k$ is a separable Banach space with the norm

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x) If $X(\langle 0,1 \rangle)$ is some function space of functions defined on the interval $\langle 0,1 \rangle$, we shall write the symbol X only.

(1.1) $$\| u \|_{k,p} = \left(\sum_{i=1}^{k} \int_0^1 |u^{(i)}(t)|^p \, dt \right)^{1/p} .$$

The Sobolev space $\overset{\circ}{W}{}_p^k$ has a usual structure (see [9] and [7, Appendix III]), moreover, it has a Schauder basis (see [3]).

If $y \in W_p^k$, let us define $\xi(y) \in [L_p]^{k+1}$ by $\xi(y) = (y, y', \dots, y^{(k)})$, $\eta(y) \in [L_p]^k$ by $\eta(y) = (y, y', \dots, y^{(k-1)})$, $\omega(y) \in R_{k+1}$ by $\omega(y) = (\int_0^1 y^2(t)\,dt, \dots, \int_0^1 (y^{(k)}(t))^2 dt)$ and $\tau(y) \in R_k$ by

$$\tau(y) = \left(\int_0^1 y^2(t)\,dt, \dots, \int_0^1 (y^{(k-1)}(t))^2 \, dt \right) .$$

We shall use the symbol $|\ |$ to denote the absolute value and the norm in the k-space R_k. Sometimes we shall write instead of $\xi = (\xi_0, \dots, \xi_k) \in R_{k+1}$ only $\xi = (\eta, \xi_k)$, where $\eta = (\xi_0, \dots, \xi_{k-1}) \in R_k$. Set

$$R_k^+ = \{ x = (x_0, \dots, x_{k-1}) \in R_k : x_i \ge 0,\ i = 0, \dots, k-1 \} ,$$

$$K_k(y) = \{ x \in R_k : |x| \le y \} ,$$

$$K_k^+(y) = K_k(y) \cap R_k^+ .$$

<u>Definition</u>. Let $p \ge 2$, $\lambda \in R_1$ and let

$$a_j(x, \xi) : \langle 0,1 \rangle \times R_{k+1} \longrightarrow R_1 , \quad j = 0, \dots, k ,$$

$$b_j(x, \omega) : \langle 0,1 \rangle \times R_{k+1}^+ \longrightarrow R_1 , \quad j = 0, \dots, k ,$$

$c_j(x,\eta): \langle 0,1\rangle \times R_k \longrightarrow R_1, \quad j=0,\dots,k-1,$

$d_j(x,\eta,\tau): \langle 0,1\rangle \times R_k \times R_k^+ \rightarrow R_1, \; j=0,\dots,2k-1$

and $w \in L_q \; (p^{-1}+q^{-1}=1)$.

The function $u \in \mathring{W}^k_p$ is said to be a weak solution of the homogeneous Dirichlet boundary value problem for ordinary integrodifferential equation

$$(1.2)_w \quad \lambda\{\sum_{j=0}^{k}(-1)^j\Big[\frac{d^j}{dx^j}(a_j(x,\xi(u)(x))) +$$
$$+2u^{(2j)}(x)\int_0^1 b_j(t,\omega(u))dt\Big]\} - \sum_{j=0}^{k-1}(-1)^j\Big[\frac{d^j}{dx^j}(c_j(x,\eta(u)(x))+$$
$$+d_j(x,\eta(u)(x),\tau(u)))+$$
$$+2u^{(2j)}(x)\int_0^1 d_{k+j}(t,\eta(u)(t),\tau(u))\,dt\Big] = w(x)$$

if for each $h \in \mathring{W}^k_p$ the following integral identity holds:

$$(1.3)_w \quad \lambda\{\sum_{j=0}^{k}[\int_0^1 a_j(x,\xi(u)(x))\, h^{(j)}(x)dx +$$
$$+2(\int_0^1 b_j(t,\omega(u))dt)(\int_0^1 u^{(j)}(x)\, h^{(j)}(x)dx)]\} -$$
$$-\sum_{j=0}^{k-1}[\int_0^1 c_j(x,\eta(u)(x))\, h^{(j)}(x)dx +$$
$$+\int_0^1 d_j(x,\eta(u)(x),\tau(u))\, h^{(j)}(x)dx +$$
$$+2(\int_0^1 u^{(j)}(x)\, h^{(j)}(x)dx)(\int_0^1 d_{k+j}(t,\eta(u)(t),\tau(u))dt)] =$$
$$=\int_0^1 w(x)\, h(x)dx$$

(provided the functions a_j, b_j, c_j, d_j satisfy such assumptions that all integrals in the relation $(1.3)_w$ have sense for arbitrary $u, h \in \overset{\circ}{W}{}^k_p$.

Remark 1. By the same way it is possible to define a weak solution of the equation $(1.2)_w$ provided the right hand side w is a bounded linear functional on the space $\overset{\circ}{W}{}^k_p$.

The type of results obtained in this paper may be illustrated best on the following theorem:

Theorem. Let p, k be positive integers. Consider the homogeneous Dirichlet boundary value problem for the equation

$$(1.4)_w \qquad (-1)^k \lambda \Big\{ \frac{d^k}{dx^k}((u^{(k)}(x))^{2p-1}) + u^{(2k)}(x)\Big(\int_0^1 (u^{(k)}(t))^2 dt\Big)^{p-1} \Big\} - \Big\{ \sum_{j=0}^{k-1} (-1)^j \frac{d^j}{dx^j}((u^{(j)}(x))^{2p-1}) \Big\} + (-1)^{k-1} u^{(2k-2)}(x)\Big(\int_0^1 (u^{(k-1)}(t))^2 dt\Big)^{p-1} = w(x) .$$

Then there exists a sequence Λ of positive numbers converging to zero such that

(i) the equation $(1.4)_w$ has a weak solution $u \in \overset{\circ}{W}{}^k_{2p}$ for any $w \in L_q$ $((2p)^{-1} + q^{-1} = 1)$ provided $\lambda \notin \Lambda \cup \{0\}$;

(ii) for each $\lambda \in \Lambda$ the equation $(1.4)_n$ has a nontrivial weak solution $u \in \overset{\circ}{W}{}^{k}_{2p}$. Moreover, $u \in C^{\infty}$ and u is a classical solution, i.e., it satisfies the equation $(1.4)_0$ in each point $x \in \langle 0, 1 \rangle$.

The assertions of the previous theorem follow from general Theorems 1 - 3 , Remark 2 and Lemma 1.

<u>Theorem 1</u> (application of the Ljusternik-Schnirelman theory). Suppose

$$\begin{aligned} a(x, \xi) \quad &: \langle 0, 1 \rangle \times R_{k+1} \longrightarrow R_1 \ , \\ b(x, \omega) \quad &: \langle 0, 1 \rangle \times R^+_{k+1} \longrightarrow R_1 \ , \\ c(x, \eta) \quad &: \langle 0, 1 \rangle \times R_{k} \longrightarrow R_1 \ , \\ d(x, \eta, \tau) &: \langle 0, 1 \rangle \times R_k \times R^+_k \longrightarrow R_1 \ . \end{aligned}$$

Let the following conditions be fulfilled:

(a 1) the function $a(x, \xi)$ satisfies the Carathéodory conditions on the interval $\langle 0, 1 \rangle$ (for definition see e.g. [10]);

(a 2) there exist a nonnegative function $g_a(x, y)$ defined on $\langle 0, 1 \rangle \times R^+_1$, $g_a(\cdot, y) \in L_1$ for each $y \in R^+_1$ and the constants $c_1, c_2 > 0$ such that

$$c_1 |\xi_k|^p \leq a(x, \eta, \xi_k) \leq g_a(x, y) + c_2 |\xi_k|^p$$

for each $\eta \in K_k(y)$, $\xi_k \in R_1$ and almost all $x \in \langle 0, 1 \rangle$;

(a 3) $a(x, \xi) = 0 \Longleftrightarrow \xi = 0, \ a(x, -\xi) = a(x, \xi)$;

(a 4) the partial derivatives $a_j(x, \xi) = \dfrac{\partial a}{\partial \xi_j}(x, \xi)$

$(j = 0, \dots, k)$ exist on $\langle 0, 1 \rangle \times R_{k+1}$ and satisfy the Carathéodory conditions on $\langle 0, 1 \rangle$;

(a 5) there exist nonnegative functions $g_{a_j}(x, y)$ defined on $\langle 0, 1 \rangle \times R_1^+$ and a constant $c > 0$ such that

$g_{a_j}(\cdot, y) \in L_1$, $j = 0, \dots, k-1$, $y \in R_1^+$,

$g_{a_k}(\cdot, y) \in L_q$, $(p^{-1} + q^{-1} = 1)$,

$|a_j(x, \eta, \xi_k)| \le g_{a_j}(x, y) + c\,|\xi_k|^p$ $(j = 0, \dots, k-1)$,

$|a_k(x, \eta, \xi_k)| \le g_{a_k}(x, y) + c\,|\xi_k|^{p-1}$

for each $\eta \in K_k(y)$, $\xi_k \in R_1$ and almost all $x \in$ $\in \langle 0, 1 \rangle$;

(a 6) there exists a constant $M > 0$ such that

$$\sum_{j=0}^{k} a_j(x, \xi)\,\xi_j \ge M a(x, \xi)$$

for each $\xi \in R_{k+1}$ and almost all $x \in \langle 0, 1 \rangle$;

(a 7) there exists a continuous nonnegative function $\gamma(t)$ defined on R_1^+ such that

$|a_j(x, \eta, \xi_k) - a_j(x, \eta', \xi'_k)| \le \gamma(|\eta| + |\eta'|)\,\cdot$
$\cdot\,(1 + |\xi_k|^{p-2} + |\xi'_k|^{p-2})\,|\xi_k - \xi'_k|$

for $j = 0, \dots, k$, each $(\eta, \xi_k), (\eta', \xi'_k) \in R_{k+1}$ and almost all $x \in \langle 0, 1 \rangle$;

(a 8) the inequality

$$(a_k(x,\eta,\xi_k) - a_k(x,\eta,\xi'_k))(\xi_k - \xi'_k) > 0$$

holds for each $\eta \in R_k$, all $\xi_k, \xi'_k \in R_1$ and almost all $x \in \langle 0,1 \rangle$;

(b 1) for each $\omega \in R^+_{k+1}$ let $b(\cdot,\omega) \in L_1$ and let

$$b(x,\omega) \geq 0, \quad b(x,0) = 0$$

for each $\omega \in R_{k+1}$ and almost all $x \in \langle 0,1 \rangle$;

(b 2) the partial derivatives $b_j(x,\omega) = \dfrac{\partial b}{\partial \omega_j}(x,\omega)$ exist on $\langle 0,1 \rangle \times R^+_{k+1}$ and $b_j(\cdot,\omega) \in L_1$ for $\omega \in R^+_{k+1}$, $j = 0,\dots,k$;

(b 3) $\sum_{j=0}^{k} b_j(x,\omega)\omega_j \geq M b(x,\omega)$,

(b 4) $b_j(x,\omega) \geq 0$;

(b 5) for each $y \in R^+_1$ there exists $b_y \in L_1$, $b_y(x) \geq 0$ such that $|b_j(x,\omega)| \leq b_y(x)$ for $j = 0,\dots,k$, $\omega \in K^+_{k+1}(y)$ and almost all $x \in \langle 0,1 \rangle$;

(b 6) there exists a continuous nonnegative function $\varphi(x,y)$ defined on R^+_2 such that

$$|b_j(x,\omega) - b_j(x,\omega')| \leq \varphi(|\omega|+|\omega'|,y)\,|\omega_k - \omega'_k|$$

for $j = 0,\dots,k$, $\omega,\omega' \in R^+_{k+1}$ and almost all $x \in \langle 0,1 \rangle$;

(c 1) the function $c(x,\eta)$ satisfies the Carathéodory conditions on $\langle 0,1\rangle$;

(c 2) there exists a nonnegative function $g_c(x,y)$ defined on $\langle 0,1\rangle \times R_1^+$, $g_c(\cdot,y)\in L_1$ such that $|c(x,\eta)|\leq g_c(x,y)$ for each $\eta\in K_k(y)$ and almost all $x\in\langle 0,1\rangle$;

(c 3) $c(x,-\eta)=c(x,\eta)\geq 0,\ c(x,\eta)=0 \Longleftrightarrow \eta=0$;

(c 4) the partial derivatives $c_j(x,\eta)=\dfrac{\partial c}{\partial \eta_j}(x,\eta)$

exist on $\langle 0,1\rangle\times R_k$, satisfy the Carathéodory conditions on $\langle 0,1\rangle$ and they are bounded on $\langle 0,1\rangle\times K_k(y)$ for any $y\in R_1^+$ $(j=0,\dots,k-1)$;

(c 5) $\sum_{j=0}^{k-1} c_j(x,\eta)\eta_j>0$, $c_j(x,0)=0$

for each $\eta\in R_k$, $\eta\neq 0$ and almost all $x\in\langle 0,1\rangle$;

(d 1) the function $d(x,\eta,\tau)$ satisfies the Carathéodory conditions on $\langle 0,1\rangle$;

(d 2) for each $\tau\in R_k^+$ there exists a nonnegative function $g_{d\tau}(x,y)$ defined on $\langle 0,1\rangle\times R_1^+$, $g_{d\tau}(\cdot,y)\in L_1$ for each $y\in R_1^+$ such that

$$|d(x,\eta,\tau)|\leq g_{d\tau}(x,y)$$

for each $\eta\in K_k(y)$ and for almost all $x\in\langle 0,1\rangle$;

(d 3) $d(x,-\eta,\tau)=d(x,\eta,\tau)\geq 0,\ d(x,0,0)=0$;

(d 4) the partial derivatives

$d_j(x,\eta,\tau) = \frac{\partial d}{\partial \eta_j}(x,\eta,\tau)$, $d_{k+j} = \frac{\partial d}{\partial \tau_j}(x,\eta,\tau)$ ($j = 0,\dots,k-1$) exist on $\langle 0,1\rangle \times R_k \times R_k^+$, satisfy the Carathéodory conditions on $\langle 0,1\rangle$ and they are bounded on $\langle 0,1\rangle \times K_k(y) \times K_k^+(y)$ for each $y \in R_1^+$;

(d 5) $\sum_{j=0}^{k-1} (d_j(x,\eta,\tau)\eta_j + 2 d_{k+j}(x,\eta,\tau)\tau_j) \geq 0$

for each $\eta \in R_k$, $\tau \in R_k^+$ and almost all $x \in \langle 0,1\rangle$;

(d 6) $d_j(x,0,0) = 0$, $j = 0,\dots,k-1$.

Let $\kappa > 0$. Then there exists a sequence $\{\lambda_m\}$ such that for $\lambda = \lambda_m$ there exists a weak solution $u_m \in \mathring{W}_p^k$ of the Dirichlet boundary value problems for the equation $(1.2)_0$ such that:

(i) $u_m \rightharpoonup 0$ (converges weakly in $\mathring{W}_p^k$);

(ii) $$\gamma_m = \int_0^1 (c(x,\eta(u_m)(x)) + d(x,\eta(u_m)(x), \tau(u_m)))\,dx \searrow 0;$$

(iii) $$\int_0^1 (a(x,\xi(u_m)(x)) + b(x,\omega(u_m)))\,dx = \kappa;$$

(iv) $$\Big[\sum_{j=0}^{k-1} \Big(\int_0^1 c_j(x,\eta(u_m)(x))\, u_m^{(j)}(x)\,dx + \int_0^1 d_j(x,\eta(u_m)(x),\tau(u_m))\, u_m^{(j)}(x)\,dx + 2\Big(\int_0^1 d_{k+j}(t,\eta(u_m)(t),\tau(u_m))\,dt\Big)\Big(\int_0^1 (u_m^{(j)}(x))^2\,dx\Big)\Big)\Big] : \Big[\sum_{j=0}^{k} \Big(\int_0^1 a_j(x,\xi(u_m)(x))\, u_m^{(j)}(x)\,dx + 2\Big(\int_0^1 b_j(t,\omega)(u_m))\,dt\Big)\Big(\int_0^1 (u_m^{(j)}(x))^2\,dx\Big)\Big)\Big] = \lambda_m.$$

<u>Theorem 2</u> (application of the Fredholm alternative). Let the notation introduced in Theorem 1 be observed. Suppose (a 1) - (a 8), (b 1), (b 2), (b 4) - (b 6), (c 1) - (c 4), (d 1) - (d 4). Moreover, let the following conditions be fulfilled:

(a 9) $\sum_{j=0}^{k-1}(a_j(x,\xi)-a_j(x,\xi'))(\xi_j-\xi'_j) \geq 0$

for each $\xi, \xi' \in R_{k+1}$ and almost all $x \in \langle 0,1\rangle$;

(a 10) $a_j(x,t\xi) = t^{p-1}a_j(x,\xi)$

for each $\xi \in R_{k+1}$, $t \in R_1^+$, $j = 0,\dots,k$;

(b 7) $\sum_{j=0}^{k}(b_j(x,\omega)-b_j(x,\omega'))(\omega_j-\omega'_j) \geq 0$

for each ω, $\omega' \in R_{k+1}^+$ and almost all $x \in \langle 0,1\rangle$;

(b 8) $b_j(x,t\omega) = t^{p/2-1}\, b_j(x,\omega)$

for each $\omega \in R_{k+1}^+$, $t \in R_1^+$ and almost all $x \in \langle 0,1\rangle$;

(c 6) $c_j(x,t\eta) = t^{p-1}\, c_j(x,\eta)$

for each $\eta \in R_k$, $t \in R_1^+$, $j=0,\dots,k-1$ and almost all $x \in \langle 0,1\rangle$;

(d 7) $d_j(x,t\eta,t^2\tau) = t^{p-1}\, d_j(x,\eta,\tau)$,

$d_{k+j}(x,t\eta,t^2\tau) = t^{p-2}\, d_{k+j}(x,\eta,\tau)$

for each $\eta \in R_k$, $\tau \in R_k^+$, $t \in R_1^+$, $j=0,\dots,k-1$ and almost all $x \in \langle 0,1 \rangle$

Let $\lambda \neq 0$. Then the equation $(1.2)_w$ has a weak solution $u \in \mathring{W}^k_p$ for any right hand side $w \in L_q$ provided the equation $(1.2)_0$ has only a trivial weak solution.

Theorem 3. Suppose (a 1), (a 2), (a 4) - (a 7), (b 1) - (b 6), (c 1), (c 2), (c 4), (d 1) - (d 4). Moreover, suppose:

(a 11) $a(x, \xi) \in C^{k+2}(\langle 0,1 \rangle \times R_{k+1})$;

(a 12) $\dfrac{\partial^2 a}{\partial \xi_k^2}(x, \xi) \geq 0$ for each $\xi \in R_{k+1}$ and all $x \in \langle 0, 1 \rangle$;

(a 13) the function $a(x, \xi)$ is a restriction of a continuous complex valued function $\tilde{a}(x, \tilde{\xi})$ defined on $\langle 0,1 \rangle \times \mathcal{O}^{k+1}$, where $\mathcal{O}$ is an open set in the complex plane $\mathbb{C}$ such that $\mathcal{O} \supset R_1$;

(a 14) the function $\tilde{a}(x, \tilde{\xi})$ and its derivatives $\dfrac{d^j}{dx^j}\left(\dfrac{\partial \tilde{a}}{\partial \tilde{\xi}_j}(x, \tilde{\xi})\right)$ are analytic on $\mathcal{O}^{k+1}$ for each $x \in \langle 0,1 \rangle$, $j = 0, \dots, k$;

(b 9) $b(x, \omega) \in C^1(\langle 0,1 \rangle \times R_{k+1})$;

(b 10) $b_k(x, \omega) > 0$ for each $x \in \langle 0,1 \rangle$ and $\omega \in R^+_{k+1}$, $\omega \neq 0$;

(b 11) the function $b(x, \omega)$ is a restriction of a continuous complex valued function $\tilde{b}(x, \tilde{\omega})$ defined on $\langle 0,1\rangle \times \mathbb{C}^{k+1}$;

(b 12) the function $\tilde{b}(x, \tilde{\omega})$ and its derivatives $\frac{\partial \tilde{b}}{\partial \tilde{\omega}_j}(x, \tilde{\omega})$ are analytic on $\mathbb{C}^{k+1}$ for each $x \in \langle 0,1\rangle$, $j = 0, \dots, k$;

(c 7) $c(x, \eta) \in C^{k+2}(\langle 0,1\rangle \times R_k)$;

(c 8) $c(x, 0) = c_j(x, 0) = 0$;

(c 9) the function $c(x, \eta)$ is a restriction of a continuous complex valued function $\tilde{c}(x, \tilde{\eta})$ defined on $\langle 0,1\rangle \times \mathbb{C}^{k}$;

(c 10) the function $\tilde{c}(x, \tilde{\eta})$ and its derivatives $\frac{d^j}{dx^j}\left(\frac{\partial \tilde{c}}{\partial \tilde{\eta}_j}(x, \tilde{\eta})\right)$ are analytic on $\mathbb{C}^{k}$ for each $x \in \langle 0,1\rangle$ and $j = 0, \dots, k-1$;

(d 8) $d(x, \eta, \tau) \in C^{1}(\langle 0,1\rangle \times R_k \times R_k^+)$;

(d 9) $d_j(x, \eta, \tau) \in C^{k+1}(\langle 0,1\rangle \times R_k)$ for each fixed $\tau \in R_k^+$, $(j = 0, \dots, k-1)$;

(d 10) $d(x, 0, 0) = d_j(x, 0, 0) = 0$ $(j = 0, \dots, k-1, x \in \langle 0,1\rangle)$;

(d 11) the function $d(x, \eta, \tau)$ is a restriction of a continuous complex valued function $\tilde{d}(x, \tilde{\eta}, \tilde{\tau})$ defined on $\langle 0,1\rangle \times \mathcal{O}^{2k}$;

(d 12) the function $\tilde{d}(x,\tilde{\eta},\tilde{\tau})$ and its derivatives $\frac{d^j}{dx^j}\left(\frac{\partial \tilde{d}}{\partial \tilde{\eta}_j}(x,\tilde{\eta},\tilde{\tau})\right)$, $\frac{\partial \tilde{d}}{\partial \tilde{\tau}_j}(x,\tilde{\eta},\tilde{\tau})$ are analytic on $\mathcal{O}^{2k}$ for each fixed $x \in \langle 0,1\rangle$ and $j = 0,\dots,k-1$.

Let $\varkappa > 0$. Denote by Γ the set of all

$$\gamma = \int_0^1 (c(x,\eta(u)(x)) + d(x,\eta(u)(x),\tau(u)))\,dx ,$$

where u is a solution of the boundary value problem for the equation $(1.2)_0$ for some $\lambda \in R_1$ and satisfying

$$(1.5) \qquad \int_0^1 (a(x,\xi(u)(x)) + b(x,\omega(u)))\,dx = \varkappa .$$

Then the set Γ is at most countable and the only possible accumulation point of this set is zero.

Remark 2. If the assumptions (a 10), (b 8), (c 6) and (d 7) are fulfilled then the set Λ of all $\lambda \in R_1$ for which there exists a solution of $(1.2)_0$ satisfying (1.5) is up to the multiplicative constant equal to the set Γ introduced in Theorem 3.

In the next sections we shall prove Theorems 1 - 3. The proofs are based on [4, Theorem 2], [2, Theorem 3] and [5, Theorem 3.2]. We do not write these theorems here and refer the readers to the cited papers.

2. Proof of Theorem 1.

Denote $(\mathring{W}^k_p)^* = W^{-k}_q$ (the dual space) and let (u^*, u) be the value of the functional $u^* \in W^{-k}_q$ at the point $u \in \mathring{W}^k_p$. For $u \in \mathring{W}^k_p$ set

$$(2.1)\quad f(u) = \int_0^1 (a(x, \xi(u)(x)) + b(x, \omega(u)))dx ,$$

$$(2.2)\quad g(u) = \int_0^1 (c(x, \eta(u)(x)) + d(x, \eta(u)(x), \tau(u)))dx .$$

The functionals f and g are even on the space $\mathring{W}^k_p$ and they have the Fréchet derivatives f' and g' defined by

$$(2.3)\quad (f'(u), h) = \sum_{j=0}^{k} (\int_0^1 a_j(x, \xi(u)(x)) h^{(j)}(x)\, dx +$$
$$2(\int_0^1 b_j(t, \omega(u))dt)(\int_0^1 u^{(j)}(x) h^{(j)}(x)\, dx)) ,$$

$$(2.4)\quad (g'(u), h) = \sum_{j=0}^{k-1} (\int_0^1 (c_j(x, \eta(u)(x)) +$$
$$+ d_j(x, \eta(u)(x), \tau'(u))) h^{(j)}(x)\, dx$$
$$+ 2(\int_0^1 d_{k+j}(t, \eta(u)(t), \tau(u))dt)(\int_0^1 u^{(j)}(x) h^{(j)}(x)dx))$$

for each $u, h \in \mathring{W}^k_p$.

We obtain immediately:

(2.5) $f(u) = 0 \Longleftrightarrow u = 0$ (from (a 2), (a 3), (b 1));

(2.6) $(f'(u), u) \geq M f(u) \geq M c_1 \|u\|^p_{k,p}$ (from (a 2), (a 6), (b 1), (b 3));

(2.7) $\lim\limits_{\|u\|_{k,p} \to \infty} f(u) = \infty$ (from (a 2));

(2.8) $g(u) \geq 0$, $g(u) = 0 \Longleftrightarrow u = 0$ (from (c 3), (d 3));

(2.9) g' is a strongly continuous mapping (i.e., it maps weakly convergent sequences in $\overset{\circ}{W}{}_p^k$ onto strongly convergent sequences in W_q^{-k}), (it follows from the complete continuity of the imbedding from the space $\overset{\circ}{W}{}_p^k$ into C^{k-1}) - thus g' is also uniformly continuous on each bounded subset of $\overset{\circ}{W}{}_p^k$;

(2.10) f' is uniformly continuous on each bounded subset of $\overset{\circ}{W}{}_p^k$ (from (a 7), (b 6));

(2.11) $g'(u) = 0 \Longleftrightarrow u = 0$ (from (c 5), (d 5), (d 6));

(2.12) $u_n \rightharpoonup u$, $f'(u_n) \longrightarrow z \Longrightarrow u_n \longrightarrow u$ (from (a 2), (a 6),(b 5), (a 8), (b 4)).

Thus all assumptions of [4, Theorem 2] are verified and from the assertion of this theorem (see also [7, Chapt. III]) we obtain the assertion of Theorem 1.

3. Proof of Theorem 2.

Denote $T = f'$, $S = g'$. Under assumptions of Theorem 2 the mappings T and S are odd and $(p-1)$-homogeneous (it follows from (a 10), (b 8), (c 6), (d 7), (d 3), (c 3), (a 3)). Moreover,

(3.1) T is strictly monotone (i.e., $(T(u)-T(v), u-v) > 0$ for each $u, v \in \overset{\circ}{W}{}^{k}_{p}$, $u \neq v$) and from (2.6) with using the main theorem about monotone operators (see e.g. [7, Chapt. II]) we obtain that T is surjective. Analogously as (2.12) we can prove that T is a homeomorphism from $\overset{\circ}{W}{}^{k}_{p}$ onto W^{-k}_{q};

(3.2) there exist two constants $K, L > 0$ such that

$L \| u \|^{p-1}_{k,p} \le \| T(u) \|_{-k,q} \le K \| u \|^{p-1}_{k,p}$ for each $u \in \overset{\circ}{W}{}^{k}_{p}$.

Thus together with (2.9) we verified all assumptions of [2, Theorem 3] (see also [7, Chapt. II]). From this we obtain the assertion of Theorem 2.

4. Proof of Theorem 3.

To prove Theorem 3 we shall apply [5, Theorem 3.2] (see also [7, Chapt. V]). Denote $X_1 = C^{2k} \cap \overset{\circ}{W}{}^{k}_{p}$, $X_2 = C$, $X_3 = \overset{\circ}{W}{}^{k}_{p}$. For $u \in X_1$, $v \in X_2$ we define

$$\langle u, v \rangle = \int_0^1 u(x)\, v(x)\, dx$$

the bilinear form on $X_1 \times X_2$ with the following properties:

(4.1) for each $u \in X_1$, $\langle u, \cdot \rangle$ is bounded linear functional on X_2 ;

(4.2) let $v \in X_2$ and $\langle u, v \rangle = 0$ for each $u \in X_1$. Then $v = 0$.

Lemma 1. Suppose (a 11),(a 12), (c 7), (d 9), (b 10). Let $\lambda \neq 0$ and let $u \in \overset{\circ}{W}{}_p^k$ be a weak solution of the equation $(1.2)_0$. Then $u \in X_1$.

Proof. Suppose that u is not identically zero on $\langle 0, 1 \rangle$. Put

$$K_j = 2\int_0^1 b_j(t, \omega(u))\,dt \quad (j = 0, \dots, k),$$

$$K_{i+k+1} = 2\int_0^1 d_{k+i}(t, \eta(u)(t), \tau(u))\,dt \quad (i = 0, \dots, k-1).$$

The identity $(1.3)_0$ can be written as follows:

$$(4.3) \qquad \int_0^1 P(x)\, h^{(k)}(x)\,dx = 0$$

for each $h \in \overset{\circ}{W}{}_p^k$, where

$$\begin{aligned} P(x) = {} & \lambda a_k(x, \xi(u)(x)) + \lambda K_k u^{(k)}(x) + \\ & + \sum_{j=0}^{k-1} (-1)^{k-j} \int_0^x \frac{(x-s)^{k-j-1}}{(k-j-1)!} [\lambda a_j(s, \xi(u)(s)) + \\ & + \lambda K_j u^{(j)}(s) - c_j(s, \eta(u)(s)) \\ & - d_j(s, \eta(u)(s), \tau(u)) - K_{k+1+j} u^{(j)}(s)]\,ds . \end{aligned}$$

The function P is of the class L_q. Let us show that the following assertion $(*)$ holds: there exist constants $p_0, \dots, p_{k-1}$ such that

$$(4.4) \qquad P(x) + p_0 + p_1 x + \dots + p_{k-1} x^{k-1} = 0$$

almost everywhere on $\langle 0,1\rangle$.

For the proof of the assertion $(*)$ denote by $p_0, \dots$ $\dots, p_{k-1}$ such constants that

$$\int_0^1 (P(x) + p_0 + p_1 x + \dots + p_{k-1} x^{k-1}) x^j \, dx = 0$$

for each $j = 0, 1, \dots, k-1$.

The relation (4.3) implies

$$\int_0^1 (P(x) + p_0 + p_1 x + \dots + p_{k-1} x^{k-1}) h^{(k)}(x) \, dx = 0$$

for each $h \in \mathring{W}_p^k$. Suppose $\varphi \in L_p$ and set

$$h(x) = \int_0^x \frac{(x-t)^{k-1}}{(k-1)!} (\varphi(t) + b_0 + b_1 t + \dots + b_{k-1} t^{k-1}) \, dt,$$

where b_j $(j = 0, \dots, k-1)$ are chosen so that $h \in \mathring{W}_p^k$. Substituting the function h into (4.3), we obtain

$$0 = \int_0^1 (P(x) + p_0 + \dots + p_{k-1} x^{k-1})(\varphi(x) + b_0 + \dots$$
$$\dots + b_{k-1} x^{k-1}) \, dx =$$
$$= \int_0^1 (P(x) + p_0 + \dots + p_{k-1} x^{k-1}) \varphi(x) \, dx .$$

Thus we proved $(*)$, since the function $\varphi \in L_p$ was arbitrary. Defining

$$\Phi(x, \xi_k) = \lambda a_k(x, \eta(u)(x), \xi_k) + \lambda K_k \xi_k +$$
$$+ \sum_{j=0}^{k-1} (-1)^{k-j} \int_0^x \frac{(x-s)^{k-j-1}}{(k-j-1)!} [\lambda a_j(s, \xi(u)(s)) + \lambda K_j u^{(j)}(s) -$$
$$- c_j(s, \eta(u)(s)) - d_j(s, \eta(u)(s), \tau(u)) -$$

$-K_{k+1+j}u^{(j)}(s)]\,ds + p_0 + p_1 x + \ldots + p_{k-1}x^{k-1}$

for $x \in \langle 0, 1 \rangle$ and $\xi_k \in R_1$, we have (for $\lambda > 0$)

$$(4.5) \qquad \frac{\partial \Phi}{\partial \xi_k}(x, \xi_k) =$$
$$= \lambda \frac{\partial^2 a}{\partial \xi_k^2}(x, \eta(u)(x), \xi_k) + \lambda K_k \geq \lambda K_k > 0 .$$

Hence for all $x \in \langle 0, 1 \rangle$ the equation

$$(4.6) \qquad \Phi(x, \xi_k) = 0$$

has at most one solution $\xi_k = \xi_k(x)$.

Since

$$(4.7) \qquad \Phi(x, u^{(k)}(x)) = 0$$

(with respect to (4.4)) almost everywhere on $\langle 0, 1 \rangle$ we choose $x_0 \in (0, 1)$ such that (4.7) holds with $x = x_0$. Thus on some neighborhood $\mathcal{U}$ of the point x_0 there exists a function $\xi_k(x)$ satisfying on $\mathcal{U}$ the equation (4.6), $\xi_k(x_0) = u^{(k)}(x_0)$ and in virtue of Implicit Function Theorem (and thus also $u^{(k)}$) is continuous on $\mathcal{U}$. So $u^{(k)} \in C$. As a consequence of (a 11), (c 7) and $u \in C^k$ we have $\Phi \in C^1(\langle 0, 1 \rangle \times R_1)$ and again from Implicit Function Theorem yields $\xi_k(x) \in C^1$, hence $u \in C^{k+1}$. In the same way step by step we obtain $u \in C^{k+2}, \ldots$
$\ldots, u \in C^{2k}$.

Lemma 2. Suppose (a 11), (a 13), (a 14), (b 9), (b 11), (b 12). Then the functional f defined by (2.1) is real analytic on the space X_1 (in the sense of [8]). Moreover, there exists a unique mapping $F : X_1 \longrightarrow X_2$ with the following properties:

(4.8) $(f'(u), h) = \langle h, F(u) \rangle$ for each $u, h \in X_1$;

(4.9) F is real analytic from X_1 into X_2 .

The mapping F is defined by

$$F(u)(x) = \sum_{j=0}^{k}(-1)^j\Big[\frac{d^j}{dx^j}\, a_j(x, \xi(u)(x)) + \\ + 2u^{(2j)}(x) \int_0^1 b_j(t, \omega(u))\,dt\Big]$$

for each $u \in X_1$.

Proof. Denote by $\tilde{X}_1$, $\tilde{X}_2$ the complectifications of the spaces X_1 and X_2. For $u \in \tilde{X}_1$ define

$$\tilde{f}(u) = \int_0^1 (\tilde{a}(x, \tilde{\xi}(u)(x)) + \tilde{b}(x, \tilde{\omega}(u)))\,dx ,$$

$$\tilde{F}(u)(x) = \sum_{j=0}^{k}(-1)^j\Big[\frac{d^j}{dx^j}\Big(\frac{\partial \tilde{a}}{\partial \xi_j}(x, \tilde{\xi}(u)(x))\Big) + \\ + 2u^{(2j)}(x) \int_0^1 \frac{\partial \tilde{b}}{\partial \tilde{\omega}_j}(t, \tilde{\omega}(u))\,dt\Big] .$$

The functional $\tilde{f}$ and the operator $\tilde{F}$ are analytic on the space $\tilde{X}_1$. From this, from [1, Theorem 5.7] an using the integration by parts we obtain our assertion.

Analogously we can prove

Lemma 3. Suppose (c 7), (c 9), (c 10), (d 8), (d 9), (d 11). Then the functional g defined by (2.2) is real analytic on the space X_1 and the mapping $G: X_1 \to X_2$ defined for $u \in X_1$ by

$$G(u)(x) = \sum_{j=0}^{k-1} (-1)^j \Big[\frac{d^j}{dx^j} (c_j(x, \eta(u)(x)) +$$
$$+ d_j(x, \eta(u)(x), \tau(u))) +$$
$$+ 2u^{(2j)}(x) \int_0^1 d_{k+j}(t, \eta(u)(t), \tau(u))\, dt \Big]$$

is a unique mapping with the following properties:

(4.10) $(g'(u), h) = \langle h, G(u) \rangle$ for each $u, h \in X_1$;

(4.11) G is real analytic on X_1 .

Moreover, the mapping G is completely continuous on X_1 .

Lemma 4. Let the assumptions of Lemma 1 be satisfied. Set

$$J(h)(x) = (-1)^k \Big[\frac{\partial}{\partial \xi_k} (a_k(x, \xi(u_0)(x))) +$$
$$+ \int_0^1 b_k(t, \omega(u_0))\, dt\Big]\, h^{(2k)}(x), \; K(h) = dF(u_0, h) - J(h)$$

for each $u_0, h \in X_1$.

Then the mapping J is an isomorphism from X_1 onto X_2 and the operator K is completely continuous.

(The proof is obvious.)

Let $\varkappa > 0$, $0 < \delta < \Delta$. Set

$Q(\delta, \Delta) = \{u \in \overset{\circ}{W}{}_p^{k}: f(u) = \varkappa$ and there exists λ , $\delta \le |\lambda| \le \Delta$ such that u is a weak solution of $(1.2)_0 \}$.

Under the assumptions of Lemma 1 we have not only

$Q(\sigma, \Delta) \subset X_1$ but, moreover, $Q(\sigma, \Delta) \subset$
$\subset C^{2k+1} \cap \mathring{W}_p^k$.

Lemma 5. Under the assumptions of Theorem 3 the set $Q(\sigma, \Delta)$ is a compact subset of the space X_1 .

Proof. The set $Q(\sigma, \Delta)$ is bounded in X_3 . So the coefficients $p_0, \dots, p_{k-1}$ from the relation (4.4) are bounded independently of $u \in Q(\sigma, \Delta)$. Analogously as in the proof of Lemma 1 consider the equation

$$(4.12)\quad \Psi(x, \xi_k) = \sum_{j=0}^{k-1} (-1)^{k-j-1} \int_0^x \frac{(x-s)^{k-j-1}}{(k-j-1)!} \Big[a_j(s, \xi(u)(s)) + K_j u^{(j)}(s) - \frac{1}{\lambda} c_j(s, \eta(u)(s)) - \frac{1}{\lambda} d_j(s, \eta(u)(s), \tau(u)) - \frac{1}{\lambda} K_{k+j+1} u^{(j)}(s)\Big] ds - p_0 - p_1 x - \dots - p_{k-1} x^{k-1} ,$$

where $\Psi(x, \xi_k) = a_k(x, \eta(u)(x), \xi_k) + K_k \xi_k$.

In virtue of the compact imbedding from $\mathring{W}_p^k$ into C^{k-1} the right hand side in (4.12) is compact (we consider $u \in Q(\sigma, \Delta)$ and $\sigma \le |\lambda| \le \Delta$). From Implicit Function Theorem there exists a unique solution $\xi_{ku}(x)$ on the interval $\langle 0, 1 \rangle$ for each $u \in Q(\sigma, \Delta)$. There exists $M_0 > 0$ such that for each $|\lambda| \in \langle \sigma, \Delta \rangle$, $u \in Q(\sigma, \Delta)$ and $x \in \langle 0, 1 \rangle$ it is $|\xi_{ku}(x)| \le M_0$. From the unicity of the implicit function we have $\|u\|_{C^k} \le M_0$

for each $u \in Q(\sigma, \Delta)$.

Differentiating the equation (4.4) we obtain the equation

$$(4.13)\quad u^{(k+1)}(x) = \frac{1}{\frac{\partial^2 a}{\partial \xi_k^2}(x, \xi(u)(x)) + K_k}$$

$$\Big\{\frac{d}{dx}\Big(\sum_{j=0}^{k-1}(-1)^{k-j}\int_0^x \frac{(x-s)^{k-j-1}}{(k-j-1)!}[a_j(s, \xi(u)(s)) +$$

$$+ K_j u^{(j)}(s) - \frac{1}{\lambda} c_j(s, \eta(u)(s)) -$$

$$- \frac{1}{\lambda} d_j(s, \eta(u)(s), \tau(u)) - \frac{1}{\lambda} K_{k+j+1} u^{(j)}(s)]\, ds +$$

$$+ p_1 + 2p_2 + \ldots + (k-1)p_{k-1} x^{k-1}\Big\} .$$

The right hand side in (4.13) contains only the derivatives of the order $\leq k$. Thus there exists a constant $M_1 > 0$ such that $\|u\|_{C^{k+1}} \leq M_1$ for each $u \in Q(\sigma, \Delta)$. Repeating this consideration k-times we obtain $\|u\|_{C^{2k+1}} \leq$ $\leq M_2$ $(M_2 > 0)$ for each $u \in Q(\sigma, \Delta)$. Thus from the last estimation and the Arzela theorem we obtain that the set $Q(\sigma, \Delta)$ is a compact subset of C^{2k} .

In this section we verified all assumptions of the general result [5, Theorem 3.2] . From this follows the assertion of our Theorem 3.

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