Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 16 (1975), No. 1, 151--160

Persistent URL: http://dml.cz/dmlcz/105613

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16,1 (1975)

EPIMORPHISMS OF REGULAR RINGS

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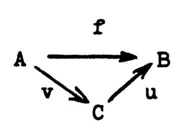
__Abstract__: It is proved that a ring epimorphism $f: A \longrightarrow B$, where A is (von Neumann) regular but does not necessarily have an identity, is surjective.

__Key words__: Epimorphism, von Neumann regular ring.

AMS: 16A30, 18A20 Ref. Ž.: 2.723.2

__Introduction__. In this paper an __epimorphism__ is a map h such that $fh = gh$ implies $f = g$. (See [1] for background.) We shall work in two categories: (i) $\mathcal{R}$, the category of (associative) rings and ring homomorphisms and (ii) $\hat{\mathcal{R}}$, the category of (associative) rings with identities and identity-preserving ring homomorphisms.

In both $\mathcal{R}$ and $\hat{\mathcal{R}}$ if f is an epimorphism, it has a factorization



where v is surjective, u is injective and both are epimorphisms. Thus in investigating the epimorphisms from a homomorphically closed class of rings, we are essentially interested only in the injective ones. Moreover, we clearly lose no generality by further specializing to the case where the map is an inclusion. If $f: A \longrightarrow B$ is such a map, we call B an _epimorphic extension_ of A.

We shall prove that all epimorphisms (in $\mathfrak{R}$) from (von Neumann) regular rings are surjective by showing that if A is a regular ring, B an epimorphic extension of A, then $A = B$. The analogous result in $\hat{\mathfrak{R}}$, which is actually a special case of out theorem, is known, and we give a short proof of this below. The generalization makes use of the Fuchs-Halperin theorem [3] concerning the embedding of a regular ring in a regular ring with identity.

Epimorphisms of rings have generally been discussed in $\hat{\mathfrak{R}}$. The non-unital versions of some standard results are given, in a somewhat condensed fashion, in Isbell's paper [4], to which we refer in the next section, though we have been a little more explicit on occasion in the interest of a more or less complete account.

1. _Preliminaries_. Let $R \subseteq S$ be rings. The _$\mathfrak{R}$-dominion_ of R in S is the set of elements d of R for which $f(d) = g(d)$ whenever f, g are homomorphisms from S such that $f(a) = g(a)$ for all $a \in R$. We shall denote this by $\mathrm{dom}(R,S)$. If $A \subseteq B$ are rings with (the same) identity, $\mathrm{dom}_1(A,B)$ denotes the $\hat{\mathfrak{R}}$ _-dominion_ of A in B (defined analogously).

Theorem 1.1. Let B be a ring with identity, A a subring with the same identity, $d \in B$. The following conditions are equivalent.

(i) $d \in \mathrm{dom}_1(A,B)$.

(ii) $1 \otimes_A d = d \otimes_A 1$ in $B \otimes_A B$.

(iii) $d = XPY$, where X is a $1 \times m$ matrix over B , P is an $m \times n$ matrix over A and Y is an $n \times 1$ matrix over B such that XP and PY are matrices over A .

Proof. (i) $\Longrightarrow$ (ii) is proved by a simple modification of a part of the proof of Proposition 1.1 in [6].

(ii) $\Longrightarrow$ (iii) is a special case of the lemma in [5].

(iii) $\Longrightarrow$ (i) can be obtained by a routine calculation when XPY is written in terms of the entries in X , P and Y .

For a ring R , let R^1 denote the ring obtained by adjoining an identity to R in the standard way.

Proposition 1.2. Let $R \subseteq S$ be rings. Then

$$\mathrm{dom}_1(R^1, S^1) = \mathrm{dom}(R,S)^1 \quad .$$

Proof. Let $f, g\colon S^1 \longrightarrow K$ be homomorphisms in $\hat{\mathfrak{R}}$ which coincide on R^1 , and let $\check{f}$, $\check{g}$ be their restrictions to S . If $d \in \mathrm{dom}(R,S)$, then

$$f(d) = \check{f}(d) = \check{g}(d) = g(d)$$

since $\check{f}$ and $\check{g}$ coincide on R . Hence $\mathrm{dom}(R,S)^1 \subseteq$ $\subseteq \mathrm{dom}_1(R^1, S^1)$. Conversely, if $(a,m) \in \mathrm{dom}_1(R^1, S^1)$ and $h, k\colon S \longrightarrow A$ are $\mathfrak{R}$-homomorphisms which agree on R , define $\hat{h}, \hat{k}\colon S^1 \longrightarrow A^1$ by $\hat{h}(s,n) = (h(s), n)$ and $\hat{k}(s,n) =$

$= (k(s), n)$. Then $\hat{h}$, $\hat{k}$ agree on R^1 so

$$(h(a), m) = \hat{h}(a,m) = \hat{k}(a,m) = (k(a), m)$$

whence $a \in dom(R,S)$ and so $dom_1(R^1, S^1) \subseteq dom(R,S)^1$.

Corollary 1.3. Let $R \subseteq S$ be rings. Then S is an epimorphic extension of R in $\mathfrak{R}$ if and only if S^1 is an epimorphic extension of R^1 in $\hat{\mathfrak{R}}$.

Theorem 1.4. Let $R \subseteq S$ be rings, $d \in S$. Then $d \in dom(R,S)$ if and only if it has a representation of the form

$$d = a + XPY$$

where $a \in R$, X is a $1 \times m$ matrix over S , P is an $m \times n$ matrix over R^1 and Y is an $n \times 1$ matrix over S such that XP and PY are matrices over R .

For the proof of this result see § 1 of [4].

Proposition 1.5. Let R be a ring with identity, S an epimorphic extension of R in $\mathfrak{R}$. Then

(i) S has an identity (namely the identity of R) and

(ii) S is an epimorphic extension of R in $\hat{\mathfrak{R}}$.

Proof. By Theorem 1.4, any $s \in S$ has the form $r + \sum_{i=1}^{m} \sum_{j=1}^{n} s_i r_{ij} u_j$, where $s_i, u_j \in S$, $r_{ij} \in R^1$ and r , $\sum_{i=1}^{m} s_i r_{ij}$, $\sum_{j=1}^{n} r_{ij} u_j \in R$ for all i, j . Let e

be the identity of R . Then

$$es = er + \sum_{j=1}^{n} e(\sum_{i=1}^{m} s_i r_{ij})u_j = r + \sum_{j=1}^{n} (\sum_{i=1}^{m} s_i r_{ij})u_j = s .$$

Similarly $se = s$. This proves (i); (ii) is straightforward.

The following result is referred to obliquely by several authors (e.g.[7]). We give a simple proof in which module-theoretic concepts do not intrude.

Theorem 1.6. Let A be a regular ring with identity, B an epimorphic extension in $\hat{\mathcal{R}}$. Then A = B .

Proof. Any element b of B has the form XPY , where the notation is as in (iii) of Theorem 1.1. By putting in zero rows and columns, we can arrange things so that P is a square matrix ($n \times n$, say) . The ring of such matrices is regular, so P = PTP for a suitable matrix T over A . But then b = XPY = (XP) T (PY) , where XP and PY are matrices over A , whence $b \in A$.

Corollary 1.7. If A is a regular ring with identity, it has no proper epimorphic extensions in $\mathcal{R}$.

2. The results. Henceforth we shall work entirely in $\mathcal{R}$, and "epimorphism" will always mean "epimorphism in $\mathcal{R}$ " . We shall call rings divisible, torsion, etc. if their additive groups have these properties. For a ring A , A_t , A_p , $d_p(A)$ are, respectively, the maximal torsion, p-primary and p-divisible ideals (where p is a prime). Ideals gene-

rally will be indicated by the symbol $\lhd$. The construction involved in the following result will be used several times.

Lemma 2.1. Let K be a commutative ring with identity, A a unital algebra over K . Let A_K be the ring defined on $A \oplus K$ (group direct sum) by

$$(a,k)\,(a', k') = (aa' + ka' + k'a, kk') \; .$$

Then $A \cong \{(a,0) \mid a \in A\} = I \lhd A_K$ and $A_K/I \cong K$. (When $K = Z$, $A_K = A^1$, of course.)

We have first to prove two special cases of our main result. We begin with divisible (or, equivalently, torsion-free) regular rings (see [2], § 124).

Proposition 2.2. Let A be a divisible regular ring, B an epimorphic extension of A . Then $A = B$.

Proof. By Theorem 1.4, any element b of B satisfies an equation

$$b = a + \sum_{i=1}^{m} b_i\, a_i \quad ,$$

where $a, a_i \in A$, $b_i \in B$ for $i = 1, \ldots, m$. For any positive integer n , there exist elements $a'', a_1'', \ldots, a_m'' \in A$ such that $na'' = a$, $na_1'' = a_1, \ldots, na_m'' = a_m$. Hence $b = n(a'' + \sum_{i=1}^{m} b_i\, a_i'')$ and B is divisible.

Now let $\bar{B} = B/B_t$, $\bar{A} = (A + B_t)/B_t \cong A/(A \cap B_t) = A/A_t \cong A$. In the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \| \wr & & \downarrow \\ \bar{A} & \longrightarrow & \bar{B} \end{array}$$

induced by the natural map $B \longrightarrow \bar{B}$, all maps are epimorphisms. Moreover, $\bar{A}$ and $\bar{B}$ are algebras over the field Q of rational numbers. If now f, g are homomorphism from $\bar{B}_Q$ which agree on $\bar{A}_Q$, their restrictions $\check{f}$, $\check{g}$ to $\bar{B}$ agree on $\bar{A}$, so $\check{f} = \check{g}$, i.e. f and g agree on $\bar{B}$. But f and g also agree on the copy of Q in $\bar{B}_Q$, so that $f = g$ and $\bar{B}_Q$ is therefore an epimorphic extension of $\bar{A}_Q$. By Corollary 1.7, $\bar{A}_Q = \bar{B}_Q$ and hence $\bar{A} = \bar{B}$. Thus $A + B_t = B$.

If $a \in A$ and $b \in B_t$, let b have order n . Then $a = na'$ for some $a' \in A$, so $ab = (na')b = a'(nb) = 0$ and $ba = b(na') = n(ba') = (nb)a' = 0$. Since also $A \cap B_t = 0$, we have $B = A \oplus B_t$ (ring direct sum). But B is an epimorphic extension of A , and the zero map and the natural projection from B to B_t agree on A . Hence $B_t = 0$ and $A = B$ as required.

We next consider p-primary regular rings. These must be p-elementary ([2], § 124) .

<u>Proposision 2.3</u>. Let A be a p-primary regular ring, B an epimorphic extension of A . Then $A = B$.

<u>Proof</u>. From Theorem 1.4 it's easily seen that $pB = 0$. Thus A and B are algebras over the field $F(p)$ with p elements. Arguing as in the previous proof, one now shows that $B_{F(p)}$ is an epimorphic extension of the regular ring

$A_{F(p)}$ with identity, whence $A_{F(p)} = B_{F(p)}$ and $A = B$.

The proof in the general case runs along similar lines, the essential problem being to describe the additive groups of epimorphic extensions of regular tings. Throughout the proofs of the following propositions, A always denotes a regular ring, B an epimorphic extension of A .

We note firstly that $A = A_p \oplus d_p(A)$ (ring direct sum) and $d_p(A) = pA$ (see [2], § 124).

<u>Proposition 2.4</u>. $B = A_p + pB$.

<u>Proof</u>. In the diagram

$$\begin{array}{ccccc} & & A & \longrightarrow & B \\ & & \downarrow & & \downarrow \\ & & (A+pB)/pB & \longrightarrow & B/pB \\ & & \wr\wr & & \\ A_p \cong A/pA & \longrightarrow & A/A\cap pB & & \end{array}$$

all maps (which are the obvious ones) are epimorphisms. Since A_p is a regular p-ring, we have $(A + pB)/pB = B/pB$. The result follows.

<u>Proposition 2.5</u>. $\mathrm{dom}(pA, B) = pB$.

<u>Proof</u>. If $b \in B$, then by Theorem 1.4

$$b = a' + a'' + \sum_{i,j} b_i\ (a'_{ij} + a''_{ij} + n_{ij})c_j$$

where a', a'_{ij} A_p, a'' , $a''_{ij} \in pA$, $n_{ij} \in Z$, b_i, $c_j \in B$. Hence $pb = pa'' + \sum_{i,j} b_i(pa''_{ij} + pn_{ij})c_j$. Using the other conditions of Theorem 1.4 (assumed to be satisfied by the given representation of b), it is straightforward to show that $pb \in dom(pA, B)$. On the other hand, it follows directly from Theorem 1.4 that $dom(pA, B) \subseteq pB$.

Proposition 2.6. $pB = d_p(B)$.

Proof. Let I be the ideal of B generated by pA . Then $pB = dom(pA, B) \subseteq I \subseteq pB$. Thus $I = pB$. But $I = $ $= pA + pAB + pBA + pBAB$ is p-divisible, since pA is. Thus pB is p-divisible, so we have $pB \subseteq d_p(B) \subseteq pB$.

Proposition 2.7. $B = A_p \oplus d_p(B)$ (ring direct sum).

Proof. By Propositions 2.4 and 2.6, $B = A_p + d_p(B)$. But $A_p\, d_p(B) = 0 = d_p(B)A_p$ (cf. the proof of Proposition 2.2). In particular, $A_p \cap d_p(B)$ is a nilpotent ideal of A_p. Hence $A_p \cap d_p(B) = 0$.

Proposition 2.8. $d_p(B)$ has no p-component.

Proof. Using Proposition 2.7, it's straightforward to show that $d_p(B) = pB$ is an epimorphic extension of $d_p(A) =$ $= pA$. Let $(pB)_T = \bigoplus_{q \neq p} (pB)_q$, $(pA)_T = \bigoplus_{q \neq p} (pA)_q$. Then as in previous similar situations, $(pB)/(pB)_T$ is an epimorphic extension of $((pA)+(pB)_T)/(pB)_T \cong (pA)/(pA)_T$. But the latter is torsion-free and regular, so by Proposition 2.2, so is $(pB)/(pB)_T$. This proves the proposition.

Corollary 2.9. $A_p = B_p$ and $B = B_p \oplus d_p(B)$ (ring direct sum).

We can now proceed as in §§ 124 and 125 of [2] to show that B is an algebra over the regular ring M of [3]. An argument like that in Proposition 2.2 shows that B_M is an epimorphic extension of A_M , whence A = B .

Thus we have proved

Theorem 2.10. All epimorphisms from regular rings are surjective.

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(Oblatum 3.12.1974)