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ON LOCAL AND GLOBAL MODULI OF CONVEXITY 1)

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Abstract: There is given a new proof of the equality of all "natural" definitions of local (global) moduli of convexity in normed linear spaces.

Key words: Normed linear space, local modulus of convexity, global modulus of convexity, monotone functions.

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The goal of the paper is to show that all "natural" definitions of local (global) moduli of convexity in normed linear spaces coincide. For global moduli of convexity this was shown by M.M. Day [2, Lemma 5.1] and for local moduli of convexity by Bui-Min-Či and V.I. Gurariĭ [1, Proposition 1] .

Bui-Min-Či and Gurariĭ's proof relies upon a lemma [1, Lemma 1] which coincides essentially with our Lemma 2. But their proof contains an inaccuracy (being a consequence of their picture 1). Indeed, they assert that, in our notation, the straight line $x(\varphi) + t(x(\psi(\varphi)) - x(\varphi))$, $t \in (-\infty,+\infty)$,

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1) The results of the paper are a part of the author's communication "Some remarks on nonlinear functional analysis" at the Summer School on "Nonlinear Functional Analysis and Mechanics", Stará Lesná, High Tatras, Czechoslovakia, Sept. 23-27 (1974).

intersects the half-line $tx(\psi(\varphi')), t \geq 1$, which is easily seen to be false; for notation see below.

Our proof of the equivalence of different definitions of local (global) moduli of convexity differs from that of Bui-Min-Či and Gurariĭ [1] (Day [2]).

We shall begin with the following lemma on real functions.

Lemma 1. Let $-\infty \leq \alpha < \beta \leq +\infty$ and f a real function defined on (α, β). If there exists an upper semi-continuous function $\psi : (\alpha, \beta) \longrightarrow (\alpha, \beta)$ such that:

(i) $\psi(t) < t$ for each $t \in (\alpha, \beta)$; and

(ii) $f(t') \geq f(t)$ for each $t \in (\alpha, \beta)$ and $t' \in (\psi(t), t)$,

then f is nonincreasing.

Proof. Let $t \in (\alpha, \beta)$ and set $s(t) = \min\{2t, \frac{t+\beta}{2}\}$, $\varepsilon(t) = -\sup\{\psi(t') - t' : t' \in [t, s(t)]\}$ and $\varkappa(t) = \min\{s(t), t + \varepsilon(t)\}$. Clearly $t < s(t) < +\infty$. As ψ is upper semi-continuous, we have $\varepsilon(t) > 0$ and $\varkappa(t) > t$. For each $t' \in (t, \varkappa(t))$, one has $\psi(t') \leq t' - \varepsilon(t) < t < t'$, so that $f(t') \leq f(t)$ by (ii). This and (ii) easily imply that f is nonincreasing on (α, β). The author thanks to J. Reif for a simplification of the original proof of the lemma.

Remark 1. Lemma 1 remains true with (α, β) replaced by $(\alpha, \beta]$ where $-\infty \leq \alpha < \beta < +\infty$.

Remark 2. If the condition (ii) of Lemma 1 is replaced by

(ii') $f(t') > f(t)$ for each $t \in (\alpha, \beta)$ and $t' \in (\psi(t), t)$,

then f is (strictly) decreasing on (α, β).

In the following lemmas and proposition, $(X, \|\cdot\|)$ is a two-dimensional real normed linear space, B its closed unit ball at O and S the boundary of B. For $\varepsilon \in [0,2]$ and $x \in S$ let us define $\delta(x,\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : y \in S, \|y - x\| = \varepsilon\}$ (the local modulus of convexity of X at x).

Lemma 2. Let x in S be given. Consider a Euclidean system of coordinates in X such that the origin of this system coincides with that of X and the half-line tx, $t>0$, is the positive ξ-axis. For $\varphi \in [0,2\pi)$, let $x(\varphi)$ in S be defined (uniquely) by the condition: $\arg x(\varphi) = \varphi$. Then:

(1) $\|x + x(\varphi)\|$ is a nonincreasing continuous function for $\varphi \in [0,\pi]$;

(2) $\|x + x(\varphi)\|$ is a nondecreasing continuous function for $\varphi \in [\pi,2\pi]$;

(3) $\|x - x(\varphi)\|$ is a nondecreasing continuous function for $\varphi \in [0,\pi]$;

(4) $\|x - x(\varphi)\|$ is a nonincreasing continuous function for $\varphi \in [\pi,2\pi]$.

Proof. From $x(\varphi+\pi) = -x(\varphi)$ ($\varphi \in [0,\pi]$) and from the symmetry it follows that it is sufficient to prove the assertion (1).

From the equivalence of $\|.\|$ and $|.|$, where $|.|$ is the Euclidean norm corresponding to the Euclidean system of coordinates we have fixed, it follows that $x(\varphi)$ is continuous (in $(X, \|.\|)$) for $\varphi \in [0,2\pi]$, i.e. $x(.) \in C([0,2\pi], (X, \|.\|))$. As $x + x(\varphi) \neq 0$ for each $\varphi \in [0,\pi)$, the function $\psi(\varphi) = \arg(x + x(\varphi))$ is continuous on $[0,\pi)$. As $x + x(\varphi)$ ($\varphi \in [0,\pi)$) lies in the half-space $\{(\xi,\eta) : \eta \geq 0\}$ and not on the negative ξ-axis, we have that

$\psi(.): [0,\pi) \longrightarrow [0,\pi)$. For $\varphi \in [0,\pi)$, let $A(\varphi)$ be the closed convex angle (cone) with vertex at 0 and generated by $\{x, x(\varphi)\}$ and $B(\varphi)$ the closed convex angle (cone) with vertex at $-x$ and generated by $\{x, x(\varphi)\}$.

If $0 \leq \varphi' \leq \varphi < \pi$, then $A(\varphi') \subset A(\varphi)$. From this and $0 \in (-x, x)$ it follows that $B(\varphi') \subset B(\varphi)$. As $A(\psi(\varphi)) = x + B(\varphi)$, $A(\psi(\varphi')) = x + B(\varphi')$, we have $A(\psi(\varphi')) \subset A(\psi(\varphi))$, so that $\psi(\varphi') \leq \psi(\varphi)$. As $0 \in (-x, x)$, we have, for $\varphi \in (0,\pi)$, $B(\varphi) + x = A(\psi(\varphi)) \subsetneqq A(\varphi)$, i.e., $\psi(\varphi) < \varphi$. Clearly $\psi(\varphi) > 0$ for $\varphi \in (0,\pi)$. Therefore $\psi(.)$ is a non-decreasing continuous function from $[0,\pi)$ into $[0,\pi)$ with $\psi(\varphi) < \varphi$ for each $\varphi \in (0,\pi)$ and $\psi(\varphi) = 0$ iff $\varphi = 0$.

Let $\varphi \in (0,\pi)$ and $\varphi' \in (\psi(\varphi), \varphi)$ be given. Then $x(\varphi') \in \text{Int}\,(A(\varphi) \setminus A(\psi(\varphi))) \cap S$, so that, by the convexity of B, $x(\varphi') \in H$, where H is the closed halfplane with $\partial H = \{x(\varphi) + t(x(\psi(\varphi)) - x(\varphi)) : t \in (-\infty, +\infty)\}$ and $0 \notin H$. From the convexity of B it follows that $x(\varphi') \in H$ and hence also $x(\varphi) + t(x(\varphi') - x(\varphi)) \in H$ for all $t \geq 0$. As $x(\psi(\varphi)) \in H$, we have $x(\psi(\varphi)) + t(x(\varphi') - x(\varphi)) \in H$ for all $t \geq 0$. From $x(\varphi), x(\psi(\varphi)) \in S$ we conclude that $\| x(\psi(\varphi)) + t(x(\varphi') - x(\varphi)) \| \geq 1$ for all $t \geq 0$. But $u \equiv (x + x(\varphi')) \| x + x(\varphi) \|^{-1} = x(\psi(\varphi)) + t_0(x(\varphi') - x(\varphi))$, where $t_0 = \| x + x(\varphi) \|^{-1}$ (because $x(\psi(\varphi)) = (x + x(\varphi)) \| x + x(\varphi) \|^{-1}$). Hence $\| u \| \geq 1$, i.e., $\| x + x(\varphi') \| \geq \| x + x(\varphi) \|$. Setting $(\alpha, \beta) = (0, \pi)$, $f(\varphi) = \| x + x(\varphi) \|$, we see that the hypotheses of Lemma 1 are satisfied, and hence the assertion (1) follows, because

$f(0) = 2 \geq f(\varphi) \geq f(\pi) = 0$ for all $\varphi \in [0,\pi]$.

Proposition 1. $\delta(x,\varepsilon)$ is a nondecreasing function of $\varepsilon \in [0,2]$ (for each $x \in S$).

Proof. Clearly $\delta(x,0) = 0 \leq \delta(x,\varepsilon)$ for all $\varepsilon \in [0,2]$. Let $0 \leq \varepsilon < \varepsilon' \leq 2$ with $\delta(x,\varepsilon') < 1$. Let $y' \in S$ be such that $\delta(x,\varepsilon') = 1 - \left\| \frac{x+y'}{2} \right\|$ (the existence follows at once from a continuity and compactness argument). Take a Euclidean coordinate system as in Lemma 2 such that $y' = x(\varphi')$ with $\varphi' \in (0,\pi)$ (this is possible because $\delta(x,\varepsilon') < 1$). Let $y = x(\varphi)$, $\varphi \in (0,\pi)$ be such that $\|x - y\| = \varepsilon$. As $\varepsilon' > \varepsilon$, we have by Lemma 2, (3), that $0 < \varphi < \varphi'$. By Lemma 2, (1), it follows that

$$\delta(x,\varepsilon') = 1 - \left\| \frac{x+y'}{2} \right\| = 1 - \left\| \frac{x+x(\varphi')}{2} \right\| \geq 1 - \left\| \frac{x+x(\varphi)}{2} \right\| \geq$$

$$\geq \delta(x,\varepsilon).$$

As $\delta(x,\varepsilon') < 1$ for any $\varepsilon' \in [0,2)$, the proposition follows.

Lemma 3. For each x in S and $\varepsilon \in [0,2]$ we have

$$\delta_2(x,\varepsilon) \equiv \inf\left\{1 - \left\| \frac{x+y}{2} \right\| : y \in S,\ \|x - y\| \geq \varepsilon \right\} =$$

$$= \delta(x,\varepsilon).$$

Proof. This follows at once from $\delta_2(x,\varepsilon) = \inf\{\delta(x,\varepsilon') : \varepsilon' \in [\varepsilon,2]\}$ and Proposition 1.

Lemma 4. Let x in S and y in $\mathrm{Int}(B)$ be given. Then there is a point z in S such that $\|y - x\| = \|z - x\|$ and $\|z + x\| \geq \|y + x\|$.

Proof. 1) Suppose that $y \notin (-x,x)$. Let $u, v \in S$ be defined by the following conditions: $u = -x + t(y + x)$ for

some $t > 0$ and $v = x + s(y - x)$ for some $s > 0$. Take a Euclidean coordinate system as in Lemma 2 such that $x = x(0)$ and $u = x(\omega)$ for some $\omega \in (0, \pi)$. Then $v = x(\nu)$ for some $\nu \in (\omega, \pi)$ (easy). Take $\varphi \in (0, \pi)$ such that $\| x - x(\varphi) \| = \| y - x \| \equiv \varepsilon$. As $\| x - x(\nu) \| > \| x - y \| = \varepsilon$, we have, by Lemma 2, (3), $\nu > \varphi$. Suppose that $\varphi \leq \omega$. Then, by Lemma 2, (1), $\| x + x(\varphi) \| \geq \| x + x(\omega) \| > \| x + y \|$, so that we can set $z = x(\varphi)$. Now suppose that $\omega < \varphi$. As $\varphi \in (\omega, \nu)$, $y \in \mathrm{Int}(T)$, where T is the (closed) triangle with vertices x, $-x$, and $x(\varphi)$. This implies that
$\| x - x(\varphi) \| + \| x(\varphi) - (-x) \| \geq \| x - y \| + \| y - (-x) \|$.
As $\| x - x(\varphi) \| = \| x - y \| = \varepsilon$, we have $\| x + x(\varphi) \| \geq \| x + y \|$, so that we may set $z = x(\varphi)$.

2) If $y \in (-x, x)$ and z in S is such that $\| z - x \| = \| y - x \|$, then $\| x - z \| + \| z + x \| \geq 2 \| x \| = 2 = \| y - x \| + \| y + x \|$, and hence z satisfies the assertion of the lemma.

Lemma 5. For $x \in S$ and $\varepsilon \in [0, 2]$,
$\delta(x, \varepsilon) = \inf \{ 1 - \| \frac{y + x}{2} \| : y \in B, \| y - x \| = \varepsilon \}$
$(\equiv \delta_3(x, \varepsilon)) = \inf \{ 1 - \| \frac{y + x}{2} \| : y \in B, \| y - x \| \geq \varepsilon \}$
$(\equiv \delta_4(x, \varepsilon))$.

Proof. It is clear that $\delta(x, \varepsilon) \geq \delta_3(x, \varepsilon) \geq \delta_4(x, \varepsilon)$. Let w in B be such that $\| w - x \| \geq \varepsilon$ and $\| w + x \| = \sup \{ \| y + x \| : y \in B, \| y - x \| \geq \varepsilon \}$. Let $\varepsilon' = \| w - x \|$ and take $z \in S$ (by Lemma 4) such that $\| z - x \| = \varepsilon'$ and $\| z + x \| \geq \| w + x \|$ (indeed, it is clear that $\| z + x \| = \| w + x \|$). Thus we have $\delta(x, \varepsilon) \leq \delta(x, \varepsilon') \leq$

$\leq 1 - \| \frac{z + x}{2} \| \leq 1 - \| \frac{w + x}{2} \| = \delta_4(x, \varepsilon).$

Lemma 6. If we define, for $\varepsilon \in [0,2]$,

$$\delta(\varepsilon) = \inf \{1 - \| \frac{x + y}{2} \| : x \in S,\ y \in S,\ \|x - y\| = \varepsilon\},$$

$$\delta_2(\varepsilon) = \inf \{1 - \| \frac{x + y}{2} \| : x \in S,\ y \in S,\ \|x - y\| \geq \varepsilon\},$$

$$\delta_3(\varepsilon) = \inf \{1 - \| \frac{x + y}{2} \| : x \in S,\ y \in B,\ \|x - y\| = \varepsilon\},$$

$$\delta_4(\varepsilon) = \inf \{1 - \| \frac{x + y}{2} \| : x \in S,\ y \in B,\ \|x - y\| \geq \varepsilon\},$$

$$\delta_5(\varepsilon) = \inf \{1 - \| \frac{x + y}{2} \| : x \in B,\ y \in B,\ \|x - y\| = \varepsilon\},$$

$$\delta_6(\varepsilon) = \inf \{1 - \| \frac{x + y}{2} \| : x \in B,\ y \in B,\ \|x - y\| \geq \varepsilon\},$$

then $\delta(\varepsilon) = \delta_2(\varepsilon) = \delta_3(\varepsilon) = \delta_4(\varepsilon) = \delta_5(\varepsilon) = \delta_6(\varepsilon)$.

Proof. It is clear that the following inequalities hold:

$$\begin{array}{ccccc} \delta(\varepsilon) & \geq & \delta_3(\varepsilon) & \geq & \delta_5(\varepsilon) \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\ \delta_2(\varepsilon) & \geq & \delta_4(\varepsilon) & \geq & \delta_6(\varepsilon). \end{array}$$

Therefore it is sufficient to prove that $\delta_6(\varepsilon) \geq \delta(\varepsilon)$. Let x and y in B be such that $\|x - y\| \geq \varepsilon$ and $\delta_6(\varepsilon) = 1 - \| \frac{x + y}{2} \|$. It is easy to see that $\{x,y\} \cap S \neq \emptyset$, i.e. $x \in S$ or $y \in S$. We may suppose that $x \in S$. Set $\varepsilon' = \|x - y\|$. Then $\delta_6(\varepsilon) = 1 - \| \frac{y + x}{2} \| \geq \delta_3(x, \varepsilon') = \delta(x, \varepsilon') \geq \delta(x, \varepsilon) \geq \delta(\varepsilon)$ (by Lemmas 3 and 5) and the lemma follows.

Let X be a normed linear space (over the field of real or complex numbers) of real dimension greater than one, B its unit (closed) ball at 0 and S the boundary of B. If $\delta_i(x, \varepsilon)$

and $\delta_i(\varepsilon)$ ($i = \emptyset$, 2,3,4, resp. $i = \emptyset$, 2,3,4,5,6) are defined as above for X of dimension two, then, clearly, $\delta_i(x,\varepsilon) = \inf \delta_i^Y(x,\varepsilon)$ and $\delta_i(\varepsilon) = \inf \delta_i^Y(\varepsilon)$, where Y runs over all two-dimensional real subspaces of X and $\delta_i^Y(x,.)$ and $\delta_i^Y(.)$ denote the corresponding $\delta_i(x,.)$ and $\delta_i(.)$ for Y.

From the above results we obtain at once the following

<u>Theorem.</u> Let X be a (real or complex) normed linear space of real dimension ≥ 2 and $x \in X$, $\|x\| = 1$. Then:

(1) $\delta(x,.)$ is a nondecreasing function;

(2) $\delta(x,.) = \delta_2(x,.) = \delta_3(x,.) = \delta_4(x,.)$;

(3) $\delta(.)$ is a nondecreasing function;

(4) $\delta(.) = \delta_2(.) = \delta_3(.) = \delta_4(.) = \delta_5(.) = \delta_6(.)$.

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